# Solutions Manual For
# ANALYSIS AND PERFORMANCE OF FIBER COMPOSITES

THIRD EDITION

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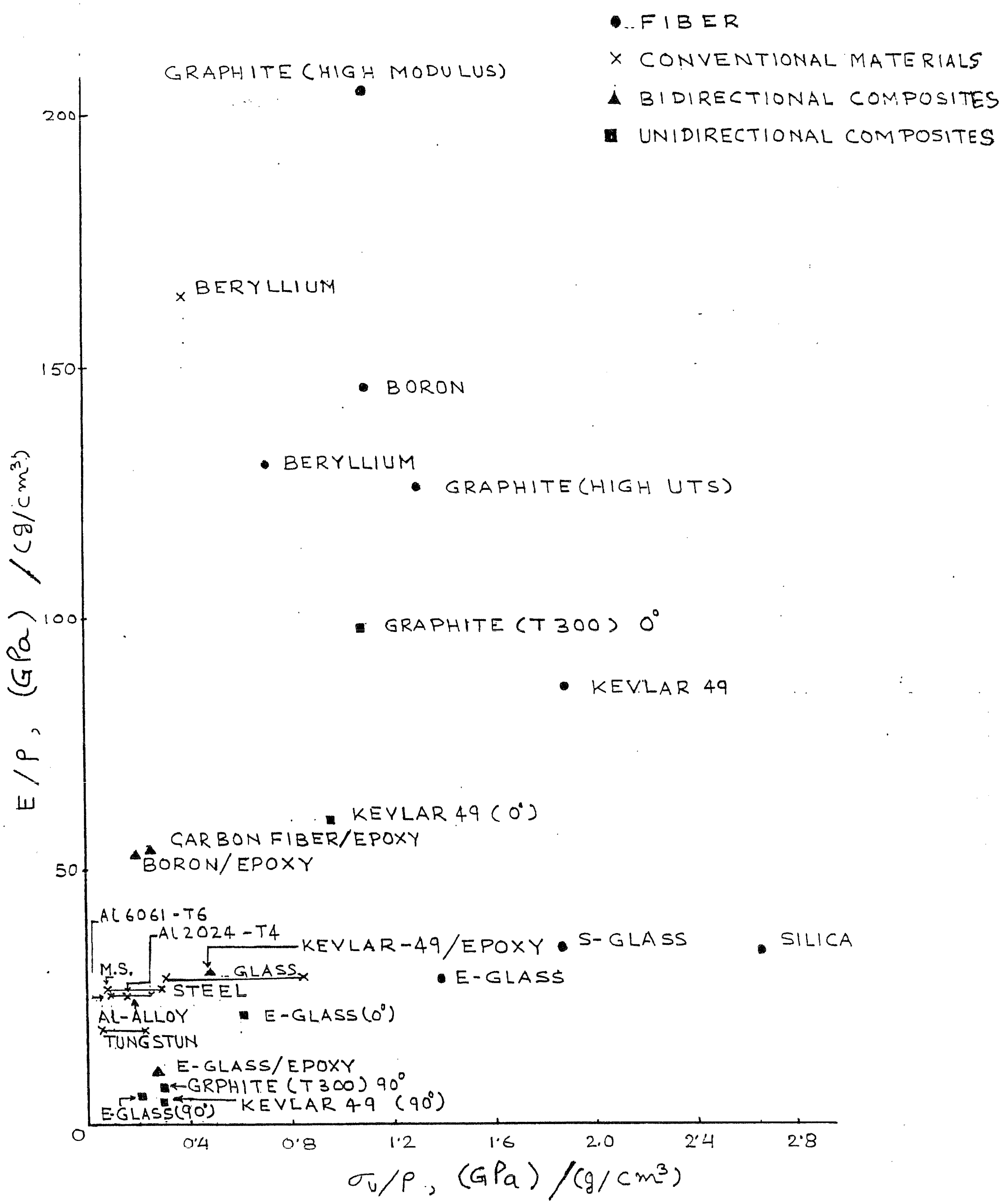
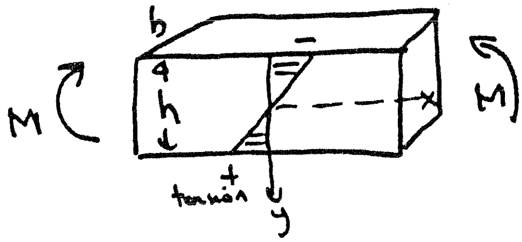
## Chapter 1

**1.3.**



● FIBER
× CONVENTIONAL MATERIALS
▲ BIDIRECTIONAL COMPOSITES
■ UNIDIRECTIONAL COMPOSITES
GRAPHITE (HIGH MODULUS)
BERYLLIUM
BORON
BERYLLIUM
GRAPHITE (HIGH UTS)
GRAPHITE (T 300) 0°
KEVLAR 49
KEVLAR 49 (0°)
CARBON FIBER/EPOXY
BORON/EPOXY
AL 6061-T6
AL 2024-T4
KEVLAR-49/EPOXY
S-GLASS
SILICA
M.S.
GLASS
STEEL
E-GLASS
AL-ALLOY
E-GLASS (0°)
TUNGSTUN
E-GLASS/EPOXY
GRPHITE (T 300) 90°
KEVLAR 49 (90°)
E-GLASS (90°)
0
50
100
150
200
0.4
0.8
1.2
1.6
2.0
2.4
2.8
E/ρ, (GPa) /(g/cm³)
σᵤ/ρ, (GPa) /(g/cm³)

1.4




$\sigma_x = My/I, \; I = \frac{1}{12} bh^3$

$\sigma_{max} = M\frac{h}{2}/I = 6\,M/bh^2$

(a-1) "EI" of steel beam = 210 GPa × $\frac{1}{12}$ × 0.1 × $0.006^3\,m^4$

= 378 N.m$^2$

"EI" of 2024-T4 Aluminum beam

= 73 GPa × ($\frac{1}{12}$ × 0.1 × $h^3\,m^4$)

= 378 Nm$^2$

h = beam thickness for equivalent "EI"

= 8.5 mm

(a-2) Bending moment on steel beam:

$(\sigma_{ULT})_{steel} = 6\,M/bh^2$

M = (0.83 GPa × .1 × $.006^2$)$m^3$/6 = 498 Nm

Aluminum beam:

$(\sigma_{ULT})_{Al} = 0.41\,GPa = \frac{6\,M}{0.1h^2} = \frac{6 \times 498\,Nm}{0.1h^2\,m}$

Beam thickness for equivalent strength:

h = 8.54 × $10^{-3}$m = 8.54 mm

(b-1) Weight of steel beam per unit length

= b h $\rho$ = 10 cm × 0.6 cm × 7.8 gm/cm$^3$

= 46.8 gm/cm

Weight of Aluminum beam for equivalent "EI"

= bh [from part (a-1)] × $\rho_{Al}$

= 10 × 0.85 × 2.7

= 23.0 gm/cm

(b-2) Weight of steel beam = 46.8 gm/cm

Weight of aluminum beam with equivalent strength

$= b\,h$ [ from part (a-2)] $\times \rho_{Al}$

$= 10 \times 0.854 \times 2.7$

$= 23.1$ gm/cm

(c-1) Weight of steel beam = 46.8 gm/cm

Weight of Aluminum beam = 46.8 gm/cm = $b \cdot h \times \rho_{Al}$

$h = 46.8/10 \times 2.7 = 1.73$ cm

"EI" of Aluminum beam with same weight as steel beam

$$EI = 73\,(GPa) \times \frac{1}{12} \times (0.1\text{ m}) \times (0.0173)^3$$

$$= 2.99 \times 10^{-6}\text{ GN - m}^2$$

$$= 299\text{ N m}^2$$

(c-2) Steel Beam:

$$\text{Bending Moment} = M = \frac{1}{6}\,\sigma_{ULT}\,b\,h^2$$

$$= \frac{1}{6} \times .83\text{ GPa} \times 0.1 \times 0.006^2\text{m}^3$$

$$= 498\text{ N m}$$

Bending Moment for aluminum beam with same weight as steel beam

$$M = \frac{1}{6} \times (\sigma_{ULT})_{Al} \times b \times h \text{ [from part (c-1)]}$$

$$= \frac{1}{6} \times .41 \times 0.1 \times 0.017^2\text{ GN-m}$$

$$= 1970\text{ N-m}$$

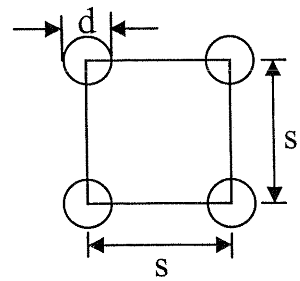
| Material | (a-1) h (mm) Stiffness | (a-2) h (mm) Strength | (b-1) wt/unit length (Normalized) Stiffness | (b-2) wt/unit length (Normalized) Strength | (c-1) EI Normalized | (c-2) Moment Normalized |
|---|---|---|---|---|---|---|
| Steel | 6 | 6 | 1 | 1 | 1.0 | 1.0 |
| Al-2024-T4 | 8.5 | 8.5 | .49 | .49 | 7.9 | 4.0 |
| Al-6061-T6 | 8.7 | 10.7 | .50 | .62 | 7.5 | 2.5 |
| E-glass epoxy ($\varphi_f = .57$) | 12.8 | 7.2 | .54 | .30 | 6.6 | 11.0 |
| Kevlar-49/epoxy ($\varphi_f = .6$) | 10.4 | 6.8 | .31 | .20 | 31.7 | 23.7 |
| Carbon fiber/epoxy ($\varphi_f = .58$) | 8.2 | 8.9 | .27 | .29 | 49.5 | 11.4 |
| Boron fiber/epoxy ($\varphi_f = .60$) | 7.5 | 8.9 | .32 | .38 | 28.3 | 6.73 |

# Chapter 3

**3.1.**

$$V_f = \frac{A_f}{A}$$

Square array

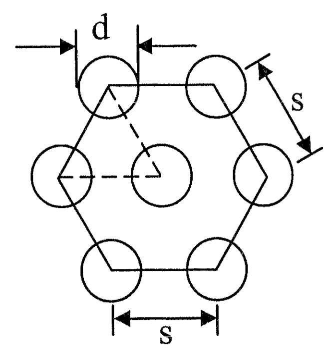



$$A_f = 4\left(\frac{1}{4}\frac{\pi d^2}{4}\right) = \frac{\pi d^2}{4}$$

$$A = s^2$$

$$V_f = \frac{\pi d^2}{4s^2}$$

Hexagonal array

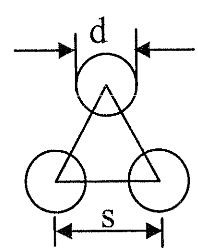



$$A_f = \frac{\pi d^2}{4} + 6\left(\frac{1}{3}\frac{\pi d^2}{4}\right) = \frac{\pi d^2}{4} + \frac{\pi d^2}{2} = \frac{3}{4}\pi d^2$$

$$A = 6\left(\frac{1}{2}\frac{\sqrt{3}\,s}{2}\,s\right) = 6\left(\frac{\sqrt{3}}{4}\,s^2\right) = \frac{3\sqrt{3}}{2}s^2$$

$$V_f = \frac{\pi d^2}{2\sqrt{3}\,s^2}$$

Alternatively, consider triangular array




$$A_f = 3 \text{ x } \frac{1}{6}\frac{\pi d^2}{4} = \frac{\pi d^2}{8}$$

$$A = \frac{1}{2}s\left(\frac{\sqrt{3}\,s}{2}\right) = \frac{\sqrt{3}}{4}s^2$$

$$V_f = \frac{\pi d^2}{2\sqrt{3}\,s^2}$$

When adjacent fibers touch each other, $s = d$

$$V_{fmax} = \frac{\pi}{4} \text{ for square array} = 78.5\%$$

$$= \frac{\pi}{2\sqrt{3}} \text{ for hexagonal array} = 90.7\ \%$$

3.2 $$W_f = \frac{49.4476 - 47.6504}{50.1817 - 47.6504} = 0.71$$

$$W_m = 1 - 0.71 = 0.29$$

$$\rho_c = \frac{1}{\frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}} = \frac{1}{\frac{0.71}{2.5} + \frac{0.29}{1.2}} = 1.902 \text{ g/cm}^3$$

$$V_f = \frac{\rho_c}{\rho_f} W_f = \frac{1.902}{2.5} \times 0.71 = 0.54$$

$$V_m = \frac{1.902}{1.2} \times 0.29 = 0.46$$

**3.3.**

$$v_c = v_f + v_m + v_v = \frac{w_c}{\rho_{ce}} \qquad (1)$$

$$v_f + v_m = \frac{w_c}{\rho_{ct}} \qquad (2)$$

(1) – (2) gives:

$$v_v = \frac{w_c}{\rho_{ce}} - \frac{w_c}{\rho_{ct}}$$

$$= \frac{w_c}{\rho_{ce}}\left(1 - \frac{\rho_{ce}}{\rho_{ct}}\right)$$

$$= v_c\left(\frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}}\right)$$

$$\boxed{V_v = \frac{v_v}{v_c} = \frac{\rho_{ct} - \rho_{ce}}{\rho_{ct}}}$$

3.4 From Problem 2, $\rho_{ct} = 1.902$

From Equation 3.8, $V_v = \dfrac{1.902 - 1.86}{1.902} = 0.0221$ or 2.21%

3.5 $\dfrac{\sigma_f}{\sigma_m} = \dfrac{E_f}{E_m} = \dfrac{400}{3.2} = 125$ for all $V_f$

| $V_f$ | $E_c$ (GPa) | $\dfrac{\sigma_f}{\sigma_c} = \dfrac{E_f}{E_c}$ |
|---|---|---|
| 10 | 42.88 | 9.33 |
| 25 | 102.40 | 3.91 |
| 50 | 201.60 | 1.98 |
| 75 | 300.80 | 1.33 |

3.6 The following values are obtained by using rule of mixtures for $E_L$ and $\upsilon_{LT}$ and the Halpin-Tsai equations for $E_T$ and $G_{LT}$:

| SYSTEM | $V_f$ (%) | Property | | | |
|---|---|---|---|---|---|
| | | $E_L$ (GPa) | $E_T$ (GPa) | $G_{LT}$ (GPa) | $\nu_{LT}$ |
| Glass/Epoxy | 25 | 20.13 | 6.39 | 2.07 | 0.313 |
| | 50 | 36.75 | 11.48 | 3.48 | 0.275 |
| | 75 | 53.38 | 22.81 | 6.96 | 0.238 |
| Graphite/Epoxy | 25 | 65.13 | 6.81 | 2.13 | 0.313 |
| | 50 | 126.75 | 13.18 | 3.76 | 0.275 |
| | 75 | 188.38 | 30.41 | 8.36 | 0.238 |
| Kevlar/Epoxy | 25 | 37.63 | 6.67 | 2.11 | 0.313 |
| | 50 | 71.75 | 12.60 | 3.67 | 0.275 |
| | 75 | 105.88 | 27.59 | 7.88 | 0.238 |
| Boron/Aluminum | 25 | 140.00 | 105.00 | 37.37 | 0.298 |
| | 50 | 210.00 | 154.00 | 54.30 | 0.265 |
| | 75 | 280.00 | 227.50 | 83.49 | 0.233 |

3.7

$$\rho_c = \frac{1}{\frac{0.35}{1.3} + \frac{0.45}{2.5} + \frac{0.20}{1.6}} = 1.74$$

Volume fractions:

$$V_m = \frac{1.74}{1.3} \times 0.35 = 0.47$$

$$V_{fA} = \frac{1.74}{2.5} \times 0.45 = 0.31$$

$$V_{fB} = \frac{1.74}{1.6} \times 0.30 = 0.22$$

Fracture strains:

$$\varepsilon_m = \frac{0.06}{3.5} \times 100 = 1.71\%$$

$$\varepsilon_{fA} = \frac{1.4}{70} \times 100 = 2.0\%$$

$$\varepsilon_{fB} = \frac{0.45}{6} \times 100 = 7.5\%$$

Failure sequence will be binder, fiber A, fiber B.

(a) Composite stress at fracture strain of binder:

$$\sigma_c = 0.0171\ (3.5 \times 0.47 + 70 \times 0.31 + 6 \times 0.22) = 0.422\ \text{GPa}$$

$$P_c = 0.422 \times 10 \times 10^{-4} = 0.422 \times 10^{-3}\ \text{GN} = 0.422\ \text{MN}$$

(b) Composite stress and load at fracture strain of fibers A:

$$\sigma_c = 0.02\ (70 \times 0.31 + 6 \times 0.22) = 0.46\ \text{GPa}$$

$$P_c = 0.46 \times 10 \times 10^{-4}\ \text{GN} = 0.46\ \text{MN}$$

Composite stress and load at fractue strain of fiber B:

$$\sigma_c = 0.075 \times 6 \times 0.22 = 0.099\ \text{GPa}$$

$$P_c = 0.099 \times 10 \times 10^{-4}\ \text{GN} = 0.099\ \text{MN}$$

Thus, the maximum load carried by the rod = 0.46 MN

(c) Fiber B will rupture last.

(d) Load maintained test

Strain in the composite required to maintain a load of 0.422 MN after failure of

$$\text{binder} = \frac{0.422 \times 100}{70 \times 0.31 + 6 \times 0.22} = 1.83\%$$

Strain in the composite required to maintain a load of 0.46 MN after failure of

$$\text{binder and fiber A} = \frac{0.46 \times 100}{6 \times 0.22} = 34.05\%$$

In view of the fracture strain of fiber B, the rod will fracture at a strain equal to 7.5%

Elongation maintained test

Load on the rod at a strain of 1.71% after failure of binder = 0.0171 (70 x 0.31 + 6 x 0.22) x 10 x $10^{-4}$ GN = 0.394 MN.

Load on the rod at a strain of 2.0% after failure of binder and fiber A = 0.02 x 6 x 0.22 x 10 x $10^{-4}$ GN = 0.0264 MN.

3.8 (a) $\sigma_c = \sigma_f V_f + \sigma_m V_m = \frac{1}{2}(\sigma_f + \sigma_m)$

$\varepsilon = 1\%$ $\quad \sigma_f = E_f \varepsilon_f = 70 \times 10^3 \times 0.01 = 700$ MPa

$\sigma_{mA} = \sigma_{mB} = 17.5$ MPa

Composite stress: $(\sigma_c)_A = (\sigma_c)_B = 1/2\ (700 + 17.5) = 358.75$ MPa

$\varepsilon = 4\%$ $\quad \sigma_f = 70 \times 10^3 \times 0.04 = 2{,}800$ MPa

$\sigma_{mA} = 35$ M

$\sigma_{mB} = 35 + (70-35) \times \frac{4-2}{10-2} = 43.75$ MPa

$(\sigma_c)_A = 1/2\ (2800 + 35) = 1417.5$ MPa

$(\sigma_c)_B = 1/2\ (2800 + 43.75) = 1421.875$ MPa

(b) $$V_{crit} = \frac{\sigma_{mu} - (\sigma_m)_{\varepsilon_f^*}}{\sigma_{fu} - (\sigma_m)_{\varepsilon_f^*}}$$

$$V_{min} = \frac{\sigma_{mu} - (\sigma_m)_{\varepsilon_f^*}}{\sigma_{fu} + \sigma_{mu} - (\sigma_m)_{\varepsilon_f^*}}$$

$\varepsilon_f^* = \frac{28}{700} \times 100 = 4\%$

For composite A: $\sigma_{mu} - (\sigma_m)_{\varepsilon_f^*} = 0$

Thus, $V_{crit} = V_{min} = 0$

For composite B: $\sigma_{mu} = 70$ MPa

$$(\sigma_m)_{\epsilon_{f*}} = 43.75 \text{ MPa}$$

$$V_{crit} = \frac{70 - 43.75}{2800 - 43.75} = 0.0095 = 0.95\%$$

$$V_{min} = \frac{70 - 43.75}{2800 + 70 - 43.75} = 0.0093 = 0.93\%$$

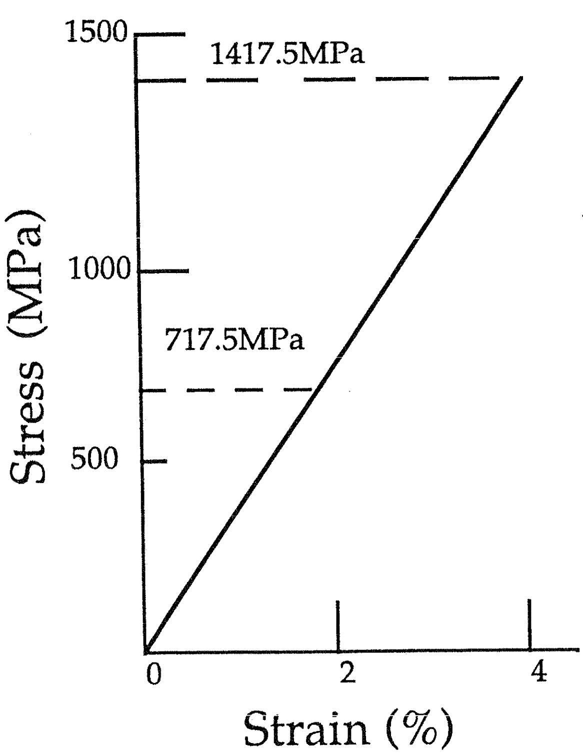
3.9 $\underline{\varepsilon = 2\%}$

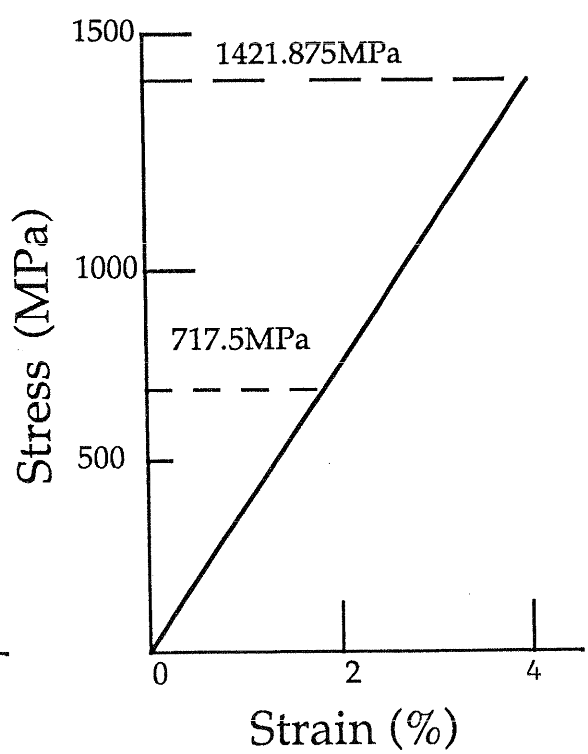
$$(\sigma_c)_A = (\sigma_c)_B = \frac{1}{2}(70 \times 10^3 \times 0.02 + 35) = 717.5 \text{ MPa}$$

$\underline{\varepsilon = 4\%\text{ (At Fracture)}}$

$$(\sigma_c)_A = 1417.5 \text{ MPa}$$

$$(\sigma_c)_B = 1421.875 \text{ MPa}$$



3.10 <u>*High modulus fiber* ($E_f = 390$ *GPa*, $\sigma_{fu} = 2100$ *MPa*)</u>

$$70 = 390\, V_f + 3.5\,(1 - V_f)$$

$V_f = 0.172$ or 17.2%

$\rho_c = 0.172 \times 1.90 + 0.828 \times 1.2 = 1.3204$

$$\text{Weight savings} = \frac{2.7 - 1.3204}{2.7} \times 100 = 51.1\%$$

<u>*High tensile strength fiber* ($E_f = 240$ *GPa*, $\sigma_{fu} = 2500$)</u>

$$70 = 240\, V_f + 3.5\,(1 - V_f)$$

$V_f = 0.281$ *or* 28.1%

$\rho_c = 0.281 \times 1.90 + 0.719 \times 1.2 = 1.3967$

$$\text{Weight savings} = \frac{2.7 - 1.3967}{2.7} \times 100 = 48.3\%$$

3.11 <u>High modulus fibers</u>

$\sigma_{cu} = V_f\ \sigma_{fu} = 0.172 \times 2100 = 361.2$ MPa

<u>High tensile strength fibers</u>

$\sigma_{cu} = 0.281 \times 2500 = 702.5$ MPa

3.12 $70 = 130\, V_f + 3.5\,(1 - V_f)$

$V_f = 0.526$ *or* 52.6%

$\rho_c = 0.526 \times 1.5 + 0.474 \times 1.2 = 1.3578$

$$\text{Weight savings} = \frac{2.7 - 1.3578}{2.7} \times 100 = 49.7\%$$

$\sigma_{cu} = 0.526 \times 2800 = 1472.8$ MPa

**3.13.**

Cross-sectional area of the steel rod, $A_s$, that can carry a load of 2000 N:

$$A_s = \frac{2{,}000}{450} = 4.44 \text{ mm}^2$$

Assume the following properties of graphite fibers and epoxy resin (From Table 2-3 and 2-11, respectively):

$$\sigma_{fu} = 2{,}200 \text{ MPa}$$
$$\rho_f = 1.8 \text{ g/cc}$$
$$\rho_m = 1.2 \text{ g/cc}$$

If the load carried by the epoxy matrix is neglected, the strength of the composite red becomes:

$$\sigma_{cu} = 0.65 \text{ x } 2{,}200 = 1{,}430 \text{ MPa}$$

Cross-sectional area of the composite rod, $A_c$, that can carry a load of 2000 N:

$$A_s = \frac{2{,}000}{1{,}430} = 1.4 \text{ mm}^2$$

Specific gravity of composite

$$\rho_c = 0.65 \text{ x } 1.8 + 0.35 \text{ x } 1.2 = 1.59$$

Weight and cost of the two rods can now be compared as follows:

$$\text{Weight ratio} = \frac{\rho_s A_s}{\rho_c A_c} = \frac{7.8 \text{ x } 4.44}{1.59 \text{ x } 1.4} = 15.56$$

$$\text{Cost ratio} = \frac{\rho_s A_s}{5\rho_c A_c} = \frac{15.56}{5} = 3.11$$

Thus, for both criteria, composite rod will be chosen.

3.14 <u>Glass/Epoxy</u>

$V_m = 60\%$, $V_f = 40\%$

$$\sigma_{cu} = 700 \times 0.4 + \frac{700}{70 \times 10^3} \times 3.5 \times 10^3 \times 0.6 = 301 \text{ MPa}$$

<u>Glass/Carbon/Epoxy</u>

Glass/epoxy systems acts like the matrix material.

For strengthening,

$$V_{crit} = \frac{\sigma_{mu} - (\sigma_m)_{\varepsilon_f^*}}{\sigma_{fu} - (\sigma_m)_{\varepsilon_f^*}}$$

$$\varepsilon_f^* = \frac{700 \times 10^{-3}}{350} = 2 \times 10^{-3}$$

$\sigma_{fu} = 700$ MPa
$\sigma_{mu} = 301$ MPa

$$(\sigma_m)_{\varepsilon_f^*} = E_{glass/epoxy} \cdot \varepsilon_f^*$$

$$= (3.5 \times 10^3 \times 0.6 + 70 \times 10^3 \times 0.4) \times 2 \times 10^{-3}$$

$$= 60.2 \text{ MPa}$$

$$V_{crit} = \frac{301 - 60.2}{700 - 60.2} \times 100 = 37.6\%$$

3.15 <u>Glass/Epoxy</u>

$E = 3.5 \times 0.3 + 70 \times 0.7 = 50.05$ GPa

$$\sigma_{cu} = 700 \times 0.7 + \frac{700}{70 \times 10^3} \times 3.5\ 10^3 \times 0.3 = 500.5 \text{ MPa}$$

<u>Carbon/Glass/Epoxy</u>

$E = 2 \times 50.05 = 3.5 \times 0.3 + 70 \times (0.7 - V_{cf}) + 350\ V_{cf}$

$$V_{cf} = \frac{50.05}{350 - 70} = 17.9\%$$

Composite stress at fracture of carbon fibers ($\varepsilon_f$) = $2 \times 10^{-3}$

$$\sigma_c = 700 \times 0.179 + 70 \times 10^3 \times 2 \times 10^{-3} (0.7 - 0.179) + 3.5 \times 10^3 \times 2 \times 10^{-3} \times 0.3$$
$$= 200.34 \text{ MPa}$$

Composite stress at fracture of glass fibers $\left(\varepsilon = \frac{700 \times 10^{-3}}{70} = 10^{-2}\right)$

$$\sigma_c = 70 \times 10^3 \times 10^{-2} \times (0.7 - 0.179) + 3.5 \times 10^3 \times 10^{-2} \times 0.3$$
$$= 375.2 \text{ MPa}$$
$$\sigma_{cu} = 375.2 \text{ MPa}$$

3.16

$$\varepsilon'_{LU} = \frac{0.0036}{0.25} = 0.0144$$
$$\sigma'_{LU} = 40 \times 10^3 \times 0.0144 = 576 \text{ MPa}$$

3.17 Combining Equations 3.39 and 3.40

$$\frac{\sigma_{TU}}{\sigma_{mU}} = \frac{1}{SCF} = \frac{1 - (4 V_f/\pi)^{1/2} (1 - E_m/E_f)}{1 - V_f (1 - E_m/E_f)} = \frac{1 - 1.072 V_f^{1/2}}{1 - 0.95 V_f}$$

Combining Equations 3.39 and 3.41

$$\frac{\sigma_{TU}}{\sigma_{mU}} = \frac{1}{SMF} = 1 - (4 V_f/\pi)^{1/2} (1 - E_m/E_f) = 1 - 1.072 V_f^{1/2}$$

| $V_f$ | $\sigma_{TU}/\sigma_{mu}$ | |
|---|---|---|
| | $1/SCF$ | $1/SMF$ |
| 0 | 1 | 1 |
| 0.1 | 0.730 | 0.661 |
| 0.2 | 0.643 | 0.521 |
| 0.3 | 0.577 | 0.413 |
| 0.4 | 0.519 | 0.322 |
| 0.5 | 0.461 | 0.242 |
| 0.6 | 0.395 | 0.170 |
| 0.7 | 0.308 | 0.103 |
| 0.8 | 0.172 | 0.041 |
| 0.87 | 0 | 0 |

3.18 Equation 3.67 can be written as

$$\alpha_L = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m}$$

After substitution of values $\alpha_f$, $\alpha_m$, $E_f$ and $E_m$, the equation can be written in a simpler form as

$$\alpha_L = \frac{9 + V_f}{1 + 19 V_f} \times 10^{-5}$$

Substitution of values of $\alpha_f$, $\alpha_m$, $\nu_f$, $\nu_m$ in equation 3.68 gives

$$\alpha_T = [12.15 - 11.55 V_f - (0.35 - 0.15 V_f) \alpha_L] \times 10^{-5}$$

Equation 3.69 gives

$$\alpha_T = (12.15 - 11.65 V_f) \times 10^{-5}$$

| $V_f$ | $\alpha_L$ (Eq. 3.67) $10^{-5}/°C$ | $\alpha_T$ (Eq. 3.68) $10^{-5}/°C$ |
|---|---|---|
| 0 | 9.0 | 9.0 |
| 0.05 | 4.64 | 9.98 |
| 0.10 | 3.14 | 9.94 |
| 0.20 | 1.92 | 9.23 |
| 0.30 | 1.39 | 8.26 |
| 0.40 | 1.09 | 7.21 |
| 0.50 | 0.91 | 6.13 |
| 0.60 | 0.77 | 5.02 |
| 0.70 | 0.68 | 3.90 |
| 0.80 | 0.61 | 2.77 |
| 0.90 | 0.55 | 1.64 |
| 1.00 | 0.50 | 0.50 |

3.19 Glass fiber/epoxy composite

$$K_L = 1.05\,V_f + 0.25\,(1 - V_f)$$

or $K_L = 0.25 + 0.80\,V_f$

$\xi$ and $\eta$ are same as in example 3.6. Therefore,

$$\frac{K_T}{K_m} = \frac{1 + 0.615\,V_f}{1 - 0.615\,V_f}$$

Carbon fiber/epoxy composite

$$K_L = 80\,V_f + 0.25\,(1 - V_f)$$

or $K_L = 0.25 + 79.75\,V_f$

$$\frac{K_T}{K_m} = \frac{1 + 0.961\,V_f}{1 - 0.961\,V_f}$$

| $V_f$ | $K_T$ (W/m °C) | |
|---|---|---|
| | Glass/epoxy | Carbon/epoxy |
| 0 | 0.250 | 0.25 |
| 0.1 | 0.283 | 0.303 |
| 0.2 | 0.320 | 0.369 |
| 0.3 | 0.363 | 0.452 |
| 0.4 | 0.413 | 0.562 |
| 0.5 | 0.472 | 0.712 |
| 0.6 | 0.543 | 0.931 |
| 0.7 | 0.628 | 1.278 |
| 0.8 | 0.735 | 1.913 |
| 0.9 | 0.871 | 3.451 |
| 1.0 | 1.050 | 12.500 |

3.20

$$\alpha_L = \frac{1}{E_L}(\alpha_f\ E_f\ V_f + \alpha_m\ E_m\ V_m)$$

$$E_L = 294\ x\ 0.55 + 3.5\ x\ 0.45 = 163.3$$

$$-0.61\ x\ 10^{-6} = \frac{1}{163.3}(\alpha_f\ x\ 294\ x\ 0.55 + 54\ x\ 10^{-6}\ x\ 3.5\ x\ 0.45)$$

$$\alpha_f = -1.14\ x\ 10^{-6}\ /°C$$

3.21 If fibers do not absorb moisture, the moisture in composite is ($W_m \cdot C_{matrix}$).

$$W_m = \frac{\rho_m}{\rho_c} V_m = \frac{1.2}{1.6}\ x\ 0.3 = 0.225$$

Maximum moisture in composite = 0.225 x 6
= 1.35%

# CHAPTER 4

4.1 For a continuous fiber composite:

$$E_c = E_f V_f + E_m V_m = 0.4 \times 70 + 0.6 \times 3.5 = 30.1 \text{ GPa}$$

For matrix:

Yield stress in shear = 1/2 x yield stress in tension

$$\therefore \quad \tau_y = 14 \text{ MPa}$$

(a) $(\sigma_f)_{max} = \frac{E_f}{E_c} \sigma_c$

$\sigma_c = 70$ MPa

$$(\sigma_f)_{max} = \frac{70}{30.1} \times 70 = 162.8 \text{ MPa}$$

$$\frac{\ell_t}{d} = \frac{162.8}{2 \times 14} = 5.814$$

$$\ell_t = 5.814 \times 0.03 = 0.1744 \text{ mm}$$

$\sigma_c = 210$ MPa

$$(\sigma_f)_{max} = \frac{70}{30.1} \times 210 = 488.4 \text{ MPa}$$

$$\frac{\ell_t}{d} = \frac{488.4}{2 \times 14} = 17.44$$

$$\ell_t = 17.44 = 0.5232 \text{ mm}$$

(b) $\sigma_c = 70$ MPa

Using equation .10 for calculating average stress:

$$\ell = 1/2 \quad \ell_t; \quad \frac{\ell}{d} = 1/2 \times 5.814 = 2.907$$

$$\overline{\sigma}_f = \frac{\tau_y \ell}{d} = 14 \times 2.907 = 40.7 \text{ MPa}$$

$\ell = 4\ \ell_t;\ \frac{\ell_t}{2\ell} = \frac{1}{8} = 0.125;\ (\sigma_f)_{max} = 162.8\ MPa$

$\overline{\sigma}_f = (\sigma_f)_{max}\ (1 - \frac{\ell_t}{2\ell}) = 162.8\ (1 - 0.125) = 142.45\ MPa$

$\underline{\sigma_c = 210\ MPa}$

$\ell = \frac{1}{2}\ \ell_t;\ \frac{\ell}{d} = 1/2 \times 17.44 = 8.72$

$\overline{\sigma}_f = 14 \times 8.72 = 122.1\ MPa$

$\ell = 4\ \ell_t;\ \frac{\ell}{2\ell} = 0.125;\ (\sigma_f)_{max} = 488.4\ MPa$

$\overline{\sigma}_f = 488.4\ (1 - 0.125) = 427.35\ MPa$

(c) Strain varies linearly from 0 at the fiber ends to a maximum value at the middle of the fiber since fiber length is less than or equal to load transfer length. Maximum strain can be calculated as follows:

$\underline{\sigma_c = 70\ MPa}$

$\ell = \ell_t \qquad (\epsilon_f)_{max} = \frac{162.8}{70 \times 10^3} = 0.00233$ or $0.233\%$

$\ell = 1/2\ \ell_t \quad (\epsilon_f)_{max} = 1/2 \times 0.233 = 0.1165\%$

$\underline{\sigma_c = 210\ MPa}$

$\ell = \ell_t \qquad (\epsilon_f)_{max} = \frac{488.4}{70 \times 10^3} = 0.00698$ or $0.698\%$

$\ell = 1/2\ \ell_t \quad (\epsilon_f)_{max} = 1/2 \times 0.698 = 0.349\%$

4.2 Using equation 4.8

$\frac{\ell_c}{d} = \frac{1.4 \times 10^3}{2 \times 14} = 50$

$\ell \leq \ell_c$

Equation 4.20 can be written as

$$\sigma_{cu} = \frac{\tau_y \ell_c}{d} \frac{\ell}{\ell_c} V_f + \sigma_{mu} V_m$$

$\tau_y = 14$ MPa , $\frac{\ell_c}{d} = 50$, $V_f = 0.4$, $V_f = 0.6$

$\sigma_{mu}$ may be taken as 28 MPa

$$\sigma_{cu} = 14 \times 50 \times 0.4 \frac{\ell}{\ell_c} + 28 \times 0.6$$

$$\sigma_{cu} = 16.8 + 280 \frac{\ell}{\ell_c}$$

$\ell \geq \ell_c$

Values may be directly substituted in equation 4.21

$$\sigma_{cu} = 1400 \times (1 - \frac{\ell_c}{2\ell}) \times 0.4 + 28 \times 0.6$$

$$\sigma_{cu} = 576.8 - 280 \frac{\ell_c}{\ell}$$

| $\frac{\ell}{\ell_c}$ | $\sigma_{cu}$ | |
|---|---|---|
| | $16.8 + 280\frac{\ell}{\ell_c}$ | $576.8 - 280\frac{\ell_c}{\ell}$ |
| 0.1 | 44.8 | - |
| 0.2 | 72.8 | - |
| 0.5 | 156.8 | - |
| 1.0 | 296.8 | 296.8 |
| 2.0 | - | 436.8 |
| 5.0 | - | 520.8 |
| 10.0 | - | 548.8 |
| 20 | - | 562.8 |
| 50 | - | 571.2 |
| 100 | - | 574.0 |
| 200 | - | 575.4 |
| 500 | - | 576.24 |
| 1000 | - | 576.52 |

4.3 Using the given expression for the interfacial shear stress, Eq. (4.3) becomes:

$$\sigma_f = (2/r)\int_0^z(\tau_y - az)\,dz \qquad (1)$$

Integration yields

$$\sigma_f = (2/r)(\tau_y \cdot z - 1/2\,az^2) \qquad (2)$$

For $z = \ell_t/2$,

$$\sigma_f = (\sigma_f)_{max} \qquad (3)$$

and $\tau = 0 = \tau_y - a\,.\,\ell_t/2$

or $$\ell_t/2 = \tau_y/a \qquad (4)$$

Substitution of Eq. (3) and $z = \ell_t/2 = \tau_{ya}$ in Eq. (2) gives

$$\ell_t/d = (\sigma_f)_{max}/\tau_y \qquad (5)$$

Comparison of Eq. (5) above with Eq. (3.7) shows that with linear variation of interfacial shear stress the load transfer length is twice the load transfer length with constant interfacial shear stress.

4.4

$$\rho_c = \frac{1}{\frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}} = \frac{\rho_f \, \rho_m}{\rho_m W_f + \rho_f W_m} = \frac{2.5 \ x \ 1.4}{2.5 \ x \ 0.8 + 1.4 \ x \ 0.2} = 1.535 \ g/cm^3$$

$$V_f = \frac{\rho_c}{\rho_f} W_f = \frac{1.535}{2.5} \ x \ 0.2 = 0.1228$$

$$\frac{E_T}{E_m} = \frac{1 + 2 \ \eta_T V_f}{1 - \eta_T V_f}, \ \eta_T = \frac{(E_f/E_m) - 1}{(E_f/E_m) + 2} = \frac{20 - 1}{20 + 2} = \frac{19}{22}$$

$$E_T = 3.5 \ x \ \frac{1 + 2 \ x \ (19/22) \ x \ 0.1228}{1 - (19/22) \ x \ 0.1228} = 4.746 \ GPa$$

$$7 = (3/8) \ E_L + (5/8) \ x \ 4.746$$

$$E_L = 10.76 \ GPa$$

$$\frac{E_L}{E_m} = \frac{1 + (2\ell/d)\eta_L V_f}{1 - \eta_L V_f}, \ \eta_L = \frac{(E_f/E_m) - 1}{(E_f/E_m) + (2\ell/d)} = \frac{20 - 1}{20 + a}$$

$$\frac{10.76}{3.5} = \frac{1 + a \ x \ [19/(20 + a)] \ x \ 0.1228}{1 - [19/(20 + a)] \ x \ 0.1228} = \frac{20 + a + 2.3332a}{20 + a - 2.3332}$$

$$a = 2\ell/d = 132.5$$

$$\ell = \frac{132.5 \ x \ 20}{2} = 1325 \ \mu m$$

$$or \ 1.325 \ mm$$

4.5

$$\rho_c = \frac{1.8 \ x \ 1.4}{1.4 \ x \ 0.2 + 1.8 \ x \ 0.8} = 1.465 \ g/cm^3$$

$$V_f = \frac{1.465}{1.8} \ x \ 0.2 = 0.163$$

$$\eta_T = \frac{(210/3.5) - 1}{(210/3.5) + 2} = \frac{59}{62}$$

$$E_T = 3.5 \ x \ \frac{1 + 2 \ x \ (59/62) \ x \ 0.163}{1 - (59/62) \ x \ 0.163} = 5.428 \ GPa$$

$$14 = (3/8) \ E_L + (5/8) \ x \ 5.428$$
$$E_L = 28.29 \ GPa$$

$$\frac{28.29}{3.5} = \frac{1 + 2\ell/d \cdot \eta_L \cdot V_f}{1 - \eta_L \cdot V_f}$$

$$\eta_L = \frac{(E_f/E_m) - 1}{(E_f/E_m) + 2\ell/d} = \frac{59}{60+a} \quad Where \ 2\ell/d = a$$

$$\frac{28.29}{3.5} = 8.08 = \frac{1 + a \ x \ [59/(60 + a)] \ x \ 0.163}{1 - [59/(60 + a)] \ x \ 0.163}$$
$$a = 136.8$$
$$2\ell/d = 136.8$$

$$\ell = \frac{136.8 \ x \ 15}{2} = 1026 \ \mu m$$

or 1.026 mm

4.6 For $\ell > \ell_c$, composite strength can be written as

$$\sigma_{cu} = \bar{\sigma}_f V_f + (\sigma_m)_{\epsilon_f^*} (1 - V_f)$$

For $V_f < V_{min}$, strength is controlled by matrix and can be calculated using equation 4.25. Thus, for a limiting case with $V_f = V_{min}$

$$\bar{\sigma}_f V_{min} + (\sigma_m)_{\epsilon_f^*} (1 - V_{min}) = \sigma_{mu} (1 - V_{min})$$

$$V_{min} = \frac{\sigma_{mu} - (\sigma_m)_{\epsilon_f^*}}{\sigma_f + \sigma_{mu} - (\sigma_m)_{\epsilon_f^*}}$$

At $V_f = V_{crit}$, $\sigma_{cu} = \sigma_{mu}$. Thus,

$$\bar{\sigma}_f V_{crit} + (\sigma_m)_{\epsilon_f^*} (1 - V_{crit}) = \sigma_{mu}$$

$$V_{crit} = \frac{\sigma_{mu} - (\sigma_m)_{\epsilon_f^*}}{\bar{\sigma}_f - (\sigma_m)_{\epsilon_f^*}}$$

4.7 $$\frac{(\sigma_{cu})\text{ disc.}}{(\sigma_{cu})\text{ cont.}} = \frac{\sigma_{fu}\left(1 - \frac{\ell_c}{2\ell}\right) V_f + (\sigma_m)_{\epsilon_f^*}(1 - V_f)}{\sigma_{fu} V_f + (\sigma_m)_{\epsilon_f^*}(1 - V_f)}$$

$$\frac{(\sigma_{cu})\text{ disc.}}{(\sigma_{cu})\text{ cont.}} = 1 - \frac{\frac{\ell_c}{2\ell}\sigma_{fu} V_f}{\sigma_{fu} V_f + (\sigma_m)_{\epsilon_f^*}(1 - V_f)}$$

$$\frac{(\sigma_{cu})\text{ disc.}}{(\sigma_{cu})\text{ cont.}} = 1 - \frac{1}{\frac{2\ell}{\ell_c}\left\{1 + \left[(\sigma_m)_{\epsilon_f^*}/\sigma_{fu}\right]\left(\frac{1}{V_f} - 1\right)\right\}}$$

For a limiting of $V_f = 1$

$$\frac{(\sigma_{cu})\text{ disc.}}{(\sigma_{cu})\text{ cont.}} = 1 - \frac{\ell_c}{2\ell}$$

For practical values of $V_f$, the ratio of two strengths will increase and hence lie in the shaded region.

4.8 $t_m = 0.1\, t_r,\ W_m = t_r$

Substituting in equation 4.27

$$V_r = \frac{1}{\left(1 + \frac{0.5\, t_r}{W_r}\right)\left(1 + \frac{0.1\, t_r}{t_r}\right)} = \frac{1}{1.1\left(1 + \frac{t_r}{2W_r}\right)}$$

| $\frac{W_r}{t_r}$ | $V_r$ |
|---|---|
| 1 | 0.606 |
| 2 | 0.727 |
| 5 | 0.826 |
| 10 | 0.866 |
| 20 | 0.887 |
| 50 | 0.900 |
| 100 | 0.905 |
| 200 | 0.907 |
| 500 | 0.908 |
| 1000 | 0.909 |

4.9 For square array: $V_f = \frac{\frac{\pi}{4} d^2}{(d + 0.1d)^2} = \frac{\pi}{4 \times 1.1^2} = 0.649$

For hexagonal array: $V_f = \frac{\frac{1}{2}\frac{\pi}{4} d^2}{\frac{1}{2}(1.1d)\,(1.1\, d \sin 60)} = \frac{\pi}{4 \times 1.1^2 \sin 60} = 0.7495$

# Chapter 5

**5.1.**

When $\sigma_y$ is the only non-zero stress:

$$\sigma_L = \sigma_y \sin^2\theta, \quad \sigma_T = \sigma_y \cos^2\theta, \quad \tau_{LT} = \sigma_y \sin\theta\cos\theta$$

$$\epsilon_L = \sigma_y \left(\frac{\sin^2\theta}{E_L} - \nu_{TL}\frac{\cos^2\theta}{E_T}\right)$$

$$\epsilon_T = \sigma_y \left(\frac{\cos^2\theta}{E_T} - \nu_{LT}\frac{\sin^2\theta}{E_L}\right)$$

$$\gamma_{LT} = \frac{\sigma_y \sin\theta\cos\theta}{G_{LT}}$$

By transformation of strains

$$\epsilon_x = -\sigma_y \left[\frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)\sin^2 2\theta\right]$$

$$\epsilon_y = \sigma_y \left[\frac{\sin^4\theta}{E_L} + \frac{\cos^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta\right]$$

$$\gamma_{xy} = -\sigma_y \sin 2\theta \left[\frac{\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{2G_{LT}} - \sin^2\theta\left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)\right]$$

Thus,

$$\frac{1}{E_y} \equiv \frac{\epsilon_y}{\sigma_y} = \frac{\sin^4\theta}{E_L} + \frac{\cos^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta$$

$$\frac{\nu_{yx}}{E_y} \equiv -\frac{\epsilon_x}{\epsilon_y E_y} = -\frac{\epsilon_x}{\sigma_y} = \frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)\sin^2 2\theta$$

$$m_y \equiv -\frac{\gamma_{xy}}{\sigma_y/E_L} = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \sin^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right]$$

5.2. When $\tau_{xy}$ is the only non-zero stress, strains in the L-T coordinates are given by equation 5.34. Strains in the x-y coordinates can be obtained by tensor transformation:

$$\epsilon_x = -\tau_{xy}\sin 2\theta\left[\frac{\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{2G_{LT}} - \cos^2\theta\left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)\right]$$

$$\epsilon_y = -\tau_{xy}\sin 2\theta\left[\frac{\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{2G_{LT}} - \cos^2\theta\left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)\right]$$

Therefore, by definition of $m_x$ and $m_y$

$$m_x = -\frac{\epsilon_x}{\tau_{xy}/E_L} = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \cos^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right]$$

$$m_y = -\frac{\epsilon_y}{\tau_{xy}/E_L} = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \sin^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right]$$

5.3.

| $\theta$ | $E_x$ (GPa) | $G_{xy}$ (GPa) | $\nu_{xy}$ | $m_x$ | $m_y$ |
|---|---|---|---|---|---|
| 0 | 35 | 4.00 | 0.45 | 0 | 0 |
| 10 | 29.68 | 3.84 | 0.303 | 1.033 | 2.045 |
| 20 | 20.26 | 3.48 | 0.072 | 2.117 | 3.668 |
| 30 | 13.16 | 3.15 | -0.053 | 3.215 | 4.579 |
| 40 | 8.85 | 2.96 | -0.079 | 4.162 | 4.701 |
| 45 | 7.43 | 2.94 | -0.072 | 4.500 | 4.500 |
| 50 | 6.34 | 2.96 | -0.057 | 4.701 | 4.162 |
| 60 | 4.89 | 3.15 | -0.020 | 4.579 | 3.215 |
| 70 | 4.06 | 3.48 | 0.014 | 3.668 | 2.117 |
| 80 | 3.63 | 3.84 | 0.037 | 2.045 | 1.033 |
| 90 | 3.50 | 4.00 | 0.045 | 0 | 0 |

$\nu_{xy} = 0$ at $\theta = 24.55$ and $65.45$

5.4.

| θ | $E_x$ (GPa) | $G_{xy}$ (GPa) | $\nu_{xy}$ | $m_x$ | $m_y$ |
|---|---|---|---|---|---|
| 0 | 15 | 2.50 | 0.200 | 0 | 0 |
| 10 | 13.57 | 2.69 | 0.276 | -0.579 | 0.579 |
| 20 | 10.93 | 3.32 | 0.417 | -0.886 | 0.886 |
| 22.5 | - | - | - | -0.900 | 0.900 |
| 30 | 8.96 | 4.55 | 0.523 | -0.779 | 0.779 |
| 40 | 8.01 | 5.98 | 0.573 | -0.308 | 0.308 |
| 45 | 7.89 | 6.25 | 0.579 | 0 | 0 |
| 50 | 8.01 | 5.98 | 0.573 | 0.308 | -0.308 |
| 60 | 8.96 | 4.55 | 0.523 | 0.779 | -0.779 |
| 67.5 | - | - | - | 0.900 | -0.900 |
| 70 | 10.93 | 3.32 | 0.417 | 0.886 | -0.886 |
| 80 | 13.57 | 2.69 | 0.276 | 0.579 | -0.579 |
| 90 | 15 | 2.50 | 0.200 | 0 | 0 |

5.5.

| θ = | $30^\circ$ | $45^\circ$ | $60^\circ$ |
|---|---|---|---|
| $E_x$ (GPa) | 10.87 | 7.43 | 5.02 |
| $G_{xy}$ (GPa) | 2.70 | 2.41 | 2.70 |
| $\nu_{xy}$ | -0.0485 | -0.115 | -0.0224 |
| $m_x$ | 0.765 | 1.500 | 1.833 |
| $m_y$ | 1.833 | 1.500 | 0.765 |

5.6 Transformation equation:

$$E'_{ijk\ell} = a_{mi}\, a_{nj}\, a_{rk}\, a_{sl}\, E_{mnrs}$$

Under the transformation given by equation 5.52 with $a_{ij}$ given by equation 5.53, elastic constants indicated in equation 5.55 can be written as

$$E'_{1113} = a_{m1}\, a_{n1}\, a_{r1}\, a_{s3}\, E_{mnrs} = -E_{1113}$$

$$E'_{2223} = a_{m2}\, a_{n2}\, a_{r2}\, a_{s3}\, E_{mnrs} = -E_{2223}$$

$$E'_{1123} = a_{m1}\, a_{n1}\, a_{r2}\, a_{s3}\, E_{mnrs} = -E_{1123}$$

$$E'_{2213} = a_{m2}\, a_{n2}\, a_{r1}\, a_{s3}\, E_{mnrs} = -E_{2213}$$

$$E'_{1213} = a_{m1}\, a_{n2}\, a_{r1}\, a_{s3}\, E_{mnrs} = -E_{1213}$$

$$E'_{1223} = a_{m1}\, a_{n2}\, a_{r2}\, a_{s3}\, E_{mnrs} = -E_{1223}$$

$$E'_{1333} = a_{m1}\, a_{n3}\, a_{r3}\, a_{s3}\, E_{mnrs} = E_{1333}$$

$$E'_{2333} = a_{m2}\, a_{n3}\, a_{r3}\, a_{s3}\, E_{mnrs} = -E_{2333}$$

In view of the invariance condition, the above constants should be zero.

5.7 Under the transformation given by equation 5.56 with $a_{ij}$ given by equation 5.57, the elastic constants indicated in equation 5.58 can be written as

$$E'_{1233} = a_{m1}\, a_{n2}\, a_{r3}\, a_{s3}\, E_{mnrs} = -E_{1233}$$

$$E'_{1323} = a_{m1}\, a_{n3}\, a_{r2}\, a_{s3}\, E_{mnrs} = -E_{1323}$$

$$E'_{1222} = a_{m1}\, a_{n2}\, a_{r2}\, a_{s2}\, E_{mnrs} = E_{1222}$$

$$E'_{1112} = a_{m1}\, a_{n1}\, a_{r1}\, a_{s2}\, E_{mnrs} = -E_{1112}$$

In view of the invariance condition, the above constants should be zero.

5.8 If $x_1x_3$- plane is the plane of symmetry, elastic constants should not change under the following coordinate transformation:

$$x'_1 = x_1,\ x'_2 = -x_2,\ x'_3 = x_3$$

The direction cosines corresponding to the above transformation are

| | $x'_1$ | $x'_2$ | $x'_3$ |
|---|---|---|---|
| $x_1$ | $a_{11} = 1$ | $a_{12} = 0$ | $a_{13} = 0$ |
| $x_2$ | $a_{21} = 0$ | $a_{22} = -1$ | $a_{23} = 0$ |
| $x_3$ | $a_{31} = 0$ | $a_{32} = 0$ | $a_{33} = 1$ |

It may be observed that the transformations of elastic constants in which subscript Z appears precisely once or thrice, will lead to the contradictions of the type shown in equation 5.54. To be consistent with the condition of invariance of elastic constants, the following eight elastic constants must be set equal to zero:

$$E_{1213},\ E_{2333},\ E_{2223},\ E_{1123},\ E_{1222},\ E_{1112},\ E_{1233},\ E_{1323}$$

First four of these constants are common with equation 5.55 and remaining with equation 5.58.

5.9

$$\nu_{TL} = 0.35 \times \frac{2}{20} = 0.035$$

$$Q_{11} = \frac{20}{1-0.35 \times 0.035} = 20.25 \text{ GPa}$$

$$Q_{22} = \frac{2}{1-0.35 \times 0.035} = 2.025 \text{ GPa}$$

$$Q_{12} = \frac{0.35 \times 2}{1-0.35 \times 0.035} = 0.709 \text{ GPa}$$

$$Q_{66} = 0.7 \text{ GPa}$$

Thus, stiffness matrix is

$$[Q] = \begin{bmatrix} 20.25 & 0.709 & 0 \\ 0.709 & 2.025 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \text{ GPa}$$

Compliance matrix can be easily found to be

$$[S] = \begin{bmatrix} 0.05 & -0.0175 & 0 \\ -0.0175 & 0.5 & 0 \\ 0 & 0 & 1.429 \end{bmatrix} \text{ 1/GPa}$$

5.10 Equation 5.82 is identically staisfied.

$1 - \nu_{LT} \quad \nu_{TL} = 1 - 1.97 \times 0.22 = 0.5666 > 0$, thus, equation 5.83 is also satisfied.

$$\left(\frac{E_L}{E_T}\right)^{1/2} = 2.996$$

Thus, condition $\nu_{LT} < \left(\frac{E_L}{E_T}\right)^{1/2}$ is also satisfied. Accordingly, $\nu_{LT} = 1.97$ is a reasonable number. It will not be an admissible number for an isotropic material in which Poisson's ratio should be less than 0.5.

$\nu_{TL}$ satisfies symmetry requirements.

5.11 For isotropic materials

$$\nu_{LT} = \nu_{TL} = \nu_{LT'} = \nu_{T'L} = \nu_{TT'} = \nu_{T'T} = \nu \text{ (say)}$$

Equation 5.83 reduces to

$$1 - \nu^2 > 0$$

or $-1 < \nu < 1$

5.12 For isotropic material, equation 5.84 reduces to

$$1 - 3\nu^2 - 2\nu^3 > 0$$

or $1 - 2\nu > 0 \quad \Rightarrow \quad \nu < \frac{1}{2}$

5.13. Using equation 5.95, elements of the $\bar{Q}$ matrix can be obtained as follows:

| $\theta =$ | $30^\circ$ | $45^\circ$ | $60^\circ$ |
|---|---|---|---|
| $Q_{11}$ | 12.16 | 6.55 | 3.16 |
| $Q_{22}$ | 3.16 | 6.55 | 12.16 |
| $Q_{12}$ | 4.04 | 5.15 | 4.04 |
| $Q_{66}$ | 4.04 | 5.15 | 4.04 |
| $Q_{16}$ | 5.82 | 4.50 | 1.97 |
| $Q_{26}$ | 1.97 | 4.50 | 5.82 |

5.14 (a)

$$\sigma_x = \frac{500}{12.5 \times 4} = 10 \text{ N/mm}^2 = 10 \text{ MPa}$$

$$\sigma_L = \sigma_x \cos^2\theta = 10 \cos^2 45 = 5.0 \text{ MPa}$$

$$\sigma_T = \sigma_x \sin^2\theta = 10 \sin^2 45 = 5.0 \text{ MPa}$$

$$\tau_{LT} = -\sigma_x \sin\theta\cos\theta = -10 \sin 45 \cos 45 = -5.0 \text{ MPa}$$

$$\epsilon_L = \frac{5 \times 10^{-3}}{14} - \frac{0.10}{3.5} \times 5 \times 10^{-3} = 0.214 \times 10^{-3}$$

$$\epsilon_T = \frac{5 \times 10^{-3}}{3.5} - \frac{0.4}{14} \times 5 \times 10^{-3} = 1.286 \times 10^{-3}$$

$$\gamma_{LT} = -\frac{5.0 \times 10^{-3}}{4.2} = -1.190 \times 10^{-3}$$

By transformation equation 5.88

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 1.0 & 1.0 & 0 \end{bmatrix} \begin{Bmatrix} 0.214 \\ 1.286 \\ -1.190 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 1.345 \\ 0.155 \\ -1.072 \end{Bmatrix}$$

(b) $$\epsilon_x = \frac{\sigma_x}{E_x} = \frac{10 \times 10^{-3}}{1.345 \times 10^{-3}} = 7.43 \text{ GPa}$$

$$\nu_{xy} = -\frac{\epsilon_y}{\epsilon_x} = -\frac{0.155}{1.345} = -0.115$$

$$m_x = -\frac{\gamma_{xy}}{\sigma_x/E_L} = -\frac{-1.072 \times 10^{-3}}{10 \times 10^{-3}/14} = 1.5$$

The values of $E_x$ $\nu_{xy}$ and $m_x$ exactly match those obtained in problem 5.

5.15 $$\sigma_L = 10 \cos^2 30 = 7.5 \text{ MPa}$$

$$\sigma_T = 10 \sin^2 30 = 2.5 \text{ MPa}$$

$$\tau_{LT} = -10 \sin 30 \cos 30 = -4.33 \text{ MPa}$$

$$\epsilon_L = \frac{7.5 \times 10^{-3}}{14} - \frac{0.1}{3.5} \times 2.5 \times 10^{-3} = 0.464 \times 10^{-3}$$

$$\epsilon_T = \frac{2.5 \times 10^{-3}}{3.5} - \frac{0.4}{14} \times 7.5 \times 10^{-3} = 0.500 \times 10^{-3}$$

$$\gamma_{LT} = -\frac{4.33 \times 10^{-3}}{4.2} = -1.031 \times 10^{-3}$$

By transformation equation 5.88:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0.75 & 0.25 & -0.433 \\ 0.25 & 0.75 & 0.433 \\ 0.866 & -0.866 & 0.5 \end{bmatrix} \begin{Bmatrix} 0.464 \\ 0.500 \\ -1.032 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 0.920 \\ 0.044 \\ -0.546 \end{Bmatrix} \times 10^{-3}$$

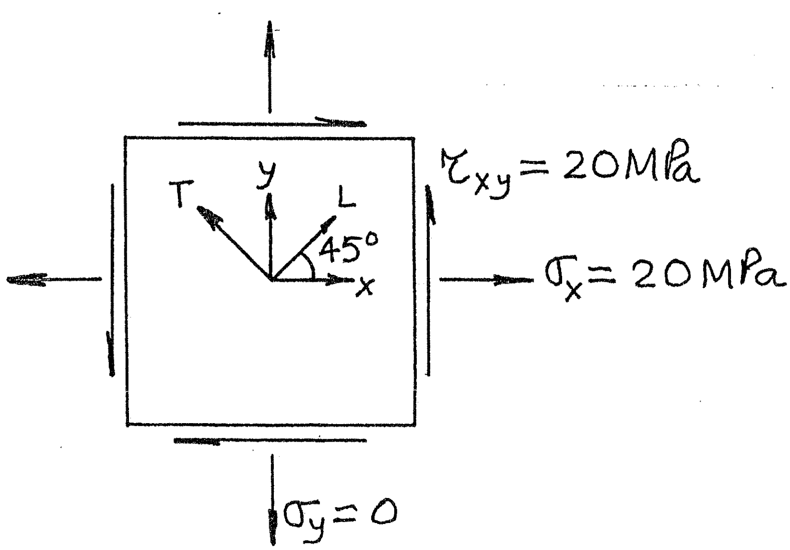
(b) $$E_x = \frac{10 \times 10^{-3}}{0.920 \times 10^{-3}} = 10.87 \text{ GPa}$$

$$\nu_{xy} = -\frac{\epsilon_y}{\epsilon_x} = -\frac{0.044}{0.920} = -0.048$$

$$m_x = -\frac{\gamma_{xy}}{\sigma_x/E_L} = -\frac{-0.546 \times 10^{-3}}{10 \times 10^{-3}/14} = 0.764$$

The values of $E_x$, $\nu_{xy}$ and $m_x$ compare favorably with those obtained in problem 5.

5.16 (a)



(b) $\sigma_L = 20 \cos^2 45 + 2 \times 20 \cos 45 \sin 45 = 30$ MPa
$\sigma_\tau = 20 \cos^2 45 - 2 \times 20 \times \cos 45 \sin 45 = -10$ MPa
$\tau_{LT} = 20 \cos^2 45 + (\cos^2 45 - \sin^2 45)\ -10$ MPa

(c) $$\left(\frac{30}{500}\right)^2 - \left(\frac{30}{500}\right)\left(\frac{10}{350}\right) + \left(\frac{10}{75}\right)^2 + \left(\frac{10}{35}\right)^2 = 0.1013$$

Lamina does not fail.

(d) When sign of $\tau_{xy}$ is reversed, the stresses are

$\sigma_L = -10$ MPa
$\sigma_\tau = 30$ MPa
$\tau_{LT} = -10$MPa

$$\left(\frac{10}{350}\right)^2 - \left(\frac{10}{350}\right)\left(\frac{30}{500}\right) + \left(\frac{30}{10}\right)^2 + \left(\frac{10}{35}\right)^2 = 9.08$$

Lamina will fail.

5.17

$\sigma_L = 20 \cos^2 30 + 2 \times 20 \cos 30 \sin 30 = 32.32$ MPa

$\sigma_T = 20 \sin^2 30 - 2 \times 20 \cos 30 \sin 30 = -12.32$ MPa

$\tau_{LT} = -20 \cos 30 \sin 30 + 20 (\cos^2 30 - \sin^2 30) = 1.34$ MPa

Failure Criterion

$$\left(\frac{32.32}{500}\right)^2 - \left(\frac{32.32}{500}\right)\left(\frac{12.32}{350}\right) + \left(\frac{12.32}{75}\right)^2 + \left(\frac{1.34}{35}\right)^2 = 0.0304$$

Therefore, no failure.

When sign of $\tau_{xy}$ is reversed, the stresses are

$\sigma_L = -2.32$ MPa
$\sigma_T = 22.32$ MPa
$\tau_{LT} = 18.34$ MPa

Failure criterion

$$\left(\frac{2.32}{350}\right)^2 - \left(\frac{2.32}{350}\right)\left(\frac{22.32}{500}\right) + \left(\frac{22.32}{10}\right)^2 \left(\frac{18.34}{35}\right)^2 = 5.256$$

Therefore, failure occurs.

5.18

$\sigma_L = \tau_{xy} \sin 2\theta$, $\sigma_T = -\tau_{xy} \sin 2\theta$, $\tau_{LT} = \tau_{xy} \cos 2\theta$.

If $\tau_{xy}$ is positive, $\sigma_L$ is positive and $\sigma_T$ is negative.

Therefore, at failure, $\tau_{xy}$ is given by

$$\frac{1}{\tau_{xy}^2} = \frac{\sin^2 2\theta}{(1725)^2} - \frac{\sin^2 2\theta}{(1725x1350)} + \frac{\sin^2 2\theta}{(275)^2} + \frac{\cos^2 2\theta}{(95)^2}$$

If $\tau_{xy}$ is negative, signs of $\sigma_L$ and $\sigma_T$ are reversed and at failure $\tau_{xy}$ is given by

$$\frac{1}{\tau_{xy}^2} = \frac{\sin^2 2\theta}{(1350)^2} - \frac{\sin^2 2\theta}{(1350x1725)} + \frac{\sin^2 2\theta}{(40)^2} + \frac{\cos^2 2\theta}{(95)^2}$$

| $\theta$ = | Off-axis Shear Strength (MPa) | |
|---|---|---|
| | $\tau_{xy}$ (+) | $\tau_{xy}$ (-) |
| 30° | 163.2 | 44.9 |
| 45° | 276 | 40 |

# CHAPTER 6

6.1 For aluminum: $E = 70$ GPa, $\nu = 0.33$

$$[Q]_{Al} = \begin{bmatrix} 78.55 & 25.92 & 0 \\ 25.92 & 78.55 & 0 \\ 0 & 0 & 26.32 \end{bmatrix} \text{GPa}$$

For steel· $E = 210$ GPa, $\nu = 0.3$

$$[Q]_{St} = \begin{bmatrix} 224 & 56 & 0 \\ 56 & 224 & 0 \\ 0 & 0 & 84 \end{bmatrix} \text{GPa}$$

$$A_{ij} = 5[(Q_{ij})_{Al} + (Q_{ij})_{St}]$$

$$[A] = \begin{bmatrix} 1512.75 & 409.60 & 0 \\ 409.60 & 1512.75 & 0 \\ 0 & 0 & 551.60 \end{bmatrix}$$

$$B_{ij} = \frac{25}{2}[(Q_{ij})_{St} - (Q_{ij})_{Al}]$$

$$[B] = \begin{bmatrix} 1818 & 376 & 0 \\ 376 & 1818 & 0 \\ 0 & 0 & 721 \end{bmatrix}$$

$$D_{ij} = \frac{125}{3}[(Q_{ij})_{Al} + (Q_{ij})_{St}]$$

$$D = \begin{bmatrix} 12606 & 3413 & 0 \\ 3413 & 12606 & 0 \\ 0 & 0 & 4597 \end{bmatrix}$$

Nonzero value of coupling matrix [B] indicates that for the bimetallic strip of aluminum and steel (both isotropic materials), coupling between extension and bending exists.

6.2

$$\left[\bar{Q}\right]_{\theta} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}$$

$$\left[\bar{Q}\right]_{-\theta} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & -\bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & -\bar{Q}_{26} \\ -\bar{Q}_{16} & -\bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}$$

Assume laminate thickness to be h.

$$[A] = h \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix}$$

$$N_x = h\sigma_{xo},\ N_y = N_{xy} = 0$$

Mid-plane strains can be obtained from

$$\begin{Bmatrix} h\sigma_{xo} \\ 0 \\ 0 \end{Bmatrix} = h \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$$

Solution of this equation gives

$$\varepsilon_x^0 = \frac{\bar{Q}_{22}}{\bar{Q}_{11}\,\bar{Q}_{22} - \bar{Q}_{12}^2}\,\sigma_{xo}$$

$$\varepsilon_y^0 = \frac{\bar{Q}_{12}}{\bar{Q}_{11}\,\bar{Q}_{22} - \bar{Q}_{12}^2}\,\sigma_{xo}$$

$$\gamma_{xy}^0 = 0$$

All the plate curvatures will be zero since no bending moments are applied and the coupling matrix [B] is zero. In the absence of plate curvatures, mid-plane strains are also the laminae strains. Therefore, the laminae stresses are

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}_\theta = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ 0\end{Bmatrix} = \begin{Bmatrix}\sigma_{xo}\\ 0\\ \bar{Q}_{16}\ \varepsilon_x^0 + \bar{Q}_{26}\ \varepsilon_y^0\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}_{-\theta} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & -\bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & -\bar{Q}_{26}\\ -\bar{Q}_{16} & -\bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ 0\end{Bmatrix} = \begin{Bmatrix}\sigma_{xo}\\ 0\\ -\bar{Q}_{16}\ \varepsilon_x^0 - \bar{Q}_{26}\ \varepsilon_y^0\end{Bmatrix}$$

Stresses along L and T directions can be obtained through transformation as follows:

$$\begin{Bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{Bmatrix}_\theta = \begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\ \cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\ \cos\theta\\ -\sin\theta\ \cos\theta & \sin\theta\ \cos\theta & \cos^2\ -\sin^2\theta\end{bmatrix}\begin{Bmatrix}\sigma_{x0}\\ 0\\ \tau_{xy}\end{Bmatrix}$$

$$\textit{where}\quad \tau_{xy} = \bar{Q}_{16}\ \varepsilon_x^0 + \bar{Q}_{26}\ \varepsilon_y^0 = \frac{\bar{Q}_{16}\ \bar{Q}_{22} - \bar{Q}_{26}\ \bar{Q}_{12}}{\bar{Q}_{11}\ \bar{Q}_{22} - \bar{Q}_{12}^2}\ \sigma_{xo}$$

$$\begin{Bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{Bmatrix}_\theta = \begin{Bmatrix}\sigma_{xo}\ \cos^2\theta + 2\tau_{xy}\ \sin\theta\ \cos\theta\\ \sigma_{xo}\ \sin^2\theta - 2\tau_{xy}\ \sin\theta\ \cos\theta\\ -\sigma_{xo}\ \sin\theta\ \cos\theta + \tau_{xy}(\cos^2 - \sin^2\theta)\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{Bmatrix}_{-\theta} = \begin{bmatrix}\cos^2\theta & \sin^2\theta & -2\sin\theta\ \cos\theta\\ \sin^2\theta & \cos^2\theta & 2\sin\theta\ \cos\theta\\ \sin\theta\cos\theta & -\sin\theta\ \cos\theta & \cos^2\theta - \sin^2\theta\end{bmatrix}\begin{Bmatrix}\sigma_{xo}\\ 0\\ -\tau_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_L\\ \sigma_T\\ \tau_{LT}\end{Bmatrix}_{-\theta} = \begin{Bmatrix}\sigma_{xo}\ \cos^2\theta + 2\tau_{xy}\ \sin\theta\ \cos\theta\\ \sigma_{xo}\ \sin^2\theta - 2\tau_{xy}\ \sin\theta\ \cos\theta\\ \sigma_{xo}\ \sin\theta\ \cos\theta - \tau_{xy}\ (\cos^2\theta - \sin^2\theta)\end{Bmatrix}$$

6.3 Substituting $\theta = 45$, stresses are

$$\sigma_L = \frac{1}{2}(\sigma_{xo} + 2\tau_{xy})$$

$$\sigma_T = \frac{1}{2}(\sigma_{xo} - 2\tau_{xy})$$

$$\tau_{LT} = -\frac{1}{2}\sigma_{xo}$$

Strains

$$\varepsilon_L = \frac{1}{2}\left(\varepsilon_x^o + \varepsilon_y^o\right)$$
$$\varepsilon_T = \frac{1}{2}\left(\varepsilon_x^o + \varepsilon_y^o\right)$$
$$\gamma_{LT} = \left(\varepsilon_y^o - \varepsilon_x^o\right)$$

Lamina shear stress $\tau_{LT}$ and lamina shear strain $\gamma_{XO}$, are related to only laminate stress, $\sigma_{XO}$, and laminate strains, $\varepsilon_x^0$ and $\varepsilon_y^0$ which can be easily measured during a tension test. Thus shear modulus $G_{LT}$ of a lamina can be determined from a tension test on [±45]s laminate.

6.4 Let the stiffness matrix of the $0^\circ$ plies be

$$[Q]_{0^\circ} = [\bar{Q}_{0^\circ}] = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

where $Q_{11}$, $Q_{22}$, $Q_{12}$ and $Q_{66}$ are independent constants. Stiffness matrix for the $90^\circ$ plies will be

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} Q_{22} & Q_{12} & 0 \\ Q_{12} & Q_{11} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

For unit thickness of each lamina, extensional stiffness matrix of the laminate will be

$$A = \begin{bmatrix} Q_{11} + Q_{22} & 2Q_{12} & 0 \\ 2Q_{12} & Q_{11} + Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix}$$

For the laminate to be quasi-isotropic

$$A_{11} - A_{12} = 2A_{66} \quad \text{or}$$

$$Q_{11} + Q_{22} - 2Q_{12} = 4Q_{66}$$

Since $Q_{11}$, $Q_{22}$, $Q_{12}$ and $Q_{66}$ are independent constants, this condition, in general, will not be satisfied.

6.5 For the assumed stiffness matrix of the $0^\circ$ plies in Problem 6.4, elements of the stiffness matrix for the $+60^\circ$ plies will be

$$\bar{Q}_{11} = \frac{1}{16}(Q_{11} + 9Q_{22} + 6Q_{12} + 12Q_{66})$$

$$\bar{Q}_{22} = \frac{1}{16}(9Q_{11} + Q_{22} + 6Q_{12} + 12Q_{66})$$

$$\bar{Q}_{12} = \frac{1}{16}(3Q_{11} + 3Q_{22} + 10Q_{12} - 12Q_{66})$$

$$\bar{Q}_{66} = \frac{1}{16}(3Q_{11} + 3Q_{22} - 6Q_{12} + 4Q_{66})$$

$$\bar{Q}_{16} = \frac{\sqrt{3}}{16}(Q_{11} - 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$\bar{Q}_{26} = \frac{\sqrt{3}}{16}(3Q_{11} - Q_{22} - 2Q_{12} - 4Q_{66})$$

$\bar{Q}_{11}$, $\bar{Q}_{22}$, $\bar{Q}_{12}$ and $\bar{Q}_{66}$ for the $-60^\circ$ plies will be the same as those for the $+60^\circ$ plies. Following are the $\bar{Q}_{16}$ and $\bar{Q}_{26}$ for the $-60^\circ$ plies:

$$\bar{Q}_{16} = -\frac{\sqrt{3}}{16}(Q_{11} - 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$\bar{Q}_{26} = -\frac{\sqrt{3}}{16}(3Q_{11} - Q_{22} - 2Q_{12} - 4Q_{66})$$

For unit thickness of each lamina the elements of extensional stiffness matrix are:

$$A_{11} = A_{22} = \frac{1}{16}[18(Q_{11} + Q_{22}) + 12Q_{12} + 24Q_{66}]$$

$$A_{12} = \frac{1}{16}[6(Q_{11} + Q_{22}) + 36Q_{12} - 24Q_{66}]$$

$$A_{66} = \frac{1}{16}[6(Q_{11} + Q_{22}) - 12Q_{12} + 24Q_{66}]$$

$$A_{16} = A_{26} = 0$$

Thus, it is observed that the condition $A_{11} - A_{12} = 2A_{66}$ is also identically satisfied to make the laminate quasi-isotropic.

6.6 The stiffness matrices for the $0^\circ$ and $90^\circ$ laminae may be assumed the same as in Problem 6.4. Elements of the stiffness matrix for $+45^\circ$ lamina will be:

$$\bar{Q}_{11} = \bar{Q}_{22} = \frac{1}{4}(Q_{11} + Q_{22} + 2Q_{12} + 4Q_{66})$$

$$\bar{Q}_{12} = \frac{1}{4}(Q_{11} + Q_{22} + 2Q_{12} - 4Q_{66})$$

$$\bar{Q}_{66} = \frac{1}{4}(Q_{11} + Q_{22} - 2Q_{12})$$

$$\bar{Q}_{16} = \bar{Q}_{26} = \frac{1}{4}(Q_{11} - Q_{22})$$

$\bar{Q}_{11}$, $\bar{Q}_{22}$, $\bar{Q}_{12}$ and $\bar{Q}_{66}$ for the -45° ply will be the same as those for the +45° ply. Following are the $\bar{Q}_{16}$ and $\bar{Q}_{26}$ for the -45° ply:

$$\bar{Q}_{16} = \bar{Q}_{26} = -1/4\ (Q_{11} - Q_{22})$$

For unit thickness of each lamina, the elements of extensional stiffness matrix are:

$$A_{11} = A_{22} = 1/4\ (6Q_{11} + 6Q_{22} + 4Q_{12} + 8Q_{66})$$

$$A_{12} = 1/4\ (2Q_{11} + 2Q_{22} + 12Q_{12} - 8Q_{66})$$

$$A_{66} = 1/4\ (2Q_{11} + 2Q_{22} - 4Q_{12} + 8Q_{66})$$

$$A_{16} = A_{26} = 0$$

In this case also, the condition $A_{11} - A_{12} = 2A_{66}$ is identically satisfied so that the laminate is quasi-isotropic.

6.7 (a)
$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$$

Since the laminate is symmetric, plate curvatures will be zero and the strains in the laminate will be constant through the thickness and equal to the mid-plane strains.

Consider that the laminate is subjected to the following stress state:

$$\sigma_x = \sigma_{xo},\ \sigma_y = \tau_{xy} = 0$$

Therefore $N_x = t \cdot \sigma_{xo},\ N_y = N_{xy} = 0$
where t is the laminate thickness.

$$\begin{Bmatrix} t \cdot \sigma_{xo} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_y^0 \end{Bmatrix}$$

*solution gives*

$$\varepsilon_x^0 = \frac{t \cdot A_{22}}{A_{11}\ A_{22} - A_{12}^2}\ \sigma_{xo}$$

$$\varepsilon_y^0 = \frac{-t \cdot A_{12}}{A_{11}\ A_{22} - A_{12}^2}\ \sigma_{xo}$$

*Therefore*

$$E_x = \frac{\sigma_{xo}}{\varepsilon_x^0} = \frac{A_{11}\ A_{22} - A_{12}^2}{t \cdot A_{22}}$$

$$\upsilon_{xy} = \frac{\varepsilon_y^0}{\varepsilon_x^0} = \frac{A_{12}}{A_{22}}$$

Similarly by assuming $\sigma_y$ to be the only non-zero stress acting only on the laminate, it can be easily shown that

$$E_y = \frac{A_{11}\ A_{22} - A_{12}^2}{t \cdot A_{11}}$$

$$\upsilon_{yx} = \frac{A_{12}}{A_{11}}$$

By assuming $\tau_{xy}$ to be the only non-zero stress acting on the laminate, it can be shown that

$$G_{xy} = \frac{A_{66}}{t}$$

(b) For quasi-isotropic laminate

$$A_{11} = A_{22}$$

$$A_{66} = \frac{A_{11} - A_{12}}{2}$$

Therefore, for quasi-isotropic laminate

$$E_x = E_y = \frac{A_{11}^2 - A_{12}^2}{t \cdot A_{11}}$$

$$\upsilon_{xy} = \upsilon_{yx} = \frac{A_{12}}{A_{11}}$$

$$G_{xy} = \frac{A_{66}}{t} = \frac{E_x}{2(1+\upsilon_{xy})}$$

6.8 From Example 6.5

$$[\bar{Q}_{ij}]_{0^\circ} = \begin{bmatrix} 16.74 & 1.49 & 0 \\ 1..49 & 4.66 & 0 \\ 0 & 0 & 1.49 \end{bmatrix}$$

Transformation equations [Eq. (5.61)] give

$$[\bar{Q}_{ij}]_{\pm 60^\circ} = \begin{bmatrix} 5.34 & 3.83 & \pm\bar{Q}_{16} \\ 3.83 & 11.38 & \pm\bar{Q}_{26} \\ \pm\bar{Q}_{16} & \pm\bar{Q}_{26} & 3.83 \end{bmatrix}$$

Assuming unit thickness of the laminate, the [A] matrix is obtained as follows:

$$A_{ij} = \frac{1}{3}[(\bar{Q}_{ij})_{0^\circ} + (\bar{Q}_{ij})_{60^\circ} + (\bar{Q}_{ij})_{-60^\circ}]$$

$$[A] = \begin{bmatrix} 9.14 & 3.05 & 0 \\ 3.05 & 9.14 & 0 \\ 0 & 0 & 3.05 \end{bmatrix}$$

The [A] matrix obtained here is same as that obtained in Example 6.5 with the numbers rounded off to the second decimal place. Thus, the elastic constants predicted by the [0/±60] laminate are the same as those predicted by [0/±45/90] laminate.

6.9 [Qij] matrices for 0° and ±60° laminae are the same as obtained in Exercise 6.8 [$Q_{ij}$] matrices for 90° and ±30° laminae are obtained by transformation equations [Eq. (5.95)] as follows:

$$[\bar{Q}_{ij}]_{90^\circ} = \begin{bmatrix} 4.66 & 1.49 & 0 \\ 1.49 & 16.74 & 0 \\ 0 & 0 & 1.49 \end{bmatrix}$$

$$[\bar{Q}_{ij}]_{\pm 30^\circ} = \begin{bmatrix} 11.38 & 3.83 & \pm\bar{Q}_{ij} \\ 3.83 & 5.34 & \pm\bar{Q}_{26} \\ \pm\bar{Q}_{16} & \pm\bar{Q}_{26} & 3.83 \end{bmatrix}$$

Assuming unit thickness of the laminate, the [A] matrix is obtained as follows:

$$A_{ij} = \frac{1}{6}[(\bar{Q}_{ij})_{0^\circ} + (\bar{Q}_{ij})_{30} + (\bar{Q}_{ij})_{-30} + (\bar{Q}_{ij})_{60} + (\bar{Q}_{ij})_{-60^\circ} + (\bar{Q}_{ij})_{90^\circ}]$$

$$[A] = \begin{bmatrix} 9.14 & 3.05 & 0 \\ 3.05 & 9.14 & 0 \\ 0 & 0 & 3.05 \end{bmatrix}$$

Thus prediction of all three quasi-isotropic laminates [0/±45/90], [0/±60], and [0/±30/±60/±90] are identical.

6.10 Q matrices can be evaluated first:

$$[\bar{Q}]_{0^\circ} = \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{GPa}$$

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} 6.67 & 3.33 & 0 \\ 3.33 & 16.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{GPa}$$

For unit thickness of the laminate extensional stiffness matrix can be obtained as

$$[A] = \begin{bmatrix} 11.67 & 3.33 & 0 \\ 3.33 & 11.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{GPa}$$

For a unit cross-section of the laminate, stresses can be taken to be resultant forces in the corresponding direction, so that

$$\begin{Bmatrix} 15 \\ 0 \\ 1 \end{Bmatrix} \times 10^{-3} = \begin{bmatrix} 11.67 & 3.33 & 0 \\ 3.33 & 11.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Solving this matrix equation gives strains:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 1.4 \\ -0.4 \\ 0.33 \end{Bmatrix} \times 10^{-3}$$

Stresses can be obtained using Equation 5.94:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{0^\circ} = 10^{-3} \times \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1.4 \\ -0.4 \\ 0.33 \end{Bmatrix} \text{G Pa} = \begin{Bmatrix} 22 \\ 2 \\ 1 \end{Bmatrix} \text{M Pa}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{90^\circ} = 10^{-3} \begin{bmatrix} 6.67 & 3.33 & 0 \\ 3.33 & 16.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1.4 \\ -0.4 \\ 0.33 \end{Bmatrix} \text{G Pa} = \begin{Bmatrix} 8 \\ -2 \\ 1 \end{Bmatrix} \text{M Pa}$$

6.11 With reference to the given loading direction, the plies become $\pm 45^\circ$ plies. $\bar{Q}$ matrices can be evaluated as:

$$[\bar{Q}]_{45^\circ} = \begin{bmatrix} 10.5 & 4.5 & 2.5 \\ 4.5 & 10.5 & 2.5 \\ 2.5 & 2.5 & 4.17 \end{bmatrix} \text{G Pa}$$

$$[\bar{Q}]_{-45^\circ} = \begin{bmatrix} 10.5 & 4.5 & -2.5 \\ 4.5 & 10.5 & -2.5 \\ -2.5 & -2.5 & 4.17 \end{bmatrix} \text{ GPa}$$

For unit thickness of the laminate, extensional stiffness matrix becomes:

$$[A] = \begin{bmatrix} 10.5 & 4.5 & 0 \\ 4.5 & 10.5 & 0 \\ 0 & 0 & 4.17 \end{bmatrix} \text{ GPa}$$

For a unit cross-section of the laminate, stresses can be taken to be resultant forces in the corresponding directions, so that

$$10^{-3} \times \begin{Bmatrix} 30 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 10.5 & 4.5 & 0 \\ 4.5 & 10.5 & 0 \\ 0 & 0 & 4.17 \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Solving this matrix equation gives strains:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}_{45^\circ} = \begin{Bmatrix} 3.5 \\ -1.5 \\ 0 \end{Bmatrix} \times 10^{-3}$$

Stresses can be obtained using Equation 5.94:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{45^\circ} = 10^{-3} \times \begin{bmatrix} 10.5 & 4.5 & 2.5 \\ 4.5 & 10.5 & 2.5 \\ 2.5 & 2.5 & 4.17 \end{bmatrix} \begin{Bmatrix} 3.5 \\ -1.5 \\ 0 \end{Bmatrix} \text{ GPa} = \begin{Bmatrix} 30 \\ 0 \\ 5 \end{Bmatrix} \text{ MPa}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{-45^\circ} = 10^{-3} \times \begin{bmatrix} 10.5 & 4.5 & -2.5 \\ 4.5 & 10.5 & -2.5 \\ -2.5 & -2.5 & 4.17 \end{bmatrix} \begin{Bmatrix} 3.5 \\ -1.5 \\ 0 \end{Bmatrix} \text{ GPa} = \begin{Bmatrix} 30 \\ 0 \\ -5 \end{Bmatrix} \text{ MPa}$$

6.12

$$[\bar{Q}]_{0^\circ} = \begin{bmatrix} 30 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ GPa}$$

$$[\bar{Q}]_{45^\circ} = \begin{bmatrix} 9.75 & 7.75 & 6.75 \\ 7.75 & 9.75 & 6.75 \\ 6.75 & 6.75 & 7.75 \end{bmatrix} \text{ GPa}$$

$$[\bar{Q}]_{-45^\circ} = \begin{bmatrix} 9.75 & 7.75 & -6.75 \\ 7.75 & 9.75 & -6.75 \\ -6.75 & -6.75 & 7.75 \end{bmatrix} \text{ GPa}$$

<u>Symmetric Laminate $[45/\overline{0}]_s$</u>

$$A_{ij} = 8\,(\bar{Q}_{ij})_{45^\circ} + 4\,(\bar{Q}_{ij})_{0^\circ}$$

$$[A] = \begin{bmatrix} 198 & 66 & 54 \\ 66 & 90 & 54 \\ 54 & 54 & 66 \end{bmatrix} \text{ GPa. mm}$$

$$[B] = 0$$

$$D_{ij} = 2 \times 1/3\,(6^3 - 2^3)\,(\bar{Q}_{ij})_{45^\circ} + 1/3\,[2^3 - (-2)^3]\,(\bar{Q}_{ij})_{0^\circ}$$

$$= \frac{16}{3}[26\,(\bar{Q}_{ij})_{45^\circ} + (\bar{Q}_{ij})_{0^\circ}]$$

$$[D] = \begin{bmatrix} 1512 & 1080 & 936 \\ 1080 & 1368 & 936 \\ 936 & 936 & 1080 \end{bmatrix} \text{ GPa. mm}^3$$

Midplane strains and plate curvatures can be calculated independently

$$\begin{Bmatrix} 4000 \\ 4000 \\ 0 \end{Bmatrix} = 10^3 \times \begin{bmatrix} 198 & 66 & 54 \\ 66 & 90 & 54 \\ 54 & 54 & 66 \end{bmatrix} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$$

$$\begin{Bmatrix} 25000 \\ 0 \\ 0 \end{Bmatrix} = 10^3 \times \begin{bmatrix} 1512 & 1080 & 936 \\ 1080 & 1368 & 936 \\ 936 & 936 & 1080 \end{bmatrix} \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Solving above matrix equations, the following midplane strains and plate curvatures are obtained:

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 14.61 \\ 80.35 \\ -77.69 \end{Bmatrix}$$

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 43.78 \\ -21.14 \\ -19.63 \end{Bmatrix} \ mm^{-1}$$

Strains in the laminate can be written as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = 10^{-3} \begin{Bmatrix} 14.61 + 43.78Z \\ 80.35 - 21.14Z \\ -77.69 - 19.63Z \end{Bmatrix}$$

Stresses in the 45° laminae can be calculated by substituting appropriate values of Z ($-6 \leq Z \leq -2$ or $2 \leq Z \leq 6$):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{45^\circ} = \begin{bmatrix} 9.75 & 7.75 & 6.75 \\ 7.75 & 9.75 & 6.75 \\ 6.75 & 6.75 & 7.75 \end{bmatrix} \begin{Bmatrix} 14.61 + 43.78Z \\ 80.35 - 21.14Z \\ -77.69 - 19.63Z \end{Bmatrix} \text{MPa}$$

Stresses in the 0° lamina can be calculated by substituting appropriate values of Z ($-2 \leq Z \leq 2$)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{0^\circ} = \begin{bmatrix} 30 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 14.61 + 43.78Z \\ 80.35 - 21.14Z \\ -77.69 - 19.63Z \end{Bmatrix} \text{MPa}$$

Unsymmetrical Laminate (45/0/-45)

Assume that the +45° lamina is placed on the negative Z-axis and the -45° lamina on positive Z-axis. Laminate stiffness matrices can be calculated as follows:

$$[A] = \begin{bmatrix} 198 & 66 & 0 \\ 66 & 90 & 0 \\ 0 & 0 & 66 \end{bmatrix} \text{GPa} \cdot \text{mm}$$

$$[B] = \begin{bmatrix} 0 & 0 & -216 \\ 0 & 0 & -216 \\ -216 & -216 & 0 \end{bmatrix} \text{GPa} \cdot \text{mm}^2$$

$$[D] = \begin{bmatrix} 1512 & 1080 & 0 \\ 1080 & 1368 & 0 \\ 0 & 0 & 1080 \end{bmatrix} \text{GPa} \cdot \text{mm}^3$$

Midplane strains and plate curvatures can be calculated from the following matrix equation:

$$\begin{Bmatrix} 4000 \\ 4000 \\ 0 \\ 25000 \\ 0 \end{Bmatrix} = 10^3 \times \begin{bmatrix} 198 & 66 & 0 & 0 & 0 & -216 \\ 66 & 90 & 0 & 0 & 0 & -216 \\ 0 & 0 & 66 & -216 & -216 & 0 \\ 0 & 0 & -216 & 1512 & 1080 & 0 \\ 0 & 0 & -216 & 1080 & 1368 & 0 \\ -216 & -216 & 0 & 0 & 0 & 1080 \end{bmatrix} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Solutions of the above matrix equation gives midplane strains and plate curvatures are:

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy} \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 14.28 \\ 78.52 \\ 59.95 \\ 42.05 \\ -23.72 \\ 18.56 \end{Bmatrix}$$

Strains in the laminate can be written as:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 14.28 + 42.05Z \\ 78.52 - 23.72Z \\ 59.95 + 18.56Z \end{Bmatrix}$$

Stresses in the 45° lamina are ($-6 \leq Z \leq -2$):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{45^o} = \begin{bmatrix} 9.75 & 7.75 & 6.75 \\ 7.75 & 9.75 & 6.75 \\ 6.75 & 6.75 & 7.75 \end{bmatrix} \begin{Bmatrix} 14.28 + 42.05Z \\ 78.52 - 23.72Z \\ 59.95 + 18.56Z \end{Bmatrix} \text{ MPa}$$

Stresses in the 0° lamina are ($-2 \leq Z \leq 2$):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{0^\circ} = \begin{bmatrix} 30 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 14.28 + 42.05Z \\ 78.52 - 23.72Z \\ 59.95 + 18.56Z \end{Bmatrix} \text{MPa}$$

Stress in the -45° lamina are ($2 \leq Z \leq 6$):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{-45^\circ} = \begin{bmatrix} 9.75 & 7.75 & -6.75 \\ 7.75 & 9.75 & -6.75 \\ -6.75 & -6.75 & 7.75 \end{bmatrix} \begin{Bmatrix} 14.28 + 42.05Z \\ 78.52 - 23.72Z \\ 59.95 + 18.56Z \end{Bmatrix} \text{MPa}$$

6.13 Comparison of 6.48 and 6.49 shows:

$$\frac{\sigma}{E_e} = \frac{\sigma_A}{E} + \frac{\sigma - \sigma_A}{E_S}$$

$$\frac{1}{E_e} = \frac{1}{E}\left[\frac{\sigma_A}{\sigma} + \frac{E}{E_S}\left(1 - \frac{\sigma_A}{\sigma}\right)\right].$$

$$\frac{1}{E_e} = \frac{1}{E}\left[1 + \left(\frac{E}{E_S} - 1\right)\left(1 - \frac{\sigma_A}{\sigma}\right)\right]$$

Thus

$$E_e = \frac{E}{1 + \left(\frac{E}{E_S} - 1\right)\left(1 - \frac{\sigma_A}{\sigma}\right)}$$

6.14 Primary Modulus:

$$E = \frac{4}{7} \times 40 + \frac{3}{7} \times 8.5 = 26.5 \text{ GPa}$$

Secondary Modulus:

$$E_S = \frac{4}{7} \times 40 = 22.86 \text{ GPa}$$

Composite stress at knee:

$$\sigma_A = 26.5 \times \frac{0.38}{100} = 0.1007 \text{ GPa} = 100.7 \text{ MPa}$$

Composite stress at 1% strain:

$$\sigma = 100.7 + 22.86 \times 1000 \times (0.01 - 0.0038)$$

$$\sigma = 242.4 \text{ MPa}$$

Effective Modulus at 1% strain:

$$E_e = \frac{242.4}{0.01} = 24.24 \times 10^3 \text{ MPa} = 24.24 \text{ GPa}$$

(a) Stress strain curve will be a bilinear one with the slopes of the two lines being the primary and secondary modulus. Knee occurs at $\epsilon = 0.38\%$. Strain or stress will show no discontinuity at the knee. Fracture of the composite will occur at $\epsilon = 2.75\%$. Fracture stress will be:

$$\sigma_F = 100.7 + (0.0276 - 0.0038) \times 22.86 \times 10^3 = 642.4 \text{ MPa}$$

(b) In this case, strain will show a sudden change at the knee while maintaining the $\sigma = 100.7$ MPa . Increased strain at the knee will be

$$\varepsilon = \frac{100.7 \times 100}{22.86 \times 10^3} = 0.44\%$$

Stress at fracture

$$\sigma_F = \frac{2.75}{100} \times 22.86 = 0.6286 \text{ GPa}$$

$$\sigma_F = 628.6 \text{ MPa}$$

(c) In this case there will be a sudden drop in stress at the knee while maintaining $\varepsilon = 0.38\%$. Reduced stress at the knee will be

$$\sigma = 22.86 \times 10^3 \times 0.0038 = 86.86 \text{ MPa}$$

Fracture stress is same as in case (b).

6.15

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{Bmatrix} = 10^{-6} \times \begin{Bmatrix} 7 \\ 23 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{90^\circ} = \begin{Bmatrix} \alpha_T \\ \alpha_L \\ 0 \end{Bmatrix} = 10^{-6} \times \begin{Bmatrix} 23 \\ 7 \\ 0 \end{Bmatrix}$$

For a unit thickness of the laminate, thermal forces may be obtained as follows (Q matrices may be directly taken from Problem 6.10):

$$\begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix} = -100 \times 1/2 \times \left\{ \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} 7 \times 10^{-6} \\ 23 \times 10^{-6} \\ 0 \end{matrix} + \begin{bmatrix} 6.67 & 3.33 & 0 \\ 3.33 & 16.67 & 0 \\ 0 \quad 3 \quad 0 & & 3 \end{bmatrix} \begin{Bmatrix} 23 \times 10^{-6} \\ 7 \times 10^{-6} \\ 0 \end{Bmatrix} \right\}$$

$$\begin{Bmatrix} N_x^t \\ N_y^t \\ N_{xy}^t \end{Bmatrix} = -10^{-3} \begin{Bmatrix} 18.5 \\ 18.5 \\ 0 \end{Bmatrix}$$

Resultant thermal moments are zero due to midplane symmetry of the laminate.

Midplane strains may be obtained using the A matrix obtained in Problem 6·10, so that

$$-10^{-3} \times \begin{Bmatrix} 18.5 \\ 18.5 \\ 0 \end{Bmatrix} = \begin{bmatrix} 11.67 & 3.33 & 0 \\ 3.33 & 11.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix}$$

Solution of this matrix equation gives

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = -10^{-3} \begin{Bmatrix} 1.23 \\ 1.23 \\ 0 \end{Bmatrix}$$

Since plate curvatures are zero due to symmetry of the laminate, mechanical strains which cause thermal stresses can be evaluated as

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{0^o} = 10^{-3} \times \begin{Bmatrix} -1.23 + 0.7 \\ -1.23 + 2.3 \\ 0 \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} -0.53 \\ 1.07 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{90^o} = 10^{-3} \times \begin{Bmatrix} -1.23 + 2.3 \\ -1.23 + 0.7 \\ 0 \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 1.07 \\ -0.53 \\ 0 \end{Bmatrix}$$

Therefore, thermal stresses are

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_{0^\circ} = 10^{-3} \times \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} -0.53 \\ 1.07 \\ 0 \end{Bmatrix} \text{GPa} = \begin{Bmatrix} -5.33 \\ 5.33 \\ 0 \end{Bmatrix} \text{MPa}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_{90^\circ} = 10^{-3} \times \begin{bmatrix} 6.67 & 3.33 & 0 \\ 3.33 & 16.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1.07 \\ -0.53 \\ 0 \end{Bmatrix} \text{GPa} = \begin{Bmatrix} 5.33 \\ -5.33 \\ 0 \end{Bmatrix} \text{MPa}$$

Above calculations are independent of positions of the plies. Therefore, residual stresses will not be affected by interchanging position of $0^\circ$ and $90^\circ$ plies.

6.16 For a unit thickness of the laminate, [A] matrix and thermal forces are the same as in Problem 6.15:

$$[A] = \begin{bmatrix} 11.67 & 3.33 & 0 \\ 3.33 & 11.67 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = -\,10^{-3} \times \begin{Bmatrix} 18.5 \\ 18.5 \\ 0 \end{Bmatrix}$$

B and D matrices and thermal forces may be obtained as follows:

$$[B] = \begin{bmatrix} -1.25 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 0.972 & 0.278 & 0 \\ 0.278 & 0.972 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = -\frac{100}{2} \times \left\{ \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 7 \times 10^{-6} \\ 23 \times 10^{-6} \\ 0 \end{Bmatrix} (0 - \tfrac{1}{4}) - \begin{bmatrix} 6.67 & 3.33 & 0 \\ 3.33 & 16.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 23 \times 10^{-6} \\ 7 \times 10^{-6} \\ 0 \end{Bmatrix} (\tfrac{1}{4} - 0) \right\}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 0.2083 \\ -0.2083 \\ 0 \end{Bmatrix}$$

Midplane strains and plate curvatures can now be calculated using Equations 6·63 and 6·64

$$10^{-3} \times \begin{Bmatrix} -18.5 \\ -18.5 \\ 0 \\ 0.2083 \\ -0.2083 \\ 0 \end{Bmatrix} = \begin{bmatrix} 11.67 & 3.33 & 0 & -1.25 & 0 & 0 \\ 3.33 & 6.67 & 0 & 0 & 1.25 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ -1.25 & 0 & 0 & 0.972 & 0.278 & 0 \\ 0 & 1.25 & 0 & 0.278 & 0.972 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 \end{bmatrix} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Solving above matrix equation gives midplane strains and plate curvatures:

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} -1.4216 \\ -1.4216 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} -2.2589 \\ 2.2589 \\ 0 \end{Bmatrix}$$

Mechanical strains in the plies are:

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{0^\circ} = 10^{-3} \times \begin{Bmatrix} -1.4216 - 2.2589Z + 0.7 \\ -1.4216 + 2.2589Z + 2.3 \\ 0 \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} -0.7216 - 2.2589Z \\ 0.8784 + 2.2589Z \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{90^\circ} = 10^{-3} \times \begin{Bmatrix} -1.4216 - 2.2589Z + 2.3 \\ -1.4216 + 2.2589Z + 0.7 \\ 0 \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 0.8784 - 2.2589Z \\ -0.7216 + 2.2589Z \\ 0 \end{Bmatrix}$$

Thermal stresses are:

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \gamma_{xy}^T \end{Bmatrix}_{0^\circ} = \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} -0.7216 - 2.2589Z \\ 0.8784 + 2.2589Z \\ 0 \end{Bmatrix} \text{MPa}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_{90^\circ} = \begin{bmatrix} 6.67 & 3.33 & 0 \\ 3.33 & 16.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 0.8784 - 2.2589Z \\ -0.7216 + 2.2589Z \\ 0 \end{Bmatrix} \text{MPa}$$

6.17 $\bar{Q}$ matrices for 0 and $\pm$ 60 laminae can be calculated to be:

$$[\bar{Q}]_{0^\circ} = \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$[\bar{Q}]_{60^\circ} = \begin{bmatrix} 8.2917 & 4.2083 & 1.66 \\ 4.2083 & 13.2917 & 2.67 \\ 1.66 & 2.67 & 3.875 \end{bmatrix}$$

$$[\bar{Q}]_{-60^\circ} = \begin{bmatrix} 8.2917 & 4.2083 & -1.66 \\ 4.2083 & 13.2917 & -2.67 \\ -1.66 & -2.67 & 3.875 \end{bmatrix}$$

For unit thickness of each lamina and assuming that the $+60^\circ$ lamina is placed on the negative Z-axis and $-60^\circ$ lamina on the positive Z-axis, A, B and D matrices can be calculated as:

$$[A] = \begin{bmatrix} 33.25 & 11.75 & 0 \\ 11.75 & 33.25 & 0 \\ 0 & 0 & 10.75 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 & 0 & -3.32 \\ 0 & 0 & -5.34 \\ -3.32 & -5.34 & 0 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 19.3542 & 9.3958 & 0 \\ 9.3958 & 29.3542 & 0 \\ 0 & 0 & 8.6458 \end{bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 23 \\ 0 \end{Bmatrix} \times 10^{-6} \quad /^\circ C$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{60^\circ} = \begin{bmatrix} 1/4 & 3/4 & -\sqrt{3}/2 \\ 3/4 & 1/4 & \sqrt{3}/2 \\ \sqrt{3}/4 & -\sqrt{3}/4 & -1/2 \end{bmatrix} \begin{Bmatrix} 7 \\ 23 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 19 \\ 11 \\ -6.928 \end{Bmatrix} \times 10^{-6} \quad /^\circ C$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{-60^\circ} = \begin{Bmatrix} 19 \\ 11 \\ 6.928 \end{Bmatrix} \times 10^{-6} \quad /^\circ C$$

Thermal forces and moments can be calculated as follows:

$$\Delta T = 25 - 125 = -100^\circ C$$

$$\Delta T \times \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{0^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} -19.33 \\ -17.67 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\Delta T \times \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{60^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{60^\circ} = \begin{Bmatrix} -19.23 \\ -20.77 \\ -3.41 \end{Bmatrix} \times 10^{-3}$$

$$\Delta T \times \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_{-60^\circ} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{-60^\circ} = \begin{Bmatrix} -19.23 \\ -20.77 \\ +3.41 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \begin{Bmatrix} -19.33 \\ -17.67 \\ 0 \end{Bmatrix} \times 10^{-3} + \begin{Bmatrix} -19.23 \\ -20.77 \\ -3.41 \end{Bmatrix} \times 10^{-3} + \begin{Bmatrix} -19.23 \\ -20.77 \\ 3.41 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \begin{Bmatrix} -57.79 \\ -59.21 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = 1/2\ (1.5^2 - 0.5^2) \times 10^{-3} \times \begin{Bmatrix} -19.23 \\ -20.77 \\ 3.41 \end{Bmatrix} + 1/2\ (0.5^2 - 1.5^2) \times 10^{-3} \times \begin{Bmatrix} -19.23 \\ -20.77 \\ -3.41 \end{Bmatrix}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 0 \\ 0 \\ 6.82 \end{Bmatrix}$$

Midplane strains and plate curvatures can now be calculated using Equations 6.63 and 6.64:

$$10^{-3} \times \begin{Bmatrix} -57.79 \\ -59.21 \\ 0 \\ 0 \\ 0 \\ 6.82 \end{Bmatrix} = \begin{bmatrix} 33.25 & 11.75 & 0 & 0 & 0 & -3.32 \\ 11.75 & 33.25 & 0 & 0 & 0 & -5.34 \\ 0 & 0 & 10.75 & -3.32 & -5.34 & 0 \\ 0 & 0 & -3.32 & 19.3542 & 9.3958 & 0 \\ 0 & 0 & -5.34 & 9.3958 & 29.3542 & 0 \\ -3.32 & -5.34 & 0 & 0 & 0 & 8.6458 \end{bmatrix} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Solving above matrix equation gives midplane strains and plate curvatures as:

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} -1.2957 \\ -1.4166 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 0 \\ 0 \\ -0.5837 \end{Bmatrix}$$

Mechanical strains in the plies are:

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{0^o} = 10^{-3} \times \begin{Bmatrix} -1.2957 + 0.7 \\ -1.4166 + 2.3 \\ -0.5837\,Z + 0 \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} -0.5957 \\ 0.8834 \\ -0.5837\,Z \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{60^o} = 10^{-3} \times \begin{Bmatrix} -1.2957 + 1.9 \\ -1.4166 + 1.1 \\ -0.5837\,Z - .6928 \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 0.6043 \\ -0.3166 \\ -0.6928 - 0.5837\,Z \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = 10^{-3} \times \begin{Bmatrix} 0.6043 \\ -0.3166 \\ 0.6928 - 0.5837Z \end{Bmatrix}$$

Thermal stresses are:

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_{0^\circ} = \begin{bmatrix} 16.67 & 3.33 & 0 \\ 3.33 & 6.67 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} -0.5957 \\ 0.8834 \\ -0.5837Z \end{Bmatrix} \text{ MPa}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_{60^\circ} = \begin{bmatrix} 8.2917 & 4.2083 & 1.66 \\ 4.2083 & 13.2917 & 2.67 \\ 1.66 & 2.67 & 3.875 \end{bmatrix} \begin{Bmatrix} 0.6043 \\ -0.3166 \\ -0.6928 - 0.5837Z \end{Bmatrix} \text{ MPa}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix}_{-60^\circ} = \begin{bmatrix} 8.2917 & 4.2083 & -1.66 \\ 4.2083 & 8.2917 & -2.67 \\ -1.66 & -2.67 & 3.875 \end{bmatrix} \begin{Bmatrix} 0.6043 \\ -0.3166 \\ 0.6928 - .5837Z \end{Bmatrix} \text{ MPa}$$

6.18 (a) Taking x and y axes in the axial and circumferential directions the stresses can be written as

$$\sigma_x(\theta) = \sigma_x(-\theta) = \sigma_L \cos^2\theta$$
$$\sigma_y(\theta) = \sigma_y(-\theta) = \sigma_L \sin^2\theta$$
$$\tau_{xy}(\theta) = -\tau_{xy}(-\theta) = -\sigma_L \sin\theta \cos\theta$$

Integration of stresses over the thickness of the laminate gives

$$\sigma_{axial} = \frac{1}{2}\left[\sigma_x(\theta) + \sigma_x(-\theta)\right] = \sigma_L \cos^2\theta$$
$$\sigma_{hoop} = \frac{1}{2}\left[\sigma_y(\theta) + \sigma_x(-\theta)\right] = \sigma_L \sin^2\theta$$

(b)

$$\tau_{xy} = \frac{1}{2}\left[\tau_{xy}(\theta) + \tau_{xy}(-\theta)\right] = 0$$

(c) For a thin-walled, closed-ends pressure vessel

$$\sigma_{hoop} = 2\sigma_{axial}$$

Therefore,

$$\sigma_L \sin^2\theta = 2\sigma_L \cos^2\theta$$

or

$$\tan^2\theta = 2$$
$$\theta = 54.7°$$

6.19 0° plies may be assumed to be the outermost ply so that hoop directions coincides with the x-axis. Due to bending of plate into the tube, the plate curvatures can be written as

$$k_x = \frac{1}{50} = -0.02\text{mm}^{-1},\ k_y = k_{xy} = 0$$

where mean radius of the tube is assumed to be 5cm.

It can be easily visualized that applications of a torque on a thin tube is equivalent to a resultant shear force on the plate. Thus, the resultant forces are

$$N_x = N_y = 0,$$

$$N_{xy} = \frac{0.5 \times 1000}{50 \times 2\pi \times 50} = \frac{0.1}{\pi} = 0.0318 \text{ N / mm}$$

With the above loading and deformation conditions, analysis can be carried out using semi-inverted form of constitutive equation [Eq. (6.29)]. Calculation can be carried out in the following sequence.

$$[Q]_0 = [\bar{Q}]_{0^\circ} = \begin{bmatrix} 138.81 & 2.70 & 0 \\ 2.70 & 9.01 & 0 \\ 0 & 0 & 7.1 \end{bmatrix} GPa$$

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} 9.01 & 2.70 & 0 \\ 2.70 & 138.81 & 0 \\ 0 & 0 & 7.1 \end{bmatrix} GPa$$

$$[A] = \begin{bmatrix} 147.82 & 5.40 & 0 \\ 5.40 & 147.82 & 0 \\ 0 & 0 & 14.2 \end{bmatrix} GPa \cdot mm$$

$$[B] = \begin{bmatrix} -32.45 & & 0 & 0 \\ 0 & 32.45 & 0 & \\ 0 & 0 & 0 & \end{bmatrix} GPa \cdot mm^2$$

$$[D] = \begin{bmatrix} 49.27 & 1.80 & 0 \\ 1.80 & 49.27 & 0 \\ 0 & 0 & 4.73 \end{bmatrix} GPa \cdot mm^3$$

$$[A^{-1}] = 10^{-3} \begin{bmatrix} 6.7740 & -0.2475 & 0 \\ -0.2475 & 6.7740 & 0 \\ 0 & 0 & 70.4225 \end{bmatrix} \frac{1}{GPa \cdot mm}$$

$$[A^{-1}]\,[B] = 10^{-3} \begin{bmatrix} -219.82 & -8.03 & 0 \\ 8.03 & 219.82 & 0 \\ 0 & 0 & 0 \end{bmatrix} mm$$

Mid-plane strains are obtained by substituting values of {N}, {k}, $[A^{-1}]$, and $[A^{-1}]$ [B] in Eq. (6.27) with appropriate units:

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -4396.4 \\ 160.6 \\ 2.2 \end{Bmatrix}$$

From the knowledge of midplane strains and plate curvatures, strains can be calculated using Eq. (6.6) and stresses using (6.9). Strains and stresses on the outermost surface (z = -1) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -4396.4 \\ 160.6 \\ 2.2 \end{Bmatrix} + (-1) \begin{Bmatrix} -0.02 \\ 0 \\ 0 \end{Bmatrix} = 10^{-6} \begin{Bmatrix} 15{,}603.6 \\ 160.6 \\ 2.2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 10^{-6} \begin{bmatrix} 138.81 & 2.70 & 0 \\ 2.70 & 9.01 & 0 \\ 0 & 0 & 7.1 \end{bmatrix} \begin{Bmatrix} 15{,}603.6 \\ 160.6 \\ 2.2 \end{Bmatrix} GPa = \begin{Bmatrix} 2166.4 \\ 43.6 \\ 0.02 \end{Bmatrix} MPa$$

Strains and stresses on the innermost surface (Z = 1) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -4396.4 \\ 160.6 \\ 2.2 \end{Bmatrix} + \begin{Bmatrix} -0.02 \\ 0 \\ 0 \end{Bmatrix} = 10^{-6} \begin{Bmatrix} -24{,}396.4 \\ 160.6 \\ 2.2 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 10^{-6} \begin{bmatrix} 9.01 & 2.70 & 0 \\ 2.70 & 138.81 & 0 \\ 0 & 0 & 7.1 \end{bmatrix} \begin{Bmatrix} -24{,}396.4 \\ 160.6 \\ 2.2 \end{Bmatrix} GPa = \begin{Bmatrix} -219.4 \\ -43.6 \\ 0.02 \end{Bmatrix} MPa$$

**6.20.**

Obtain [Q] matrix for the lamina using Eq. (5.78)

$$[Q]=\begin{bmatrix}138.81 & 2.70 & 0\\ 2.70 & 9.01 & 0\\ 0 & 0 & 7.10\end{bmatrix}\text{GPa}$$

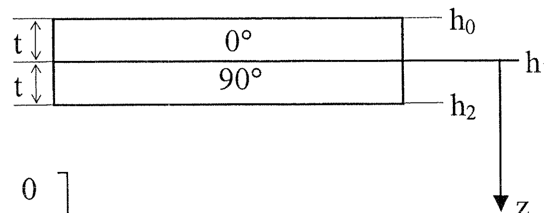
(Note: This is the same matrix as obtained in Example Problem 5.3)

$[\bar{Q}]$ matrices are:

$$[\bar{Q}]_{0^\circ}=\begin{bmatrix}138.81 & 2.70 & 0\\ 2.70 & 9.01 & 0\\ 0 & 0 & 7.10\end{bmatrix}\text{GPa}$$

$$[\bar{Q}]_{90^\circ}=\begin{bmatrix}9.01 & 2.70 & 0\\ 2.70 & 138.81 & 0\\ 0 & 0 & 7.10\end{bmatrix}\text{GPa}$$

By assuming thickness of each laminate to be t mm, obtain the following A, B and D matrices using Eq. (6.20)




$$[A]=\begin{bmatrix}147.82 & 5.41 & 0\\ 5.41 & 147.82 & 0\\ 0 & 0 & 14.2\end{bmatrix}\text{t GPa.mm}$$

$$[B]=\begin{bmatrix}-64.89 & 0 & 0\\ 0 & 64.89 & 0\\ 0 & 0 & 0\end{bmatrix}\text{t}^2\text{ GPa.mm}^2$$

$$[D]=\begin{bmatrix}49.28 & 1.80 & 0\\ 1.80 & 49.28 & 0\\ 0 & 0 & 4.73\end{bmatrix}\text{t}^3\text{ GPa.mm}^3$$

Thermal forces and moments can be calculated as follows:

$$\Delta T = -100\ ^\circ C$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = (-100)\left\{[\bar{Q}]_{0^\circ}\begin{Bmatrix} -0.30 \\ 28.10 \\ 0 \end{Bmatrix} \text{x}10^{-6} \text{x } t + [\bar{Q}]_{90^\circ}\begin{Bmatrix} 28.10 \\ -0.30 \\ 0 \end{Bmatrix} \text{x}10^{-6} \text{x } t\right\}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = \begin{Bmatrix} -286.6 \\ -286.6 \\ 0 \end{Bmatrix} \times 10^{-4} \text{t GPa.mm}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = \frac{1}{2}\text{x}(-100)\left\{[\bar{Q}]_{0^\circ}\begin{Bmatrix} -0.30 \\ 28.10 \\ 0 \end{Bmatrix} \text{x}10^{-6} \text{x } (-t^2) + [\bar{Q}]_{90^\circ}\begin{Bmatrix} 28.10 \\ -0.30 \\ 0 \end{Bmatrix} \text{x}10^{-6} \text{x } t^2\right\}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = \begin{Bmatrix} 1.09 \\ -1.09 \\ 0 \end{Bmatrix} \times 10^{-2} \text{t}^2 \text{ GPa.mm}^2$$

Plate curvatures can be obtained from Eq. (6.33) by first determining $C'$ and D' matrices as follows:

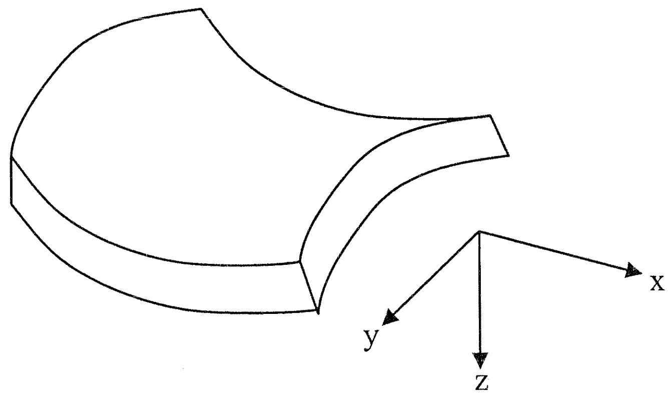
$$[C'] = \begin{bmatrix} 0.0211 & 0 & 0 \\ 0 & -0.0211 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{t}^{-2} \frac{1}{\text{GPa. mm}^2}$$

$$[D'] = \begin{bmatrix} 0.0483 & -0.0018 & 0 \\ -0.0018 & 0.0483 & 0 \\ 0 & 0 & 0.2113 \end{bmatrix} \times \text{t}^{-3} \frac{1}{\text{GPa.mm}^3}$$

The following plate curvatures are obtained from Eq. (6.33)

$$\kappa_x = -1.1535 \times 10^{-3} \text{t}^{-1}$$
$$\kappa_y = 1.1535 \times 10^{-3} \text{t}^{-1}$$
$$\kappa_{xy} = 0$$

Since $\kappa_x$ and $\kappa_y$ have opposite signs, the laminate curvatures in the x and y directions will have opposite signs. Thus, the laminate takes the form of a saddle as shown below:




Both plate curvatures, $\kappa_x$ and $\kappa_y$, increase as the lamina thickness increases, but their signs are independent of thickness. Therefore, the laminate curvatures in x and y directions will always have opposite signs, but warping will be more pronounced for thinner laminates. Further, if one of the inplane laminate dimensions (width or length) is small, the laminate may appear to take a cylindrical shape rather than a saddle shape.

## Chapter 7- Solution

**7.1:** Following are the properties of E-glass-epoxy from Appendix A.4.1:

$E_1 = 38.6$ GPa, $E_2 = 8.27$ GPa, $G_{12} = 4.14$ GPa, and $\nu_{12} = 0.26$

**(a)**

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{-\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} = \begin{bmatrix} 0.0259 & -0.006736 & 0 \\ -0.006736 & 0.121 & 0 \\ 0 & 0 & 0.21455 \end{bmatrix} 1/\text{GPa}$$

$$[Q]=[S]^{-1}=\begin{bmatrix} 39.17 & 2.18 & 0 \\ 2.18 & 8.39 & 0 \\ 0 & 0 & 4.14 \end{bmatrix} \text{GPa}$$

**(b)** For $\theta = 0^\circ$, $[\bar{Q}]_0 = [Q]$;

$$\text{For } \theta = 90^\circ,\ [\bar{Q}]_{90} = \begin{bmatrix} 8.39 & 2.18 & 0 \\ 2.18 & 39.17 & 0 \\ 0 & 0 & 4.14 \end{bmatrix} \text{GPa}$$

$$[D]=\frac{1}{3}\sum_{k=1}^{n}[\bar{Q}]_k\left(h_k^3-h_{k-1}^3\right)$$

**[0]₄ plate**

This laminate may be considered as one lamina of thickness 5mm,

so that $D = \frac{1}{3}[\bar{Q}]_{90} \times [(2.5)^3 - (-2.5)^3]$

$$[D]=\begin{bmatrix} 408 & 23 & 0 \\ 23 & 87 & 0 \\ 0 & 0 & 43 \end{bmatrix} \text{GPa.mm}^3$$

**[0/90]$_s$ plate**

$h_0 = -2.5$ mm, $h_1 = -1.25$ mm, $h_2 = 0$ mm, $h_3 = 1.25$ mm, $h_4 = 2.5$ mm,

$$[D] = \begin{bmatrix} 368 & 23 & 0 \\ 23 & 128 & 0 \\ 0 & 0 & 43 \end{bmatrix} \text{GPa.mm}^3$$

**(c)** Center deflection is obtained from Eq.(7.32). Substitution of numerical values for $D_{11}$, $D_{22}$, $D_{12}$, $D_{66}$, $p_o = 25$ Pa, and $a = b = 0.5$ m in Eq.(7.32) gives the plate center deflection for $[0]_4$ plate as:

$$w_0(0.25,0.25) = 0.416 \sum_{m=1,3,...}^{\infty} \sum_{n=1,3,...}^{\infty} \frac{(-1)^{1-\frac{m+n}{2}}}{mn\left(6{,}527.84m^4 + 3{,}487.36m^2n^2 + 1{,}399.52n^4\right)}$$

**Note:** GPa.mm$^3$ = Pa.m$^3$. Therefore, the values of $D_{11}$ etc. obtained above can be directly substituted to obtain deflection in meters.

Center deflection ($w_0 \times 10^{-5}$m) is obtained from the above series by considering a finite number of terms in the series. The results for different number of terms are as follows:

| m\n | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| 1 | 3.6452 | 3.5535 | 3.5621 | 3.5604 | 3.5609 | 3.5607 |
| 3 | 3.5360 | 3.5410 | 3.5398 | 3.5401 | 3.5400 | 3.5401 |
| 5 | 3.5421 | 3.5415 | 3.5418 | 3.5416 | 3.5417 | 3.5417 |
| 7 | 3.5413 | 3.5414 | 3.5413 | 3.5414 | 3.5414 | 3.5414 |
| 9 | 3.5414 | 3.5414 | 3.5415 | 3.5415 | 3.5415 | 3.5415 |
| 11 | 3.5414 | 3.5414 | 3.5415 | 3.5415 | 3.5415 | 3.5415 |

The table makes it obvious that the center deflection, $w_0$, converges to $3.54\times10^{-5}$ m.

By following the procedure of Example 7.1, the mid-plane strains and plate curvatures are obtained as:

$$\{\varepsilon^0\} = 0$$

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} 1.343 \\ 1.173 \\ 0 \end{Bmatrix} \times 10^{-3}$$

The maximum plate stresses are:

$\sigma_x = 138,000$ Pa , $\sigma_y = 31,900$ Pa, and $\tau_{xy} = 0$.

The center deflection for $[0/90]_s$ plate is:

$$w_0(0.25,0.25) = 0.416 \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{1-\frac{m+n}{2}}}{mn\left(5,886.72m^4 + 3,487.36m^2n^2 + 2,039.68n^4\right)}$$

Plate center deflections ($w_0 \times 10^{-5}$ m) for different number of terms are given in the following Table:

| m/n | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| 1 | 3.6452 | 3.5768 | 3.5828 | 3.5817 | 3.5820 | 3.5819 |
| 3 | 3.5547 | 3.5597 | 3.5586 | 3.5589 | 3.5588 | 3.5588 |
| 5 | 3.5610 | 3.5604 | 3.5607 | 3.5606 | 3.5606 | 3.5606 |
| 7 | 3.5602 | 3.5603 | 3.5603 | 3.5603 | 3.5603 | 3.5603 |
| 9 | 3.5604 | 3.5604 | 3.5604 | 3.5604 | 3.5604 | 3.5604 |
| 11 | 3.5604 | 3.5604 | 3.5604 | 3.5604 | 3.5604 | 3.5604 |

The center deflection converges to $3.56 \times 10^{-5}$ m.

The mid-plane strains and plate curvatures are obtained as:

$$\{\varepsilon_0\} = 0$$

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} 1.345 \\ 1.238 \\ 0 \end{Bmatrix} \times 10^{-3}$$

The maximum plate stresses are:

$\sigma_x = 138,000$ Pa, $\sigma_y = 33,300$ Pa, and $\tau_{xy} = 0$

**7.2:** $[Q]$ and $[\bar{Q}]$ matrices are the same as in Problem 7.1. Following the procedure of Problem 7.1, the following results are obtained:

**$[0]_4$ plate**

$$[D] = \begin{bmatrix} 3{,}264 & 181.8 & 0 \\ 181.8 & 699.3 & 0 \\ 0 & 0 & 345 \end{bmatrix} \quad \text{GPa} \cdot \text{mm}^3$$

$w_o = 7.08 \text{x} 10^{-5} \text{m}$

$\sigma_x = 138{,}000$ Pa, $\sigma_y = 31{,}900$ Pa and $\tau_{xy} = 0$

**$[0/90]_s$ plate**

$$[D] = \begin{bmatrix} 2{,}943 & 181.8 & 0 \\ 181.8 & 1{,}020 & 0 \\ 0 & 0 & 345 \end{bmatrix} \quad \text{GPa} \cdot \text{mm}^3$$

$w_o = 7.12 \text{x} 10^{-5} \text{m}$

$\sigma_x = 138{,}000$ Pa, $\sigma_y = 33{,}300$ Pa, and $\tau_{xy} = 0$

**7.3:** Following are the properties of Kevlar 49-epoxy from Appendix A.4.1:

$E_1 = 76.0$ GPa, $E_2 = 5.50$ GPa, $G_{12} = 2.30$ GPa and $\nu_{12} = 0.34$

Following the procedure of Problem 7.1, the following results are obtained:

$$[Q] = \begin{bmatrix} 76.44 & 1.89 & 0 \\ 1.89 & 5.54 & 0 \\ 0 & 0 & 2.3 \end{bmatrix} \quad \text{GPa}$$

<u>I. Plate thickness = 5 mm, plate edge = 0.5 m</u>

**$[0]_4$ plate**

$$[D] = \begin{bmatrix} 796 & 19.6 & 0 \\ 19.6 & 57.8 & 0 \\ 0 & 0 & 24 \end{bmatrix} \quad \text{GPa.mm}^3$$

$w_o = 2.5 \times 10^{-5}$ m

$\sigma_x = 186{,}500$ Pa, $\sigma_y = 13{,}800$ Pa, and $\tau_{xy} = 0$

**[0/90]$_s$ plate**

$$[D] = \begin{bmatrix} 704 & 19.6 & 0 \\ 19.6 & 150 & 0 \\ 0 & 0 & 24 \end{bmatrix} \text{ GPa.mm}^3$$

$w_o = 2.56 \times 10^{-5}$ m

$\sigma_x = 191{,}200$ Pa, $\sigma_y = 16{,}500$ Pa, and $\tau_{xy} = 0$

II. Plate thickness = 10 mm, plate edge = 1 m

**[0]$_4$ plate**

$$[D] = \begin{bmatrix} 6{,}370 & 157 & 0 \\ 157 & 462 & 0 \\ 0 & 0 & 192 \end{bmatrix} \text{ GPa.mm}^3$$

$w_o = 5.0 \times 10^{-5}$ m

$\sigma_x = 186{,}500$ Pa, $\sigma_y = 13{,}800$ Pa and $\tau_{xy} = 0$

**[0/90]$_s$ plate**

$$[D] = \begin{bmatrix} 5{,}632 & 157 & 0 \\ 157 & 1{,}201 & 0 \\ 0 & 0 & 192 \end{bmatrix} \text{ GPa.mm}^3$$

$w_o = 5.12 \times 10^{-5}$ m

$\sigma_x = 191{,}200$ Pa, $\sigma_y = 16{,}500$ Pa and $\tau_{xy} = 0$

**7.4:** Following are the properties of T300/N5208 Carbon-epoxy from Appendix A.4.1:

$E_1 = 181.0$ GPa, $E_2 = 10.30$ GPa, $G_{12} = 7.17$ GPa and $\nu_{12} = 0.28$

Following the procedure of Problem 7.1, the following results are obtained:

$$[Q] = \begin{bmatrix} 182 & 2.9 & 0 \\ 2.9 & 10.3 & 0 \\ 0 & 0 & 7.2 \end{bmatrix} \text{GPa.mm}^3$$

I. Plate thickness = 5 mm, plate edge = 0.5 m

**$[0]_4$ plate**

$$[D] = \begin{bmatrix} 1,894 & 30.2 & 0 \\ 30.2 & 108 & 0 \\ 0 & 0 & 74.7 \end{bmatrix} \text{GPa.mm}^3$$

$w_o = 1.04\text{x}10^{-5}$ m

$\sigma_x = 183{,}250$ Pa, $\sigma_y = 9{,}576$ Pa, and $\tau_{xy} = 0$

**$[0/90]_s$ plate**

$$[D] = \begin{bmatrix} 1,671 & 30.2 & 0 \\ 30.2 & 331 & 0 \\ 0 & 0 & 74.7 \end{bmatrix} \text{GPa.mm}^3$$

$w_o = 1.07\text{x}10^{-5}$ m

$\sigma_x = 188{,}800$ Pa, $\sigma_y = 12{,}150$ Pa, and $\tau_{xy} = 0$

II. Plate thickness = 10 mm, plate edge = 1 m

**$[0]_4$ plate**

$$[D] = \begin{bmatrix} 15,151 & 241.4 & 0 \\ 241.4 & 862.2 & 0 \\ 0 & 0 & 597.5 \end{bmatrix} \text{GPa.mm}^3$$

$w_o = 2.08\text{x}10^{-5}$ m

$\sigma_x = 183{,}250$ Pa, $\sigma_y = 9{,}580$ Pa, and $\tau_{xy} = 0$

**[0/90]$_s$ plate**

$$[D]=\begin{bmatrix} 13,365 & 241.4 & 0 \\ 241.4 & 2,648 & 0 \\ 0 & 0 & 597.5 \end{bmatrix} \text{ GPa.mm}^3$$

$w_o = 2.14 \times 10^{-5}$ m

$\sigma_x = 188,800$ Pa, $\sigma_y = 12,150$ Pa, and $\tau_{xy} = 0$

**7.5:** Procedure for obtaining center deflection and stresses in a rectangular plate is the same as that for a square plate (Example 7.1 and Problem 7.1). $[Q]$,$[\overline{Q}]$ and $[D]$ matrices for this problem have been obtained in problems 7.1 – 7.4. The following table gives the center deflection and stresses for 1 m$\times$0.5 m plates of different materials and thicknesses:

| Laminate Lay-up and Thickness | Fiber Type | $w_o$ ($10^{-5}$m) | Normal Stresses, Pa | |
|---|---|---|---|---|
| | | | $\sigma_x$ | $\sigma_y$ |
| $[0]_4$, 5 mm | Glass | 15.14 | 157,900 | 127,100 |
| | Kevlar | 18.10 | 350,500 | 101,300 |
| | Carbon | 8.08 | 362,200 | 82,500 |
| $[0/90]_S$, 5 mm | Glass | 12.30 | 123,400 | 103,700 |
| | Kevlar | 11.10 | 205,000 | 63,600 |
| | Carbon | 4.84 | 205,300 | 50,800 |
| $[0]_4$, 10 mm | Glass | 1.89 | 39,500 | 31,800 |
| | Kevlar | 2.26 | 87,600 | 25,300 |
| | Carbon | 1.01 | 90,500 | 20,600 |
| $[0/90]_S$, 10 mm | Glass | 1.54 | 30,800 | 25,900 |
| | Kevlar | 1.39 | 51,300 | 15,900 |
| | Carbon | 0.61 | 51,300 | 12,700 |

**7.6:** Uniaxial buckling load for rectangular plates is obtained from Eq.(7.43). Stiffness matrices for different plates have been obtained in Problems 7.1-7.4. Buckling loads are obtained for different sets of m and n values and the appropriate stiffness. The following table gives buckling loads for different cases. It may be pointed out that for most cases critical buckling load occurs when m = n = 1. However, for some cases (marked with an asterisk) it occurs for m = 1 and n = 2.

Uniaxial buckling loads for square plates, N/m

| | | $[0]_4$ | | $[0/90]_s$ | |
|---|---|---|---|---|---|
| | | $N_{0x}$ | $N_{0y}$ | $N_{0x}$ | $N_{0y}$ |
| E-glass-epoxy | a = b = 0.5 m<br>h = 5 mm | 28,200 | 26,430* | 28,200 | 28,160 |
| | a = b = 1 m<br>h = 10 mm | 56,300 | 52,870* | 56,300 | 56,300 |
| Kevlar-epoxy | a = b = 0.5 m<br>h = 5 mm | 39,100 | 22,320* | 39,100 | 35,980* |
| | a = b = 1 m<br>h = 10 mm | 78,100 | 44,630* | 78,100 | 72,000* |
| Carbon-epoxy | a = b = 0.5 m<br>h = 5 mm | 93,200 | 49,890* | 93,200 | 82,900* |
| | a = b = 1 m<br>h = 10 mm | 186,000 | 99,770* | 186,000 | 165,900* |

**7.7:**

Uniaxial buckling loads for rectangular plates (a =1.0 m, b = 0.5 m), N/m

| | | $N_{0x}$ | |
|---|---|---|---|
| | | $[0]_4$ | $[0/90]_s$ |
| E-glass-epoxy | h = 5 mm | 26,430<br>(m=1,n=1) | 28,160<br>(m=2,n=1) |
| | h = 10 mm | 211,500<br>(m=1,n=1) | 225,300<br>(m=2,n=1) |
| Kevlar-epoxy | h = 5 mm | 22,320<br>(m=1,n=1) | 35,980<br>(m=1,n=1) |
| | h = 10 mm | 178,500<br>(m=1,n=1) | 287,900<br>(m=1,n=1) |
| Carbon-epoxy | h = 5 mm | 49,900<br>(m=1,n=1) | 82,900<br>(m=1,n=1) |
| | h = 10 mm | 399,000<br>(m=1,n=1) | 664,000<br>(m=1,n=1) |

**7.8:**

Equibiaxial buckling loads for plates in Problems 7.1-7.4

| | | $N_0$ (N/m) | |
|---|---|---|---|
| | | $[0]_4$ | $[0/90]_s$ |
| E-glass-epoxy | a = b = 0.5 m | 14,100 (m=1,n=1) | 14,100 (m=1,n=1) |
| | a = b = 1 m | 28,200 (m=1,n=1) | 28,200 (m=1,n=1) |
| Kevlar-epoxy | a = b = 0.5 m | 17,900 (m=1,n=2) | 19,600 (m=1,n=1) |
| | a = b = 1 m | 35,700 (m=1,n=2) | 39,100 (m=1,n=1) |
| Carbon-epoxy | a = b = 0.5 m | 39,900 (m=1,n=2) | 46,600 (m=1,n=1) |
| | a = b = 1 m | 80,000 (m=1,n=2) | 93,000 (m=1,n=1) |

Equibiaxial buckling loads for plates in Problem 7.5

| | | $[0]_4$ | | $[0/90]_s$ | |
|---|---|---|---|---|---|
| | | $N_0$ (N/m) (a=1m, b=0.5m) | $N_0$ (N/m) (a=0.5m, b=1m) | $N_0$ (N/m) (a=1m, b=0.5m) | $N_0$ (N/m) (a=0.5m, b=1m) |
| E-glass-epoxy | h = 5 mm | 5,290 (m=1,n=1) | 14,100 (m=2,n=1) | 6,470 (m=1,n=1) | 13,600 (m=1,n=1) |
| | h = 10 mm | 42,300 (m=1,n=1) | 113,000 (m=2,n=1) | 51,800 (m=1,n=1) | 109,000 (m=1,n=1) |
| Kevlar-epoxy | h = 5 mm | 4,460 (m=1,n=1) | 16,900 (m=3,n=1) | 7,200 (m=1,n=1) | 19,500 (m=2,n=1) |
| | h = 10 mm | 35,700 (m=1,n=1) | 135,000 (m=3,n=1) | 57,600 (m=1,n=1) | 156,000 (m=2,n=1) |
| Carbon-epoxy | h = 5 mm | 10,000 (m=1,n=1) | 39,400 (m=3,n=1) | 16,600 (m=1,n=1) | 46,600 (m=2,n=1) |
| | h = 10 mm | 80,000 (m=1,n=1) | 316,000 (m=3,n=1) | 133,000 (m=1,n=1) | 373,000 (m=2,n=1) |

**7.9:** Natural frequencies are obtained from Eq. (7.64). Stiffness matrices for different plates have already been obtained in earlier problems. Area densities are calculated as follows:

**Glass – epoxy**

5 mm thick plate:

$$\rho_o = \rho \times h = 1800 \times 0.005 = 9 \text{ Kg/m}^2$$

10 mm thick plate:

$$\rho_o = \rho \times h = 1800 \times 0.010 = 18 \text{ Kg/m}^2$$

**Kevlar – epoxy**

5 mm thick plate:

$$\rho_o = \rho \times h = 1460 \times 0.005 = 7.3 \text{ Kg/m}^2$$

10 mm thick plate:

$$\rho_o = \rho \times h = 1460 \times 0.010 = 14.6 \text{ Kg/m}^2$$

**Carbon – epoxy**

5mm thick plate:

$$\rho_o = \rho \times h = 1600 \times 0.005 = 8 \text{ Kg/m}^2$$

10mm thick plate:

$$\rho_o = \rho \times h = 1600 \times 0.010 = 16 \text{ Kg/m}^2$$

Using Eq. (7.64), the following results are obtained:

Glass – epoxy (5 mm thick square plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 352 | 1 | 1 | 352 |
| 2 | 1 | 2 | 681 | 1 | 2 | 754 |
| 3 | 2 | 1 | 1,139 | 2 | 1 | 1,092 |
| 4 | 1 | 3 | 1,279 | 2 | 2 | 1,406 |
| 5 | 2 | 2 | 1,406 | 1 | 3 | 1,480 |

Glass – epoxy (10 mm thick square plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 176 | 1 | 1 | 176 |
| 2 | 1 | 2 | 341 | 1 | 2 | 377 |
| 3 | 2 | 1 | 569 | 2 | 1 | 546 |
| 4 | 1 | 3 | 640 | 2 | 2 | 703 |
| 5 | 2 | 2 | 703 | 1 | 3 | 740 |

Kevlar – epoxy (5 mm thick square plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 460 | 1 | 1 | 460 |
| 2 | 1 | 2 | 695 | 1 | 2 | 883 |
| 3 | 1 | 3 | 1,195 | 2 | 1 | 1,600 |
| 4 | 2 | 1 | 1,690 | 1 | 3 | 1,735 |
| 5 | 2 | 2 | 1,840 | 2 | 2 | 1,840 |

Kevlar – epoxy (10 mm thick square plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 230 | 1 | 1 | 230 |
| 2 | 1 | 2 | 442 | 1 | 2 | 442 |
| 3 | 1 | 3 | 598 | 2 | 1 | 800 |
| 4 | 2 | 1 | 845 | 1 | 3 | 868 |
| 5 | 2 | 2 | 920 | 2 | 2 | 920 |

Carbon – epoxy (5 mm thick square plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 678 | 1 | 1 | 678 |
| 2 | 1 | 2 | 992 | 1 | 2 | 1,280 |
| 3 | 1 | 3 | 1,643 | 2 | 1 | 2,356 |
| 4 | 2 | 1 | 2,491 | 1 | 3 | 2,486 |
| 5 | 2 | 2 | 2,713 | 2 | 2 | 2,713 |

Carbon – epoxy (10 mm thick square plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 339 | 1 | 1 | 339 |
| 2 | 1 | 2 | 496 | 1 | 2 | 640 |
| 3 | 1 | 3 | 822 | 2 | 1 | 1,178 |
| 4 | 2 | 1 | 1,245 | 1 | 3 | 1,243 |
| 5 | 1 | 4 | 1,310 | 2 | 2 | 1,356 |

Glass – epoxy (5 mm thick rectangular plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 170 | 1 | 1 | 188 |
| 2 | 2 | 1 | 352 | 2 | 1 | 352 |
| 3 | 1 | 2 | 533 | 1 | 2 | 629 |
| 4 | 3 | 1 | 677 | 3 | 1 | 655 |
| 5 | 2 | 2 | 681 | 2 | 2 | 754 |

Glass – epoxy (10 mm thick rectangular plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 341 | 1 | 1 | 377 |
| 2 | 2 | 1 | 703 | 2 | 1 | 703 |
| 3 | 1 | 2 | 1,067 | 1 | 2 | 1,257 |
| 4 | 3 | 1 | 1,353 | 3 | 1 | 1,311 |
| 5 | 2 | 2 | 1,362 | 2 | 2 | 1,507 |

Kevlar – epoxy (5 mm thick rectangular plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 174 | 1 | 1 | 221 |
| 2 | 2 | 1 | 460 | 2 | 1 | 460 |
| 3 | 1 | 2 | 487 | 1 | 2 | 742 |
| 4 | 2 | 2 | 695 | 2 | 2 | 882 |
| 5 | 3 | 1 | 968 | 3 | 1 | 926 |

Kevlar – epoxy (10 mm thick rectangular plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 347 | 1 | 1 | 441 |
| 2 | 2 | 1 | 919 | 2 | 1 | 919 |
| 3 | 1 | 2 | 973 | 1 | 2 | 1,485 |
| 4 | 2 | 2 | 1,390 | 2 | 2 | 1,765 |
| 5 | 3 | 1 | 1,937 | 3 | 1 | 1,852 |

Carbon – epoxy (5 mm thick rectangular plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 248 | 1 | 1 | 320 |
| 2 | 1 | 2 | 655 | 2 | 1 | 678 |
| 3 | 2 | 1 | 678 | 1 | 2 | 1,059 |
| 4 | 2 | 2 | 992 | 2 | 2 | 1,280 |
| 5 | 1 | 3 | 1,372 | 3 | 1 | 1,367 |

Carbon – epoxy (10 mm thick rectangular plate)

| Mode | $[0]_4$ | | | $[0/90]_s$ | | |
|---|---|---|---|---|---|---|
| | m | n | $\omega_{mn}$ (rad/s) | m | n | $\omega_{mn}$ (rad/s) |
| 1 | 1 | 1 | 496 | 1 | 1 | 640 |
| 2 | 1 | 2 | 1,310 | 2 | 1 | 1,356 |
| 3 | 2 | 1 | 1,356 | 1 | 2 | 2,119 |
| 4 | 2 | 2 | 1,985 | 2 | 2 | 2,559 |
| 5 | 1 | 3 | 2,743 | 3 | 1 | 2,735 |

**7.10:** Displacement $w(x, y)$ can be obtained from the series given in Eq.(7.30). However, for the given load function, $p(x, y)$, only two terms in the series are non-zero, which are as follows:

$$p_{13} = 7$$
$$p_{52} = -4$$

All other $p_{mn}$ terms are zero. Therefore, the displacement function becomes:

$$w(x, y) = w_{13} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right) + w_{52} \sin\left(\frac{5\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right)$$

Where

$$w_{13} = \frac{7}{\pi^4\left(\frac{D_{11}}{a^4} + \frac{18}{a^2b^2}(D_{12} + 2D_{66}) + \frac{81D_{22}}{b^4}\right)}$$

$$w_{52} = \frac{-4}{\pi^4\left(\frac{625D_{11}}{a^4} + \frac{200}{a^2b^2}(D_{12} + 2D_{66}) + \frac{16D_{22}}{b^4}\right)}$$

**7.11:** Deflection curve for the beam with uniformly distributed load is given by Eq.(7.89). The constants of integration are obtained by the substitution of following end conditions.

At x = 0 (clamped),

$$w_0 = 0, \quad \frac{dw_0}{dx} = 0$$

At x = L (simply supported),

$$w_0 = 0, \quad \frac{d^2w_0}{dx^2} = 0$$

Substitution of the above end conditions in Eq. (7.89) gives the constants of integration as

$$C_1 = -\frac{5p_0L}{8bD_{11}}, \; C_2 = \frac{p_0L^2}{8bD_{11}}, \; C_3 = 0, \text{ and } C_4 = 0$$

Substitution of these constants in Eq. (7.89) gives the following equation for the transverse deflection of the beam:

$$w_0(x) = \frac{p_0}{48bD_{11}}\left(2x^4 - 5Lx^3 + 3L^2x^2\right)$$

**7.12:** Constants of integration are obtained for the new boundary conditions as follows:
At x = 0 (clamped),

$$w_0 = 0, \quad \frac{dw_0}{dx} = 0$$

At x = L (free end),

$$\frac{d^2w_0}{dx^2} = 0, \quad \frac{d^3w_0}{dx^3} = 0$$

Substitution of the above end conditions in Eq. (7.89) gives the constants of integration as

$$C_1 = -\frac{p_0L}{bD_{11}}, \quad C_2 = -\frac{3p_0L^2}{2bD_{11}}, \quad C_3 = 0 \text{ and } C_4 = 0$$

Substitution of these constants in Eq. (7.89) gives the following equation for the transverse deflection of the beam:

$$w_0(x) = \frac{p_0}{24bD_{11}}\left(3x^4 - 4Lx^3 - 18L^2x^2\right)$$

**7. 13:** Natural frequencies of the beam can be calculated by using Eq. (7.111) and the mode shape by Eq. (7.110). For the given beam dimensions (l = 1 m, b = 0.050 m, and h = 0.010 m) the frequencies are as follows.

| Material and Lay-up | | $D_{11}$ (Pa.m$^3$) | ρ (Kg/ m$^3$) | $\omega_1$ (Hz) | $\omega_2$ (Hz) | $\omega_3$ (Hz) |
|---|---|---|---|---|---|---|
| Glass - epoxy | $[0]_4$ | 3,264 | 1,800 | 133 | 532 | 1,196 |
| | $[0/90]_s$ | 2,943 | 1,800 | 126 | 505 | 1,136 |
| Kevlar - epoxy | $[0]_4$ | 6,270 | 1,460 | 206 | 825 | 1,855 |
| | $[0/90]_s$ | 5,362 | 1,460 | 194 | 775 | 1,745 |
| Carbon - epoxy | $[0]_4$ | 15,151 | 1,600 | 304 | 1,215 | 2,733 |
| | $[0/90]_s$ | 13,365 | 1,600 | 285 | 1,141 | 2,567 |

Mode shapes corresponding to the first, second and third natural frequencies are given by the following functions,

$$W_x = \sin \pi x$$

$$W_x = \sin 2\pi x$$

$$W_x = \sin 3\pi x$$

**7.14:** It can be shown that for a beam clamped at both ends, the following series will satisfy the governing equation (7.102) as well as the boundary conditions:

$$w_0(x) = \sum_{m=1}^{\infty} w_m \left( \cos\frac{2m\pi x}{L} - 1 \right) \quad (m=1, 2, 3, \ldots)$$

Substituting this equation into (7.102) gives:

$$N_0 = \frac{4\pi^2 b D_{11} m^2}{L^2}$$

Critical buckling load is obtained by substituting m = 1: $(N_0)_{cri} = \frac{4\pi^2 b D_{11}}{L^2}$

Numerical values are given in the following Table:

| Material | Lay-up | $D_{11}$ (Pa.m$^3$) | $(N_0)_{cri}$ (N) |
|---|---|---|---|
| Glass-epoxy | $[0]_4$ | 3,264 | 6,443 |
| | $[0/90]_s$ | 2,943 | 5,810 |
| Kevlar-epoxy | $[0]_4$ | 6,370 | 12,570 |
| | $[0/90]_s$ | 5,362 | 11,200 |
| Carbon-epoxy | $[0]_4$ | 15,150 | 29,910 |
| | $[0/90]_s$ | 13,370 | 26,380 |

(*L = 1 m, b = 0.050 m, thickness = 10 mm).

# Chapter 8

**8.1.**

Following are the $\bar{Q}$ matrices

$$[\bar{Q}]_{0^\circ} = \begin{bmatrix} 148.43 & 3.21 & 0 \\ 3.21 & 11.07 & 0 \\ 0 & 0 & 5.3 \end{bmatrix} \text{ GPa}$$

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} 11.07 & 3.21 & 0 \\ 3.21 & 148.43 & 0 \\ 0 & 0 & 5.3 \end{bmatrix} \text{ GPa}$$

$$[\bar{Q}]_{45^\circ} = \begin{bmatrix} 46.78 & 36.18 & 34.34 \\ 36.18 & 46.78 & 34.34 \\ 34.34 & 34.34 & 38.27 \end{bmatrix} \text{ GPa}$$

$$[\bar{Q}]_{-45^\circ} = \begin{bmatrix} 46.78 & 36.18 & -34.34 \\ 36.18 & 46.78 & -34.34 \\ -34.34 & -34.34 & 38.27 \end{bmatrix} \text{ GPa}$$

$[0/\pm 45/90]_{2s}$

For unit thickness of laminae at each orientation the [A] matrix may be written as:

$$[A] = \begin{bmatrix} 253.06 & 78.78 & 0 \\ 78.78 & 253.06 & 0 \\ 0 & 0 & 87.14 \end{bmatrix}$$

$$k_T = 1 + \sqrt{\frac{2}{253.06}\left(\sqrt{253.06^2} - 78.78 + \frac{253.06^2 - 78.78^2}{2 \times 87.14}\right)}$$

$$K_T = 3$$

Note that the laminate under consideration is quasi-isotropic and hence, isotropic stress concentration factor of 3 is obtained for this laminate through Eq. 8.20.

$[0/90]_{4s}$

For unit thickness of laminae at each orientation, the [A] matrix may be obtained as:

$$[A] = \begin{bmatrix} 159.50 & 6.42 & 0 \\ 6.42 & 159.50 & 0 \\ 0 & 0 & 10.6 \end{bmatrix}$$

$$K_T = 1 + \sqrt{\frac{2}{159.50}\left(\sqrt{159.50^2} - 6.42 + \frac{159.50^2 - 6.42^2}{2 \times 10.6}\right)}$$

$$K_T = 5.116$$

8.2. For quasi-isotropic laminate $[o/\pm 45/90]_s$, $k_T$ will be same, that is 3, as in prob. 8.1;

Following are the $\bar{Q}$ matrices for 0° and 90° plies:

$$[\bar{Q}]_{0^\circ} = \begin{bmatrix} 39.17 & 2.19 & 0 \\ 2.19 & 8.42 & 0 \\ 0 & 0 & 4.1 \end{bmatrix}$$

$$[\bar{Q}]_{90^\circ} = \begin{bmatrix} 8.42 & 2.19 & 0 \\ 2.19 & 39.17 & 0 \\ 0 & 0 & 4.1 \end{bmatrix}$$

For unit thickness of laminae at each orientation, the [A] matrix may be obtained as:

$$[A] = \begin{bmatrix} 47.59 & 4.38 & 0 \\ 4.38 & 47.59 & 0 \\ 0 & 0 & 8.2 \end{bmatrix}$$

$$K_T = 1 + \sqrt{\frac{2}{47.59}\left(\sqrt{47.59^2} - 4.38 + \frac{47.59^2 - 4.38^2}{2 \times 8.2}\right)}$$

$$K_T = 3.75$$

8.3. (1). Using material from problem 8.1 : ($E_L$ = 147.5 GPa, $E_T$ = 11.0 GPa, etc.)

(a) for $(0/\pm 45/90)_{2s}$ laminate $K_T = 3$

(b) for $(0/90)_{4s}$ laminate $K_T = 5.12$

(2). Using material from problem (6.7): ($E_L$ = 38.6 GPa, $E_T$ = 8.3 GPa, etc.)

(c) for $(0/\pm 45/90)_{2s}$ laminate $K_T = 3$

(d) for $(0/90)_{4s}$ laminate $K_T = 3.75$

$d_o = 1$ mm $\quad a_o = 4$ mm

Point-Stress Criterion:

$$\frac{\sigma_N}{\sigma_o} = \frac{2}{2 + P_1^2 + 3P_1^4 - (k_T - 3)(5P_1^6 - 7P_1^8)}$$

where $P_1 = \dfrac{R}{R + d_o}$

R ranges from 0 - 15 mm

Average-Stress Criterion:

$$\frac{\sigma_N}{\sigma_o} = \frac{2(1 - P_2)}{2 - P_2^2 - P_2^4 + (k_T - 3)(P_2^6 - P_2^8)}$$

where $P_2 = \dfrac{R}{R + a_o}$

R ranges from 0 - 15 mm

(a) For $(0/\pm 45/90)_{2s}$ $K_T = 3$

| R (mm) | $\frac{\sigma_N}{\sigma_o}$ (Point-Stress Criterion) | $\frac{\sigma_N}{\sigma_o}$ (Average-Stress Criterion) |
|---|---|---|
| 0 | 1.00000 | 1.00000 |
| 1 | 0.82051 | 0.81699 |
| 2 | 0.65854 | 0.71053 |
| 3 | 0.56952 | 0.64112 |
| 4 | 0.51696 | 0.59259 |
| 5 | 0.48295 | 0.55691 |
| 6 | 0.45935 | 0.52966 |
| 7 | 0.44207 | 0.50821 |
| 8 | 0.42891 | 0.49091 |
| 9 | 0.41856 | 0.47668 |
| 10 | 0.41022 | 0.46477 |
| 11 | 0.40335 | 0.45467 |
| 12 | 0.39760 | 0.44599 |
| 13 | 0.39272 | 0.43846 |
| 14 | 0.38853 | 0.43187 |
| 15. | 0.38489 | 0.42605 |

(b) For $(0/90)_{2s}$ $K_T = 5.12$

| R (mm) | $\frac{\sigma_N}{\sigma_o}$ (Point-Stress Criterion) | $\frac{\sigma_N}{\sigma_o}$ (Average - Stress Criterion) |
|---|---|---|
| 0 | 1.00000 | 1.00000 |
| 1 | 0.85843 | 0.81694 |
| 2 | 0.74475 | 0.70955 |
| 3 | 0.64292 | 0.63729 |
| 4 | 0.55869 | 0.58399 |
| 5 | 0.49473 | 0.54227 |
| 6 | 0.44701 | 0.50836 |
| 7 | 0.41101 | 0.48010 |
| 8 | 0.38328 | 0.45618 |
| 9 | 0.36146 | 0.43566 |
| 10 | 0.34394 | 0.41790 |
| 11 | 0.32961 | 0.40239 |
| 12 | 0.31771 | 0.38875 |
| 13 | 0.30768 | 0.37668 |
| 14 | 0.29913 | 0.36594 |
| 15 | 0.29176 | 0.35633 |

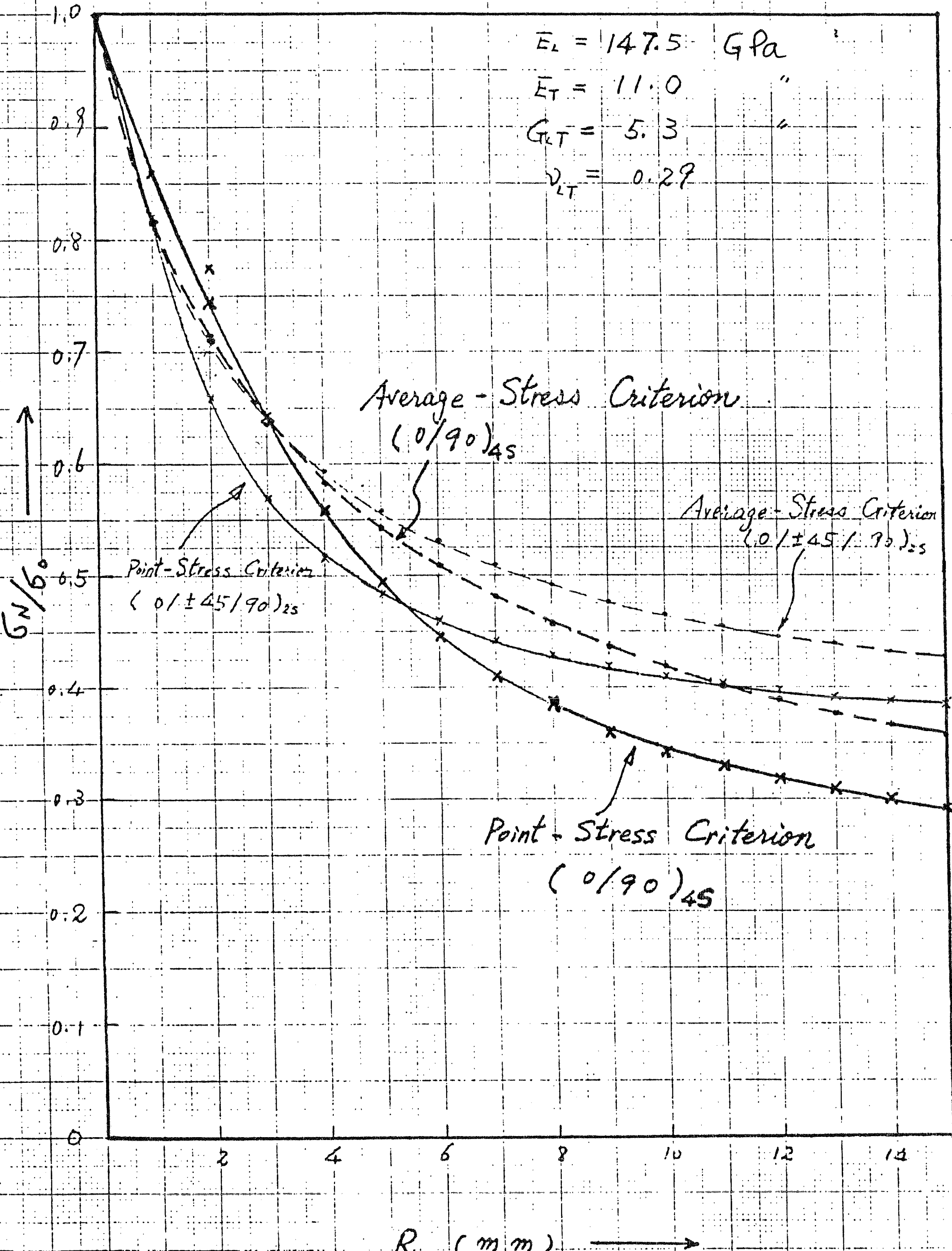
(c) For $(0/\pm 45/90)_{2s}$ $K_T = 3$

| R (mm) | $\frac{\sigma_N}{\sigma_o}$ (Point-Stress Criterion) | $\frac{\sigma_N}{\sigma_o}$ (Average-Stress Criterion) |
|---|---|---|
| 0 | 1.00000 | 1.00000 |
| 1 | 0.82051 | 0.81699 |
| 2 | 0.65854 | 0.71053 |
| 3 | 0.56952 | 0.64112 |
| 4 | 0.51696 | 0.59259 |
| 5 | 0.48295 | 0.55691 |
| 6 | 0.45935 | 0.52966 |
| 7 | 0.44207 | 0.50821 |
| 8 | 0.42891 | 0.49091 |
| 9 | 0.41856 | 0.47668 |
| 10 | 0.41022 | 0.46477 |
| 11 | 0.40335 | 0.45467 |
| 12 | 0.39760 | 0.44599 |
| 13 | 0.39272 | 0.43846 |
| 14 | 0.38853 | 0.43187 |
| 15 | 0.38489 | 0.42605 |

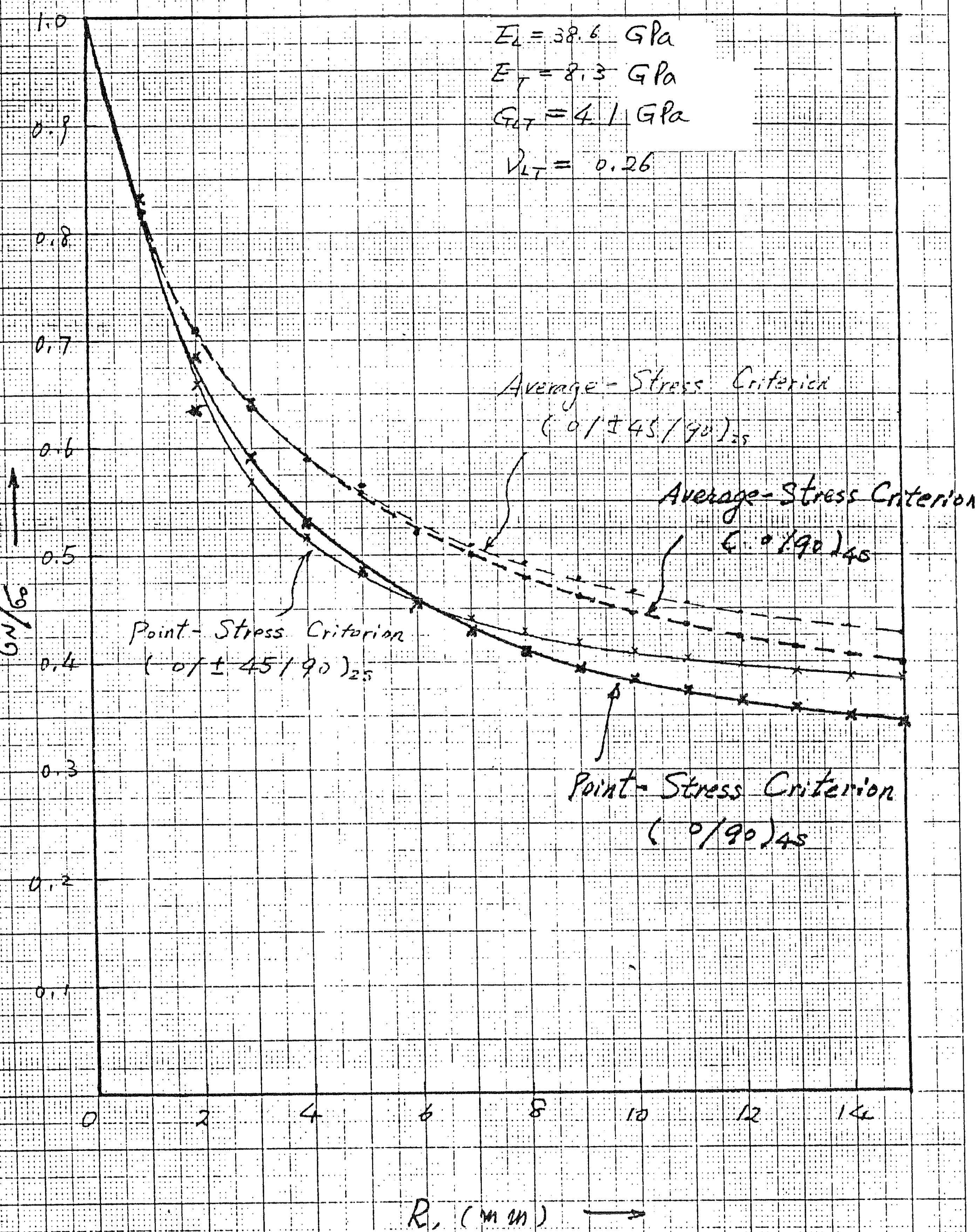
(d) For $(0/90)_{4s}$ $K_T = 3.75$

| R (mm) | $\frac{\sigma_N}{\sigma_o}$ (Point-Stress Criterion) | $\frac{\sigma_N}{\sigma_o}$ (Average-Stress Criterion) |
|---|---|---|
| 0 | 1.00000 | 1.00000 |
| 1 | 0.83354 | 0.81697 |
| 2 | 0.68666 | 0.71018 |
| 3 | 0.59349 | 0.63976 |
| 4 | 0.53099 | 0.58952 |
| 5 | 0.48705 | 0.55164 |
| 6 | 0.45491 | 0.52192 |
| 7 | 0.43056 | 0.49790 |
| 8 | 0.41157 | 0.47803 |
| 9 | 0.39640 | 0.46131 |
| 10 | 0.38403 | 0.44703 |
| 11 | 0.37377 | 0.43469 |
| 12 | 0.36512 | 0.42391 |
| 13 | 0.35774 | 0.41442 |
| 14 | 0.35138 | 0.40599 |
| 15 | 0.34583 | 0.39847 |

8-9



$E_L$ = 147.5 GPa
$E_T$ = 11.0 "
$G_{LT}$ = 5.3 "
$\nu_{LT}$ = 0.27
Average - Stress Criterion
(0/90)4S
Average - Stress Criterion
(0/±45/90)2S
Point-Stress Criterion
(0/±45/90)2S
Point - Stress Criterion
(0/90)4S
σN/σ0
R (mm)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2
4
6
8
10
12
14



$E_L = 38.6$ GPa
$E_T = 8.3$ GPa
$G_{LT} = 4.1$ GPa
$\nu_{LT} = 0.26$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
Average-Stress Criterion
$(0/\pm45/90)_{2s}$
Average-Stress Criterion
$(0/90)_{4s}$
Point-Stress Criterion
$(0/\pm45/90)_{2s}$
Point-Stress Criterion
$(0/90)_{4s}$
$\sigma_N/\sigma_0$
0
2
4
6
8
10
12
14
R, (mm)

8.4. For sharp crack:

(i) Point stress criterion:

$$\frac{\sigma_N}{\sigma_o} = \sqrt{1 - P_3^2}$$

where $P_3 = \frac{C}{C + d_o}$

and

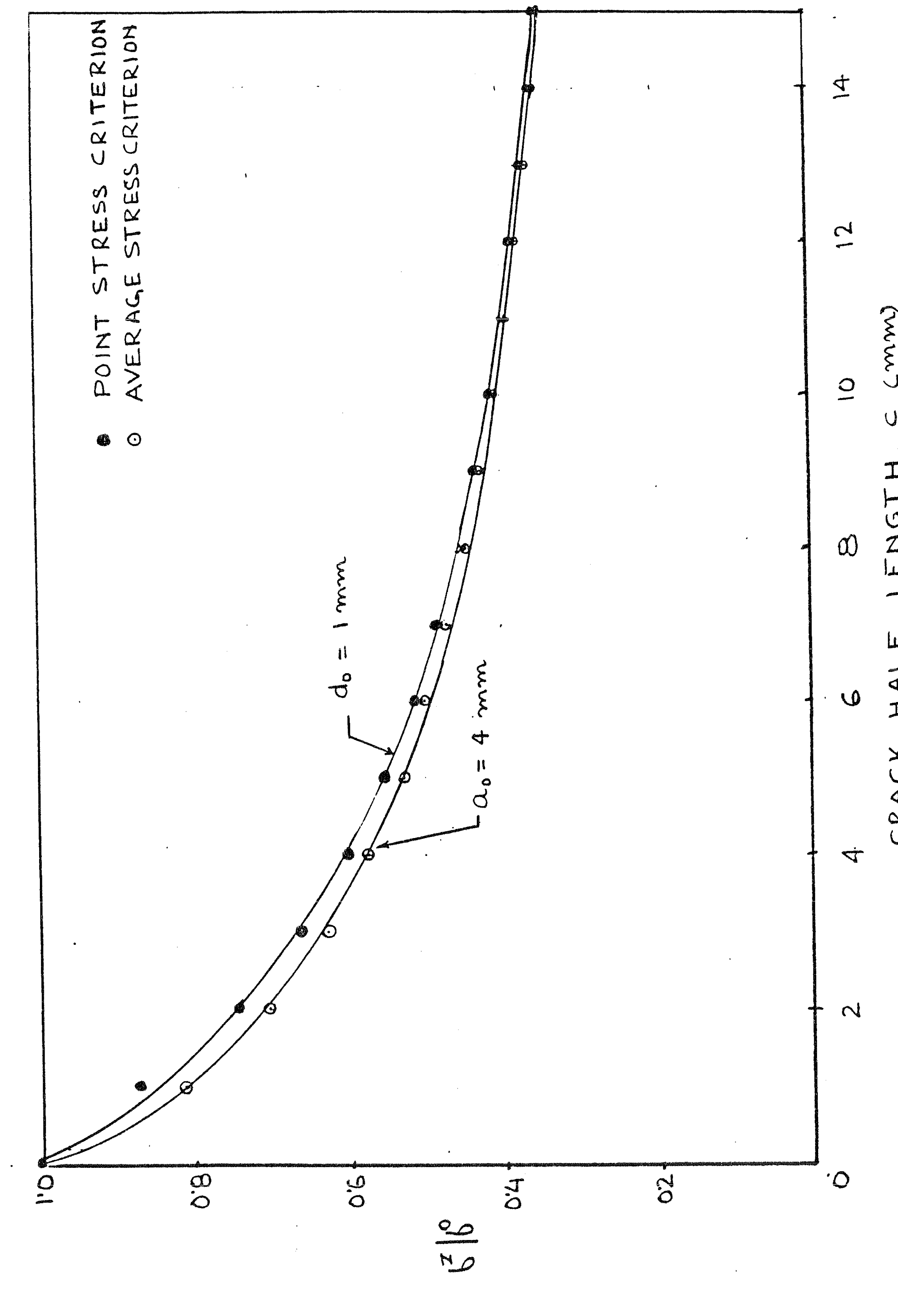
(ii) Average stress criterion

$$\frac{\sigma_N}{\sigma_o} = \sqrt{\frac{1 - P_4}{1 + P_4}}$$

where $P_4 = \frac{C}{C + a_o}$

C ranges from 0 - 15 mm

| C (mm) | $\frac{\sigma_N}{\sigma_o}$ (Point-Stress Criterion) | $\frac{\sigma_N}{\sigma_o}$ (Average - Stress Criterion) |
|---|---|---|
| 0 | 1.00000 | 1.00000 |
| 1 | 0.86602 | 0.81649 |
| 2 | 0.74535 | 0.70710 |
| 3 | 0.66143 | 0.63245 |
| 4 | 0.60000 | 0.57735 |
| 5 | 0.55277 | 0.53452 |
| 6 | 0.51507 | 0.50000 |
| 7 | 0.48412 | 0.47140 |
| 8 | 0.45812 | 0.44721 |
| 9 | 0.43588 | 0.42640 |
| 10 | 0.41659 | 0.40824 |
| 11 | 0.39965 | 0.39223 |
| 12 | 0.38461 | 0.37796 |
| 13 | 0.37115 | 0.36514 |
| 14 | 0.35901 | 0.35355 |
| 15 | 0.34798 | 0.34299 |


POINT STRESS CRITERION
AVERAGE STRESS CRITERION
$d_o$ = 1 mm
$a_o$ = 4 mm
CRACK HALF LENGTH, c (mm)
$\sigma_o / \sigma_z$
1.0
0.8
0.6
0.4
0.2
0
2
4
6
8
10
12
14

8.5. Fracture toughness for sharp crack for point and average stress criteria are respectively

$$K_Q = \sigma_o \sqrt{\pi C (1 - P_3^2)}$$

and

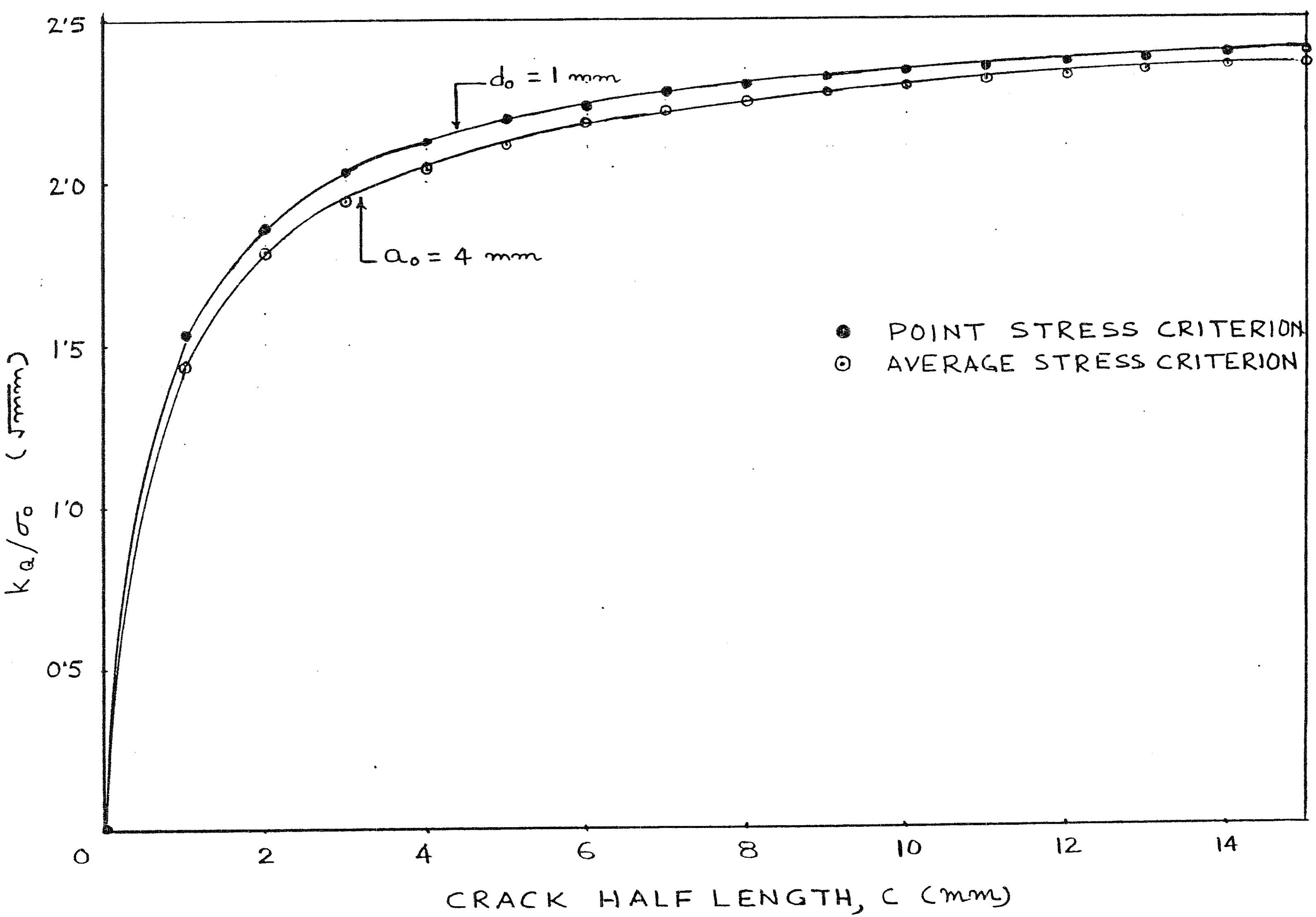
$$K_Q = \sigma_o \sqrt{\pi C \left(\frac{1 - P_4}{1 + P_4}\right)}$$

or $$\frac{K_Q}{\sigma_o} = \sqrt{\pi C (1 - P_3^2)} = \frac{\sigma_N}{\sigma_o} \sqrt{\pi C}$$

and $$\frac{K_Q}{\sigma_o} = \sqrt{\pi C \left(\frac{1 - P_4}{1 + P_4}\right)} = \frac{\sigma_N}{\sigma_o} \sqrt{\pi C}$$

C ranges from 0 - 15 mm

| C (mm) | $\frac{K_Q}{\sigma_o}$ (Point Stress Criterion) | $\frac{K_Q}{\sigma_o}$ (Average - Stress Criterion) |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1.53498 | 1.44719 |
| 2 | 1.86831 | 1.77243 |
| 3 | 2.03057 | 1.94160 |
| 4 | 2.12694 | 2.04665 |
| 5 | 2.19080 | 2.11847 |
| 6 | 2.23623 | 2.17080 |
| 7 | 2.27026 | 2.21061 |
| 8 | 2.29667 | 2.24197 |
| 9 | 2.31773 | 2.26732 |
| 10 | 2.33498 | 2.28818 |
| 11 | 2.34936 | 2.30574 |
| 12 | 2.36149 | 2.32065 |
| 13 | 2.37189 | 2.33349 |
| 14 | 2.38092 | 2.34471 |
| 15 | 2.38877 | 2.35451 |



2'5
2'0
1'5
1'0
0'5
0
2
4
6
8
10
12
14
$k_a/\sigma_o$ ($\sqrt{mm}$)
CRACK HALF LENGTH, C (mm)
$d_o$ = 1 mm
$a_o$ = 4 mm
POINT STRESS CRITERION
AVERAGE STRESS CRITERION

# CHAPTER 9

9.1. For laminate $[0]_8$

The load acting on laminate $[0]_8$ is primarily carried by fibers. Therefore, fatigue failure in the laminate would occur due to fiber break followed by matrix failure.

For laminate $[0/\pm45/90]_s$

First, the weakest lamina with $90^\circ$ fiber orientation would fail due to interfacial failure causing debonding. This would be followed by a failure in $\pm 45^\circ$ laminae caused due to a combination of interfacial shear failure and debonding. At this stage most of the fatigue load is carried by fibers in the $0^\circ$ lamina and finally the fibers would break in $0^\circ$ lamina.

9.2.

| N | $S_E$ (MPa) | t (min.) | $S_c$ (MPa) |
|---|---|---|---|
| $10^3$ | 84 | 0.5 | 23 |
| $10^4$ | 70 | 5 | 19 |
| $10^5$ | 60 | 50 | 15 |
| $10^6$ | 52 | 500 | 11 |

Eq. 9.1

$$\frac{S_A}{S_E} = 1 - \frac{S_M}{S_C}$$

Substituting the values of $S_E$ and $S_C$ in Eq. 9.1, fatigue strength at different cyclic lives can be related to the mean stress through the following equations:

| N | Relationship between $S_A$ and $S_M$ |
|---|---|
| $10^3$ | $S_A = 84 - 3.652\ S_M$ |
| $10^4$ | $S_A = 70 - 3.684\ S_M$ |
| $10^5$ | $S_A = 60 - 4.0\ S_M$ |
| $10^6$ | $S_A = 52 - 4.727\ S_M$ |

The equations can now be plotted on $S_A - S_M$ axes.

9.3. From Fig. 9.6 the following values can be obtained corresponding to 20,000 cycles:

Modulus = $3.94 \times 10^6$ psi

Relative residual strength $\cong$ 0.6

$\therefore$ Residual strength = $0.6 \times 124{,}000 = 74{,}400$ psi

Assume the material to behave linearly, the strain to failure after 20,000 cycles will be:

$$\epsilon_f = \frac{74{,}400}{3.94 \times 10^6} = 0.0189 \text{ or } 1.89\%$$

Therefore, the laminate will fail if subjected to 2% strain.

9.4. The high modulus graphite fiber sustains a very low strain at failure. The low strain could not develop stresses which are sufficient to cause delamination. Whereas the glass fiber composite with a modulus nearly 1/5 of the modulus of graphite fiber and value of breaking stress higher than of graphite, would break at much higher value of strain. The large strain would develop sufficient stress at the interlayers which in turn causes delamination.

10.2 Stress, $\sigma = \frac{500}{12.5 \times 4} = 10\ N/mm^2 = 10\ MPa$

Apparent Elastic moduli:

$$E_{30^\circ} = \frac{10}{0.0925} \times 100 = 10.81 \times 10^3\ MPa \quad = 10.81\ GPa$$

$$E_{45^\circ} = \frac{10}{0.105} \times 100 = 9.52 \times 10^3\ MPa \quad = 9.52\ GPa$$

$$E_{60^\circ} = \frac{10}{0.150} \times 100 = 6.67 \times 10^3\ MPa \quad = 6.67\ GPa$$

Moduli obtained by transformation (Prob. 5.5) are:

$E_{30^\circ} = 10.87\ GPa$

$E_{45^\circ} = 7.43\ GPa$

$E_{60^\circ} = 5.02\ GPa$

It is observed that the apparent elastic modulus for 30° specimen is close to the one obtained by transformation but for the 45° and 60° specimens, the apparent elastic moduli are different from the transformed ones. The following values of $\bar{Q}_{11}$ can be obtained using Eqs. 5.70 and 5.95

$$(\bar{Q}_{11})_{45^\circ} = 9.49, \ (\bar{Q}_{11})_{60^\circ} = 6.66$$

Thus the apparent elastic moduli for 45° and 60° specimens are actually $\bar{Q}_{11}$.

10.3. Strain transformation equations are

$$\varepsilon_L = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta$$
$$\varepsilon_T = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy} \sin\theta \cos\theta$$
$$\gamma_{LT} = 2(\varepsilon_y - \varepsilon_x) \sin\theta \cos\theta + \gamma_{xy} (\cos^2\theta - \sin^2\theta)$$

$\varepsilon_{45}$ can be obtained by substituting $\theta = 45°$ in the first of the above equations:

$$\varepsilon_{45} = \frac{1}{2} (\varepsilon_x + \varepsilon_y + \gamma_{xy})$$

or $$\gamma_{xy} = (2\varepsilon_{45} - \varepsilon_x - \varepsilon_y)$$

Substitution of value of $\gamma_{xy}$ in the strain transformation equations gives

$$\varepsilon_L = \cos\theta\ (\cos\theta) - \sin\theta)\ \varepsilon_x + \sin\theta\ (\sin\theta - \cos\theta)\ \varepsilon_y + 2\sin\theta \cos\theta\ \varepsilon_{45}$$
$$\varepsilon_T = \sin\theta\ (\cos\theta + \sin\theta)\ \varepsilon_x + \cos\theta\ (\sin\theta + \cos\theta)\ \varepsilon_y - 2 \sin\theta \cos\theta\ \varepsilon_{45}$$
$$\gamma_{LT} = -(\cos^2\theta + 2 \cos\theta \sin\theta - \sin^2\theta)\ \varepsilon_x - (\cos^2\theta - 2 \sin\theta \cos\theta - \sin^2\theta)\varepsilon_y + 2(\cos^2\theta - \sin^2\theta)\ \varepsilon_{45}$$

10.4. From Problem 6.3

$$\tau_{LT} = -\frac{1}{2}\sigma_x, \quad \gamma_{LT} = (\varepsilon_y - \varepsilon_x)$$

$$G_{LT} = \frac{\tau_{LT}}{\gamma_{LT}} = \frac{-\sigma_x}{2(\varepsilon_y - \varepsilon_x)} = \frac{\sigma_x / \varepsilon_x}{2\left(1 - \frac{\varepsilon_y}{\varepsilon_x}\right)}$$

*Substituting* $\frac{\sigma_x}{\varepsilon_x} = E_x$ *and* $-\frac{\varepsilon_y}{\varepsilon_x} = \upsilon_{xy}$

$$G_{LT} = \frac{E_x}{2(1 + \upsilon_{xy})}$$

10.5. $G_{LT}$ can be calculated using Eq. 10.18

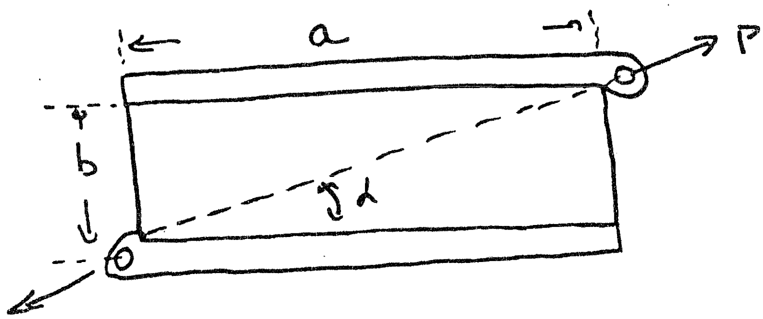
From test on $[\pm 45]_S$ laminate

$$G_{LT} = \frac{20.98}{2(1+0.69)} = 6.21\ GPa$$

From test on $[45]_n$ laminate

$$G_{LT} = \frac{15.87}{2(1+0.28)} = 6.20\ GPa$$

10.6.




Shear force (The component of force parallel to the long edge of the specimen) = P cos α

At failure:

$$\tau_{LTU} = \frac{P \cos \alpha}{a\ t}$$

where t = specimen thickness

Thus, failure load

$$P = \frac{\tau_{LTU}\ a\ t}{\cos \alpha}$$

The load angle, α, can be approximated by

$$\alpha = \tan^{-1} \frac{b}{a}$$

Therefore

$$P = \tau_{LTU} \times t \times \sqrt{a^2 + b^2}$$

10.7 In a 4 point bend test, a constant maximum bending moment occurs at different sections along the length of a beam between two loads. Probability of meeting a flaw in this case is, therefore, more than that in 3-point bend test where the maximum bending moment occurs at a section at the center of the beam. Thus, a 4-point bend test always give a lower value of flexural strength than a 3-point bend test.

10.8. Mid-span deflection in a three-point bend test is given by

$$\delta = \frac{PL^3}{48EI}$$

*so that*

$$E = \frac{PL^3}{48\ \delta I}$$

*Given*

$$P = 1\ kN$$
$$\delta = 5.1\ mm$$
$$L = 100\ mm$$
$$I = \frac{1}{12}\ x\ 12.5\ x\ (3)^3 = 28 \cdot 125\ mm^4$$

*Substitution of values gives*

$$E = \frac{1\ x\ (100)^3}{48\ x\ 5 \cdot 1\ x\ 28 \cdot 125} = 145.2\ kN/mm^2\ or\ 145.2\ GPa$$

10.9. Strain energy released by the system during crack extension at constant load

= Initial strain energy stored by the specimen + work done by the loads - final strain energy stored in the specimen

Therefore $\Delta U' = \frac{1}{2} P\delta + Pd\delta - \frac{1}{2} P(\delta + d\delta) = \frac{1}{2} Pd\delta$

Strain energy release rate:

$$G = \frac{\partial U'}{\partial c} = \frac{1}{2} P \frac{\partial \delta}{\partial c}$$

$$\delta = P/k \Rightarrow \frac{\partial \delta}{\partial c} = P \frac{\partial (1/k)}{\partial c}$$

$$G = \frac{1}{2} P^2 \frac{\partial (1/k)}{\partial c}$$

10.10. Bending moment for tension failure of the specimen (Eq. 10.19):

$$M = \frac{1}{6} \sigma_u b h^2$$

In a three point flexural test

$$M = \frac{1}{4} PL$$

Therefore, load to cause tension failure

$$P_t = \frac{2}{3} \sigma_u \frac{b h^2}{L}$$

Shear force required to cause interlaminar shear failure (Eq. 10.20):

$$F = \frac{2}{3} \tau_u b h$$

For a three point flexural test

$$F = \frac{1}{2} P$$

Therefore, load to cause interlaminar shear failure

$$P_S = \frac{4}{3} \tau_u b h$$

To ensure interlaminar shear failure

$$P_S < P_t$$

Therefore

$$\frac{L}{h} < \frac{\sigma_u}{2 \tau_u}$$

Table 3.1 shows that compression strengths of unidirectional glass-epoxy and graphite-epoxy composites ($0^{\circ}$) are lower than their tensile strengths. Therefore span to depth ratios should be calculated on the basis of compression strengths.

For glass-epoxy $0^\circ$ composite:

$$\frac{L}{h} < \frac{600}{2 \times 31} = 9.68$$

For graphite-epoxy $0^\circ$ composite:

$$\frac{L}{h} < \frac{1366}{2 \times 113} = 6.04$$

10.11. Let $l$ = distance between the notches

$t$ = specimen thickness

$b$ = specimen width

Load to cause tension failure at the weakest section:

$$P_t = \frac{1}{2} \sigma_u \, b \, t$$

Load to cause interlaminar shear failure between notches:

$$P_S = \tau_u \, l \, b$$

To assure interlaminar failure of the specimen

$$P_S < P_t$$

or $$l < \frac{\sigma_u t}{2 \tau_u}$$

10.12. Initial Kinetic Energy of the tup, $E_o = 1/2 \, MV_o^2$

Final Kinetic Energy of the tup, $E_f = 1/2 \, MV_f^2 = E_o \left(2\frac{\bar{v}}{v_o} - 1\right)^2$

Where $2\bar{v} = v_o + v_f$

Energy absorbed by the tup $E = \bar{v}\int P dt = \frac{\bar{v}}{v_o} E_a$

(Using Eqs. 10.38 and 10.39)

For energy balance

$$E = E_o - E_f$$

Substitution of expressions for E, $E_o$ and $E_f$ and simplification will yield:

$$\frac{\bar{v}}{v_o} = 1 - \frac{E_a}{4E_o}$$

Eq. 10.42 will follow immediately.

10.13. $E_o = mgh = 2 \times 9.81 \times 3 = 58.86$ Nm

$$E_i = 6.38\left(1 - \frac{6.38}{4 \times 58.86}\right) = 6.207 \text{ Nm}$$

$$E_t = 7.91\left(1 - \frac{7.91}{4 \times 58.86}\right) = 7.644 \text{ Nm}$$

$$E_p = E_t - E_i = 1.437 \text{ Nm}$$

10.14. $E_o = 0.25 \times 9.81 \times 3 = 7.3575$ Nm

$$E_i = 6.38\left(1 - \frac{6.38}{4 \times 7.3575}\right) = 4.997 \text{ Nm}$$

$$E_t = 7.91\left(1 - \frac{7.91}{4 \times 7.3575}\right) = 5.784 \text{ Nm}$$

$$E_p = 5.784 - 4.997 = 0.787 \text{ Nm}$$

10.15. $\frac{v_f}{v_o} = 0.8 \Rightarrow \frac{\bar{v}}{v_o} = 0.9$

Combining Eqs. 10.38 and 10.39,

$$\frac{E}{E_a} = \frac{\bar{v}}{v_o} = 0.9$$

$$\text{Error} = \frac{E - E_a}{E} = \frac{0.9 - 1}{0.9} = -\frac{1}{9} \text{ or } 11.1\%$$