CHAPTER

ONE

STRESS

1.1 INTRODUCTION

An experimental stress analyst must have a thorough understanding of stress, strain, and the laws relating stress to strain. For this reason, Part 1 of this text has been devoted to the elementary concepts of the theory of elasticity. The first chapter deals with stresses produced in a body due to external and body-force loadings. The second chapter deals with deformations and strains produced by the loadings and with relations between the stresses and strains. The third chapter covers plane problems in the theory of elasticity, important since a large part of a first course in experimental stress analysis deals with two-dimensional problems. Also treated is the stress-function approach to the solution of plane problems. Upon completing the subject matter of Part 1 of the text, the student should have a firm understanding of stress and strain and should be able to solve some of the more elementary two-dimensional problems in the theory of elasticity by using the Airy's-stress-function approach.

1.2 DEFINITIONS

Two basic types of force act on a body to produce stresses. Forces of the first type are called *surface forces* for the simple reason that they act on the surfaces of the body. Surface forces are generally exerted when one body comes in contact with another. Forces of the second type are called *body forces* since they act on each element of the body. Body forces are commonly produced by centrifugal, gravitational, or other force fields. The most common body forces are gravitational, being present to some degree in almost all cases. For many practical applications,

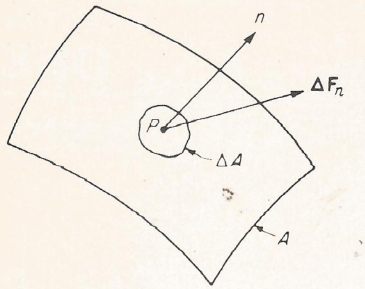

Figure 1.1 Arbitrary surface (either internal or external) showing the resultant of all forces acting over the element of area ΔA.

however, they are so small compared with the surface forces present that they can be neglected without introducing serious error. Body forces are included in the following analysis for the sake of completeness.

Consider an arbitrary internal or external surface, which may be plane or curvilinear, as shown in Fig. 1.1. Over a small area ΔA of this surface in the neighborhood of an arbitrary point P, a system of forces acts which has a resultant represented by the vector $\Delta \mathbf{F}_n$ in the figure. It should be noted that the line of action of the resultant force vector $\Delta \mathbf{F}_n$ does not necessarily coincide with the outer normal n associated with the element of area ΔA. If the resultant force $\Delta \mathbf{F}_n$ is divided by the increment of area ΔA, the average stress which acts over the area is obtained. In the limit as ΔA approaches zero, a quantity defined as the resultant stress $\mathbf{T}_n$ acting at the point P is obtained. This limiting process is illustrated in equation form below.

$$\mathbf{T}_n = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}_n}{\Delta A} \tag{1.1}$$

The line of action of this resultant stress $\mathbf{T}_n$ coincides with the line of action of the resultant force $\Delta \mathbf{F}_n$, as illustrated in Fig. 1.2. It is important to note at this point that the resultant stress $\mathbf{T}_n$ is a function of both the position of the point P in the body and the orientation of the plane which is passed through the point and identified by its outer normal n. In a body subjected to an arbitrary system of

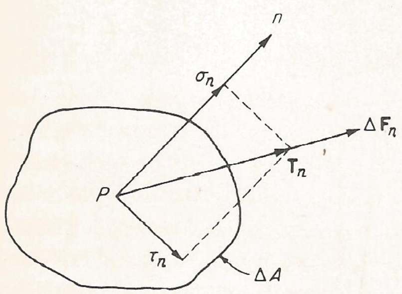

Figure 1.2 Resolution of the resultant stress T_n into its normal and tangential components σ_n and T_n.

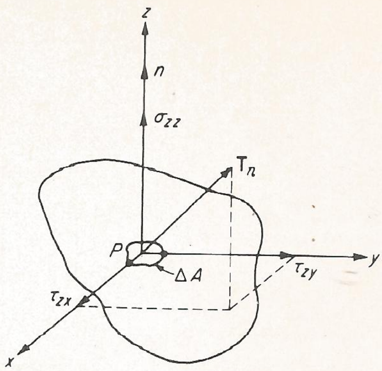

Figure 1.3 Resolution of the resultant stress T_n into its three cartesian components τ_{zx}, τ_{zy}, and σ_{zz}.

loads, both the magnitude and the direction of the resultant stress $\mathbf{T}_n$ at any point P change as the orientation of the plane under consideration is changed.

As illustrated in Fig. 1.2, it is possible to resolve $\mathbf{T}_n$ into two components: one σ_n normal to the surface is known as the resultant normal stress, while the component τ_n is known as the resultant shearing stress.

Cartesian components of stress for any coordinate system can also be obtained from the resultant stress. Consider first a surface whose outer normal is in the positive z direction, as shown in Fig. 1.3. If the resultant stress $\mathbf{T}_n$ associated with this particular surface is resolved into components along the x, y, and z axes, the cartesian stress components τ_{zx}, τ_{zy}, and σ_{zz} are obtained. The components τ_{zx} and τ_{zy} are shearing stresses since they act tangent to the surface under consideration. The component σ_{zz} is a normal stress since it acts normal to the surface.

If the same procedure is followed using surfaces whose outer normals are in the positive x and y directions, two more sets of cartesian components, τ_{xy}, τ_{xz}, σ_{xx}, and τ_{yx}, τ_{yz}, σ_{yy}, respectively, can be obtained. The three different sets of three cartesian components for the three selections of the outer normal are summarized in the array below:

$$\begin{Vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{Vmatrix} \quad \begin{matrix} \text{outer normal parallel to the } x \text{ axis} \\ \text{outer normal parallel to the } y \text{ axis} \\ \text{outer normal parallel to the } z \text{ axis} \end{matrix}$$

From this array, it is clear that nine cartesian components of stress exist. These components can be arranged on the faces of a small cubic element, as shown in Fig. 1.4. The sign convention employed in placing the cartesian stress components on the faces of this cube is as follows: if the outer normal defining the cube face is in the direction of increasing x, y, or z, then the associated normal and shear stress components are also in the direction of positive x, y, or z. If the outer normal is in the direction of negative x, y, or z, then the normal and shear stress components are also in the direction of negative x, y, or z. As for subscript convention, the first

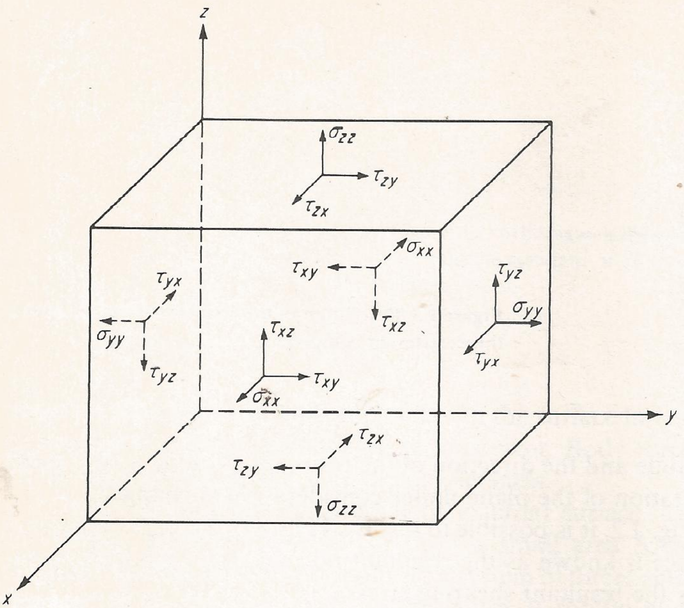

Figure 1.4 Cartesian components of stress acting on the faces of a small cubic element.

subscript refers to the outer normal and defines the plane upon which the stress component acts, whereas the second subscript gives the direction in which the stress acts. Finally, for normal stresses, positive signs indicate tension and negative signs indicate compression.

1.3 STRESS AT A POINT

At a given point of interest within a body, the magnitude and direction of the resultant stress $\mathbf{T}_n$ depend upon the orientation of the plane passed through the point. Thus an infinite number of resultant-stress vectors can be used to represent the resultant stress at each point since an infinite number of planes can be passed through each point. It is easy to show, however, that the magnitude and direction of each of these resultant-stress vectors can be specified in terms of the nine cartesian components of stress acting at the point. This can be seen by considering equilibrium of the elemental tetrahedron shown in Fig. 1.5. In this figure the stresses acting over the four faces of the tetrahedron are represented by their average values. The average value is denoted by placing a ~ sign over the stress symbol. In order for the tetrahedron to be in equilibrium, the following condition must be satisfied. First consider equilibrium in the x direction:

$$\tilde{T}_{nx} A - \tilde{\sigma}_{xx} A \cos(n, x) - \tilde{\tau}_{yx} A \cos(n, y) - \tilde{\tau}_{zx} A \cos(n, z) + \tilde{F}_x \tfrac{1}{3} h A = 0$$

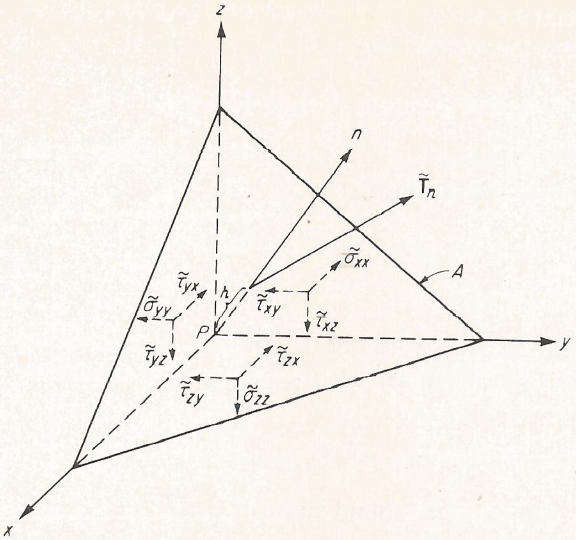

Figure 1.5 Elemental tetrahedron at point P showing the average stresses which act over its four faces.

where h = altitude of tetrahedron
A = area of base of tetrahedron
$\tilde{F}_x$ = average body-force intensity in x direction
$\tilde{T}_{nx}$ = component of resultant stress in x direction
and $A \cos(n, x)$, $A \cos(n, y)$, and $A \cos(n, z)$ are the projections of the area A on the yz, xz, and xy planes, respectively.

By letting the altitude $h \to 0$, after eliminating the common factor A from each term of the expression, it can be seen that the body-force term vanishes, the average stresses become exact stresses at the point P, and the previous expression becomes

$$T_{nx} = \sigma_{xx} \cos(n, x) + \tau_{yx} \cos(n, y) + \tau_{zx} \cos(n, z) \tag{1.2a}$$

Two similar expressions are obtained by considering equilibrium in the y and z directions:

$$T_{ny} = \tau_{xy} \cos(n, x) + \sigma_{yy} \cos(n, y) + \tau_{zy} \cos(n, z) \tag{1.2b}$$

$$T_{nz} = \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_{zz} \cos(n, z) \tag{1.2c}$$

Once the three cartesian components of the resultant stress for a particular plane have been determined by employing Eqs. (1.2), the resultant stress $\mathbf{T}_n$ can be determined by using the expression

$$\mathbf{T}_n = \sqrt{T_{nx}^2 + T_{ny}^2 + T_{nz}^2}$$

The three direction cosines which define the line of action of the resultant stress $\mathbf{T}_n$ are

$$\cos(T_n, x) = \frac{T_{nx}}{|\mathbf{T}_n|} \qquad \cos(T_n, y) = \frac{T_{ny}}{|\mathbf{T}_n|} \qquad \cos(T_n, z) = \frac{T_{nz}}{|\mathbf{T}_n|}$$

The normal stress σ_n and the shearing stress τ_n which act on the plane under consideration can be obtained from the expressions

$$\sigma_n = |\mathbf{T}_n| \cos (T_n, n) \qquad \text{and} \qquad \tau_n = |\mathbf{T}_n| \sin (T_n, n)$$

The angle between the resultant-stress vector $\mathbf{T}_n$ and the normal to the plane n can be determined by using the well-known relationship

$$\cos (T_n, n) = \cos (T_n, x) \cos (n, x) + \cos (T_n, y) \cos (n, y) + \cos (T_n, z) \cos (n, z)$$

It should also be noted that the normal stress σ_n can be determined by considering the projections of T_{nx}, T_{ny}, and T_{nz} onto the normal to the plane under consideration. Thus

$$\sigma_n = T_{nx} \cos (n, x) + T_{ny} \cos (n, y) + T_{nz} \cos (n, z)$$

Once σ_n has been determined, τ_n can easily be found since

$$\tau_n = \sqrt{T_n^2 - \sigma_n^2}$$

1.4 STRESS EQUATIONS OF EQUILIBRIUM

In a body subjected to a general system of body and surface forces, stresses of variable magnitude and direction are produced throughout the body. The distribution of these stresses must be such that the overall equilibrium of the body is maintained; furthermore, equilibrium of each element in the body must be maintained. This section deals with the equilibrium of the individual elements of the body. On the element shown in Fig. 1.6, only the stress and body-force components which act in the x direction are shown. Similar components exist and act in the y and z directions. The stress values shown are average stresses over the faces of an element which is assumed to be very small. A summation of forces in the x direction gives

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx - \sigma_{xx}\right) dy\, dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy - \tau_{yx}\right) dx\, dz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz - \tau_{zx}\right) dx\, dy + F_x\, dx\, dy\, dz = 0$$

Dividing through by $dx\, dy\, dz$ gives

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \tag{1.3a}$$

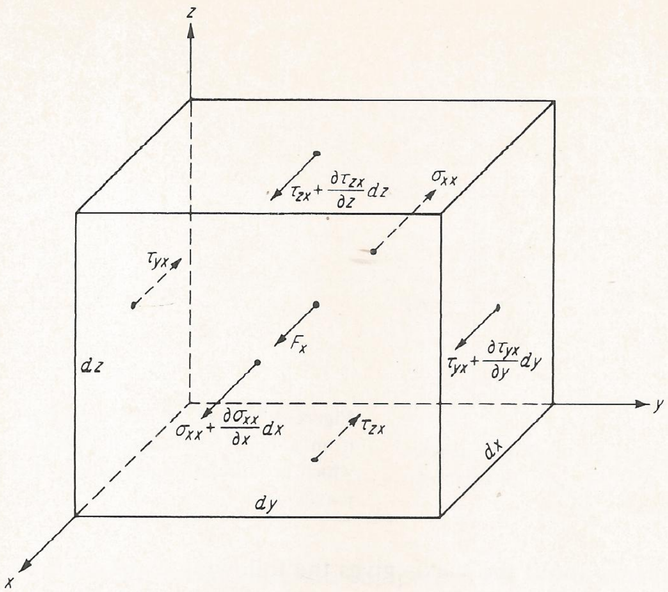

Figure 1.6 Small element removed from a body, showing the stresses acting in the x direction only.

By considering the force and stress components in the y and z directions, it can be established in a similar fashion that

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \tag{1.3b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z = 0 \tag{1.3c}$$

where F_x, F_y, F_z are body-force intensities (in lb/in^3 or N/m^3) in the x, y, and z directions, respectively.

Equations (1.3) are the well-known stress equations of equilibrium which any theoretically or experimentally obtained stress distribution must satisfy. In obtaining these equations, three of the six equilibrium conditions have been employed. The three remaining conditions can be utilized to establish additional relationships between the stresses.

Consider the element shown in Fig. 1.7. Only those stress components which will produce a moment about the y axis are shown. Since the coordinate system has been selected with its origin at the centroid of the element, the normal stress components and the body forces do not produce any moments.

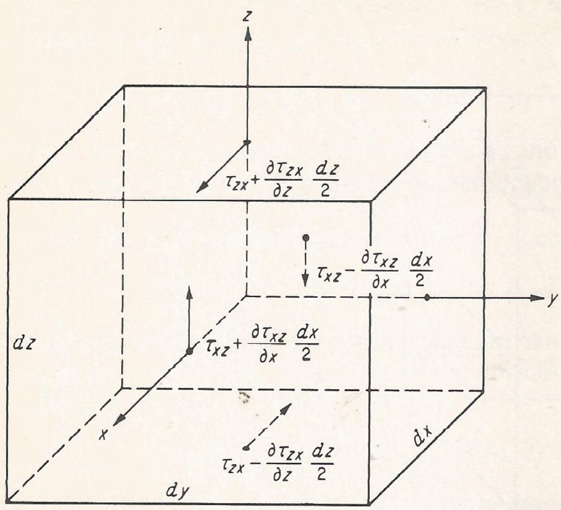

Figure 1.7 Small element removed from a body, showing the stresses which produce a moment about the y axis.

A summation of moments about the y axis gives the following expression:

$$\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy\frac{dz}{2}+\left(\tau_{zx}-\frac{\partial\tau_{zx}}{\partial z}\frac{dz}{2}\right)dx\,dy\frac{dz}{2}$$
$$-\left(\tau_{xz}+\frac{\partial\tau_{xz}}{\partial x}\frac{dx}{2}\right)dy\,dz\frac{dx}{2}-\left(\tau_{xz}-\frac{\partial\tau_{xz}}{\partial x}\frac{dx}{2}\right)dy\,dz\frac{dx}{2}=0$$

which reduces to

$$\tau_{zx}\,dx\,dy\,dz-\tau_{xz}\,dx\,dy\,dz=0$$

Therefore,

$$\tau_{zx}=\tau_{xz} \tag{1.4a}$$

The remaining two equilibrium conditions can be used in a similar manner to establish that

$$\tau_{xy}=\tau_{yx} \tag{1.4b}$$

$$\tau_{yz}=\tau_{zy} \tag{1.4c}$$

The equalities given in Eqs. (1.4) reduce the nine cartesian components of stress to six independent components, which may be expressed in the following array:

$$\begin{matrix}\sigma_{xx} & \tau_{xy} & \tau_{zx}\\ \tau_{xy} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{yz} & \sigma_{zz}\end{matrix}$$

1.5 LAWS OF STRESS TRANSFORMATION

It has previously been shown that the resultant-stress vector $\mathbf{T}_n$ acting on an arbitrary plane defined by the outer normal n can be determined by substituting

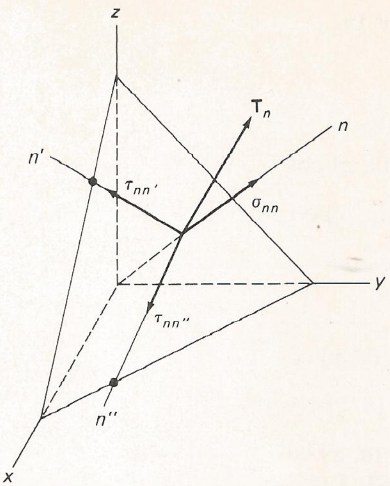

Figure 1.8 Resolution of $\mathbf{T}_n$ into three cartesian components σ_{nn}, $\tau_{nn'}$, and $\tau_{nn''}$.

the six independent cartesian components of stress into Eqs. (1.2). However, it is often desirable to make another transformation, namely, that from the stress components σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{yz}, τ_{zx}, which refer to an $Oxyz$ coordinate system, to the stress components $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\sigma_{z'z'}$, $\tau_{x'y'}$, $\tau_{y'z'}$, $\tau_{z'x'}$, which refer to an $Ox'y'z'$ coordinate system. The transformation equations commonly used to perform this operation will be developed in this section.

Consider an element similar to Fig. 1.5 with an inclined face having outer normal n. Two mutually perpendicular directions n' and n'' can then be denoted in the plane of the inclined face, as shown in Fig. 1.8. The resultant stress $\mathbf{T}_n$ acting on the inclined face can be resolved into components along the directions n, n', and n'' to yield the stresses σ_{nn}, $\tau_{nn'}$, and $\tau_{nn''}$. This resolution of the resultant stress into components can be accomplished most easily by utilizing the cartesian components T_{nx}, T_{ny}, and T_{nz}. Thus

$$\begin{aligned}
\sigma_{nn} &= T_{nx} \cos(n, x) + T_{ny} \cos(n, y) + T_{nz} \cos(n, z) \\
\tau_{nn'} &= T_{nx} \cos(n', x) + T_{ny} \cos(n', y) + T_{nz} \cos(n', z) \\
\tau_{nn''} &= T_{nx} \cos(n'', x) + T_{ny} \cos(n'', y) + T_{nz} \cos(n'', z)
\end{aligned}$$

If the results from Eqs. (1.2) and (1.4) are substituted into these expressions, the following important equations are obtained:

$$\begin{aligned}
\sigma_{nn} = \sigma_{xx} &\cos^2(n, x) + \sigma_{yy} \cos^2(n, y) + \sigma_{zz} \cos^2(n, z) \\
&+ 2\tau_{xy} \cos(n, x) \cos(n, y) + 2\tau_{yz} \cos(n, y) \cos(n, z) \\
&+ 2\tau_{zx} \cos(n, z) \cos(n, x)
\end{aligned} \tag{1.5a}$$

$$\begin{aligned}
\tau_{nn'} = \sigma_{xx} &\cos(n, x) \cos(n', x) + \sigma_{yy} \cos(n, y) \cos(n', y) \\
&+ \sigma_{zz} \cos(n, z) \cos(n', z) \\
&+ \tau_{xy}[\cos(n, x) \cos(n', y) + \cos(n, y) \cos(n', x)] \\
&+ \tau_{yz}[\cos(n, y) \cos(n', z) + \cos(n, z) \cos(n', y)] \\
&+ \tau_{zx}[\cos(n, z) \cos(n', x) + \cos(n, x) \cos(n', z)]
\end{aligned} \tag{1.5b}$$

$$\begin{aligned}
\tau_{nn''} = \sigma_{xx} &\cos(n, x) \cos(n'', x) + \sigma_{yy} \cos(n, y) \cos(n'', y) \\
&+ \sigma_{zz} \cos(n, z) \cos(n'', z) \\
&+ \tau_{xy}[\cos(n, x) \cos(n'', y) + \cos(n, y) \cos(n'', x)] \\
&+ \tau_{yz}[\cos(n, y) \cos(n'', z) + \cos(n, z) \cos(n'', y)] \\
&+ \tau_{zx}[\cos(n, z) \cos(n'', x) + \cos(n, x) \cos(n'', z)]
\end{aligned} \tag{1.5c}$$

Equations (1.5) provide the means for determining normal- and shear-stress components at a point associated with any set of cartesian reference axes provided the stresses associated with one set of axes are known.

Expressions for the stress components $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\sigma_{z'z'}$, $\tau_{x'y'}$, $\tau_{y'z'}$, $\tau_{z'x'}$ can be obtained directly from Eq. (1.5*a*) or Eq. (1.5*b*) by employing the following procedure.

In order to determine $\sigma_{x'x'}$, select a plane having an outer normal n coincident with x'. A resultant stress $\mathbf{T}_n = \mathbf{T}_{x'}$ is associated with this plane. The normal stress $\sigma_{x'x'}$ associated with this plane is obtained directly from Eq. (1.5*a*) by substituting x' for n. Thus

$$\begin{aligned}
\sigma_{x'x'} = \sigma_{xx} &\cos^2(x', x) + \sigma_{yy} \cos^2(x', y) \\
&+ \sigma_{zz} \cos^2(x', z) + 2\tau_{xy} \cos(x', x) \cos(x', y) \\
&+ 2\tau_{yz} \cos(x', y) \cos(x', z) + 2\tau_{zx} \cos(x', z) \cos(x', x)
\end{aligned} \tag{1.6a}$$

By selecting n coincident with the y' and z' axes and following the same procedure, expressions for $\sigma_{y'y'}$ and $\sigma_{z'z'}$ can be obtained as follows:

$$\begin{aligned}
\sigma_{y'y'} = \sigma_{yy} &\cos^2(y', y) + \sigma_{zz} \cos^2(y', z) \\
&+ \sigma_{xx} \cos^2(y', x) + 2\tau_{yz} \cos(y', y) \cos(y', z) \\
&+ 2\tau_{zx} \cos(y', z) \cos(y', x) + 2\tau_{xy} \cos(y', x) \cos(y', y)
\end{aligned} \tag{1.6b}$$

$$\begin{aligned}
\sigma_{z'z'} = \sigma_{zz} &\cos^2(z', z) + \sigma_{xx} \cos^2(z', x) \\
&+ \sigma_{yy} \cos^2(z', y) + 2\tau_{zx} \cos(z', z) \cos(z', x) \\
&+ 2\tau_{xy} \cos(z', x) \cos(z', y) + 2\tau_{yz} \cos(z', y) \cos(z', z)
\end{aligned} \tag{1.6c}$$

The shear-stress component $\tau_{x'y'}$ is obtained by selecting a plane having outer normal n coincident with x' and the in-plane direction n' coincident with y', as

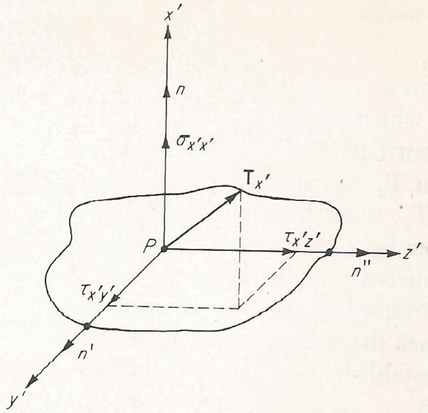

Figure 1.9 Resolution of $T_{x'}$ into three cartesian stress components $\sigma_{x'x'}$, $\tau_{x'y'}$, and $\tau_{x'z'}$.

shown in Fig. 1.9. The shear stress $\tau_{x'y'}$ is then obtained from Eq. (1.5*b*) by substituting x' for n and y' for n'. Thus

$$\begin{aligned}\tau_{x'y'} = {} & \sigma_{xx} \cos(x', x) \cos(y', x) \\ & + \sigma_{yy} \cos(x', y) \cos(y', y) + \sigma_{zz} \cos(x', z) \cos(y', z) \\ & + \tau_{xy}[\cos(x', x) \cos(y', y) + \cos(x', y) \cos(y', x)] \\ & + \tau_{yz}[\cos(x', y) \cos(y', z) + \cos(x', z) \cos(y', y)] \\ & + \tau_{zx}[\cos(x', z) \cos(y', x) + \cos(x', x) \cos(y', z)]\end{aligned} \tag{1.6d}$$

By selecting n and n' coincident with the y' and z', and z' and x' axes, additional expressions can be developed for $\tau_{y'z'}$ and $\tau_{z'x'}$, respectively, as follows:

$$\begin{aligned}\tau_{y'z'} = {} & \sigma_{yy} \cos(y', y) \cos(z', y) \\ & + \sigma_{zz} \cos(y', z) \cos(z', z) + \sigma_{xx} \cos(y', x) \cos(z', x) \\ & + \tau_{yz}[\cos(y', y) \cos(z', z) + \cos(y', z) \cos(z', y)] \\ & + \tau_{zx}[\cos(y', z) \cos(z', x) + \cos(y', x) \cos(z', z)] \\ & + \tau_{xy}[\cos(y', x) \cos(z', y) + \cos(y', y) \cos(z', x)]\end{aligned} \tag{1.6e}$$

$$\begin{aligned}\tau_{z'x'} = {} & \sigma_{zz} \cos(z', z) \cos(x', z) \\ & + \sigma_{xx} \cos(z', x) \cos(x', x) + \sigma_{yy} \cos(z', y) \cos(x', y) \\ & + \tau_{zx}[\cos(z', z) \cos(x', x) + \cos(z', x) \cos(x', z)] \\ & + \tau_{xy}[\cos(z', x) \cos(x', y) + \cos(z', y) \cos(x', x)] \\ & + \tau_{yz}[\cos(z', y) \cos(x', z) + \cos(z', z) \cos(x', y)]\end{aligned} \tag{1.6f}$$

These six equations permit the six cartesian components of stress relative to the $Oxyz$ coordinate system to be transformed into a different set of six cartesian components of stress relative to an $Ox'y'z'$ coordinate system.

1.6 PRINCIPAL STRESSES

In Sec. 1.2 it was noted that the resultant-stress vector $\mathbf{T}_n$ at a given point P depended upon the choice of the plane upon which the stress acted. If a plane is selected such that $\mathbf{T}_n$ coincides with the outer normal n, as shown in Fig. 1.10, it is clear that the shear stress τ_n vanishes and that $\mathbf{T}_n$, σ_n, and n are coincident.

If n is selected so that it coincides with $\mathbf{T}_n$, then the plane defined by n is known as a principal plane. The direction given by n is a principal direction, and the normal stress acting on this particular plane is a principal stress. In every state of stress there exist at least three principal planes, which are mutually perpendicular, and associated with these principal planes there are at most three distinct principal stresses. These statements can be established by referring to Fig. 1.10 and noting that

$$T_{nx} = \sigma_n \cos(n, x) \qquad T_{ny} = \sigma_n \cos(n, y) \qquad T_{nz} = \sigma_n \cos(n, z) \tag{a}$$

If Eqs. (1.2) are substituted into Eqs. (*a*), the following expressions are obtained:

$$\begin{aligned}
\sigma_{xx} \cos(n, x) + \tau_{yx} \cos(n, y) + \tau_{zx} \cos(n, z) &= \sigma_n \cos(n, x) \\
\tau_{xy} \cos(n, x) + \sigma_{yy} \cos(n, y) + \tau_{zy} \cos(n, z) &= \sigma_n \cos(n, y) \\
\tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_{zz} \cos(n, z) &= \sigma_n \cos(n, z)
\end{aligned} \tag{b}$$

Rearranging Eqs. (*b*) gives

$$\begin{aligned}
(\sigma_{xx} - \sigma_n) \cos(n, x) + \tau_{yx} \cos(n, y) + \tau_{zx} \cos(n, z) &= 0 \\
\tau_{xy} \cos(n, x) + (\sigma_{yy} - \sigma_n) \cos(n, y) + \tau_{zy} \cos(n, z) &= 0 \\
\tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + (\sigma_{zz} - \sigma_n) \cos(n, z) &= 0
\end{aligned} \tag{c}$$

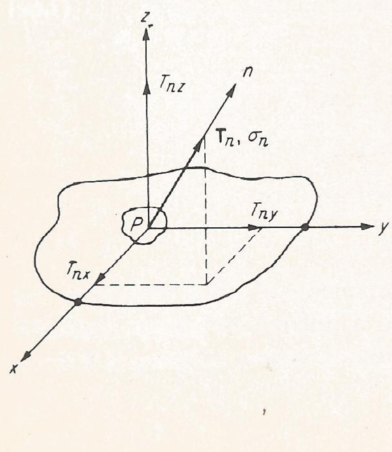

Figure 1.10 Coincidence of T_n with the outer normal n indicates that the shear stresses vanish and that σ_n becomes equal in magnitude to T_n.

Solving for any of the direction cosines, say cos (n, x), by determinants gives

$$\cos(n, x) = \frac{\begin{vmatrix} 0 & \tau_{yx} & \tau_{zx} \\ 0 & \sigma_{yy} - \sigma_n & \tau_{zy} \\ 0 & \tau_{yz} & \sigma_{zz} - \sigma_n \end{vmatrix}}{\begin{vmatrix} \sigma_{xx} - \sigma_n & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_n & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_n \end{vmatrix}} \tag{d}$$

It is clear that nontrivial solutions for the direction cosines of the principal plane will exist only if the determinant in the denominator is zero. Thus

$$\begin{vmatrix} \sigma_{xx} - \sigma_n & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_n & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_n \end{vmatrix} = 0 \tag{e}$$

Expanding the determinant after substituting Eqs. (1.4) gives the following important cubic equation:

$$\begin{aligned} \sigma_n^3 &- (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma_n^2 \\ &+ (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_n \\ &- (\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}) = 0 \end{aligned} \tag{1.7}$$

The roots of this cubic equation are the three principal stresses. By substituting the six cartesian components of stress into this equation, one can solve for σ_n and obtain three real roots. Three possible solutions exist.

1. If σ_1, σ_2, σ_3 are distinct, then n_1, n_2, and n_3 are unique and mutually perpendicular.
2. If $\sigma_1 = \sigma_2 \neq \sigma_3$, then n_3 is unique and every direction perpendicular to n_3 is a principal direction associated with $\sigma_1 = \sigma_2$.
3. If $\sigma_1 = \sigma_2 = \sigma_3$, then a hydrostatic state of stress exists and every direction is a principal direction.

Once the three principal stresses have been established, they can be substituted individually into Eqs. (*c*) to give three sets of simultaneous equations which together with the relation

$$\cos^2(n, x) + \cos^2(n, y) + \cos^2(n, z) = 1$$

can be solved to give the three sets of direction cosines defining the principal planes. A numerical example of the procedure used in computing principal stresses and directions is given in the exercises at the end of the chapter.

In treating principal stresses it is often useful to order them so that $\sigma_1 > \sigma_2 > \sigma_3$. When the stresses are ordered in this fashion, σ_1 is the normal stress having the largest algebraic value at a given point and σ_3 is the normal stress having the smallest algebraic value. It is important to recall in this ordering process that tensile stresses are considered positive and compressive stresses are considered negative.

Another important concept is that of stress invariants. It was noted in Sec. 1.5 that a state of stress could be described by its six cartesian stress components with respect to either the *Oxyz* coordinate system or the *Ox'y'z'* coordinate system. Furthermore, Eqs. (1.6) were established to give the relationship between these two systems. In addition to Eqs. (1.6), three other relations exist which are called the three invariants of stress. To establish these invariants, refer to Eq. (1.7), which is the cubic equation in terms of the principal stresses σ_1, σ_2, and σ_3. By recalling that σ_1, σ_2, and σ_3 are independent of the cartesian coordinate system employed, it is clear that the coefficients of Eq. (1.7) which contain cartesian components of the stresses must also be independent or invariant of the coordinate system. Thus, from Eq. (1.7) it is clear that

$$
\begin{aligned}
I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'} \\
I_2 &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \\
&= \sigma_{x'x'}\sigma_{y'y'} + \sigma_{y'y'}\sigma_{z'z'} + \sigma_{z'z'}\sigma_{x'x'} - \tau_{x'y'}^2 - \tau_{y'z'}^2 - \tau_{z'x'}^2 \\
I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx} \\
&= \sigma_{x'x'}\sigma_{y'y'}\sigma_{z'z'} - \sigma_{x'x'}\tau_{y'z'}^2 - \sigma_{y'y'}\tau_{z'x'}^2 - \sigma_{z'z'}\tau_{x'y'}^2 + 2\tau_{x'y'}\tau_{y'z'}\tau_{z'x'}
\end{aligned}
\tag{1.8}
$$

where I_1, I_2, and I_3 are the first, second, and third invariants of stress, respectively. If the *Oxyz* coordinate system is selected coincident with the principal directions, Eqs. (1.8) reduce to

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \qquad I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \qquad I_3 = \sigma_1\sigma_2\sigma_3 \tag{1.9}$$

1.7 MAXIMUM SHEAR STRESS

In developing equations for maximum shear stresses, the special case will be considered in which $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$. No loss in generality is introduced by considering this special case since it involves only a reorientation of the reference axes to coincide with the principal directions. In the following development n_1, n_2, and n_3 will be used to denote the principal directions. In Sec. 1.3 the resultant stress on an oblique plane was given by

$$T_n^2 = T_{nx}^2 + T_{ny}^2 + T_{nz}^2 \tag{a}$$

Substitution of values for T_{nx}, T_{ny}, and T_{nz} from Eqs. (1.2) with principal normal stresses and zero shearing stresses yields

$$T_n^2 = \sigma_1^2 \cos^2(n, n_1) + \sigma_2^2 \cos^2(n, n_2) + \sigma_3^2 \cos^2(n, n_3) \tag{b}$$

Also from Eq. (1.5*a*)

$$\sigma_n = \sigma_1 \cos^2(n, n_1) + \sigma_2 \cos^2(n, n_2) + \sigma_3 \cos^2(n, n_3) \tag{c}$$

Since $\tau_n^2 = T_n^2 - \sigma_n^2$, an expression for the shear stress τ_n on the oblique plane is obtained from Eqs. (*b*) and (*c*) after substituting $l = \cos(n, n_1)$, $m = \cos(n, n_2)$, and $n = \cos(n, n_3)$ as

$$\tau_n^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2 \tag{d}$$

The planes on which maximum and minimum shearing stresses occur can be obtained from Eq. (*d*) by differentiating with respect to the direction cosines *l*, *m*, and *n*. One of the direction cosines, *n* for example, in Eq. (*d*) can be eliminated by solving the expression

$$l^2 + m^2 + n^2 = 1 \tag{e}$$

for *l* and substituting into Eq. (*d*). Thus

$$\tau_n^2 = (\sigma_1^2 - \sigma_3^2)l^2 + (\sigma_2^2 - \sigma_3^2)m^2 + \sigma_3^2 - [(\sigma_1 - \sigma_3)l^2 + (\sigma_2 - \sigma_3)m^2 + \sigma_3]^2 \tag{f}$$

By taking the partial derivatives of Eq. (*f*), first with respect to *l* and then with respect to *m*, and equating to zero, the following equations are obtained for determining the direction cosines associated with planes having maximum and minimum shearing stresses:

$$l[\tfrac{1}{2}(\sigma_1 - \sigma_3) - (\sigma_1 - \sigma_3)l^2 - (\sigma_2 - \sigma_3)m^2] = 0 \tag{g}$$

$$m[\tfrac{1}{2}(\sigma_2 - \sigma_3) - (\sigma_1 - \sigma_3)l^2 - (\sigma_2 - \sigma_3)m^2] = 0 \tag{h}$$

One solution of these equations is obviously $l = m = 0$. Then from Eq. (*e*), $n = \pm 1$ (a principal plane with zero shear). Solutions different from zero are also possible for this set of equations. Consider first that $m = 0$; then from Eq. (*g*), $l = \pm(\frac{1}{2})^{1/2}$ and from Eq. (*e*), $n = \pm(\frac{1}{2})^{1/2}$. Also if $l = 0$, then from Eq. (*h*), $m = \pm(\frac{1}{2})^{1/2}$ and from Eq. (*e*), $n = \pm(\frac{1}{2})^{1/2}$. Repeating the above procedure by eliminating *l* and *m* in turn from Eq. (*f*) yields other values for the direction cosines which make the shearing stresses maximum or minimum. Substituting the values $l = \pm(\frac{1}{2})^{1/2}$ and $n = \pm(\frac{1}{2})^{1/2}$ into Eq. (*d*) yields

$$\tau_n^2 = \tfrac{1}{2}\sigma_1^2 + 0 + \tfrac{1}{2}\sigma_3^2 - (\tfrac{1}{2}\sigma_1 + 0 + \tfrac{1}{2}\sigma_3)^2$$

from which

$$\tau_n = \tfrac{1}{2}(\sigma_1 - \sigma_3)$$

Similarly, using the other values for the direction cosines which make the shearing stresses maximum gives

$$\tau_n = \tfrac{1}{2}(\sigma_1 - \sigma_2) \qquad \text{and} \qquad \tau_n = \tfrac{1}{2}(\sigma_2 - \sigma_3)$$

Of these three possible results, the largest magnitude will be obtained from $\sigma_1 - \sigma_3$ if the principal stresses are ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Thus

$$\tau_{max} = \tfrac{1}{2}(\sigma_{max} - \sigma_{min}) = \tfrac{1}{2}(\sigma_1 - \sigma_3) \tag{1.10}$$

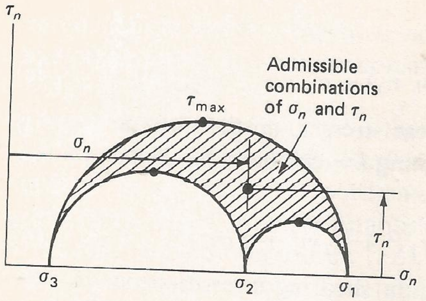

Figure 1.11 Mohr's Circle for the three-dimensional state of stress.

A useful aid for visualizing the complete state of stress at a point is the three-dimensional Mohr's circle shown in Fig. 1.11. This representation, which is similar to the familiar two-dimensional Mohr's circle, shows the three principal stresses, the maximum shearing stresses, and the range of values within which the normal- and shear-stress components must lie for a given state of stress.

1.8 THE TWO-DIMENSIONAL STATE OF STRESS

For two-dimensional stress fields where $\sigma_{zz} = \tau_{zx} = \tau_{yz} = 0$, z' is coincident with z, and θ is the angle between x and x', Eqs. (1.6a) to (1.6f) reduce to

$$\sigma_{x'x'} = \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$
$$= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \tag{1.11a}$$

$$\sigma_{y'y'} = \sigma_{yy} \cos^2 \theta + \sigma_{xx} \sin^2 \theta - 2\tau_{xy} \sin \theta \cos \theta$$
$$= \frac{\sigma_{yy} + \sigma_{xx}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \tag{1.11b}$$

$$\tau_{x'y'} = \sigma_{yy} \cos \theta \sin \theta - \sigma_{xx} \cos \theta \sin \theta + \tau_{xy}(\cos^2 \theta - \sin^2 \theta)$$
$$= \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \tag{1.11c}$$

$$\sigma_{z'z'} = \tau_{z'x'} = \tau_{y'z'} = 0 \tag{1.11d}$$

The relationships between stress components given in Eqs. (1.11) can be graphically represented by using Mohr's circle of stress, as indicated in Fig. 1.12. In this diagram, normal-stress components σ are plotted horizontally, while shear-stress components τ are plotted vertically. Tensile stresses are plotted to the right of the τ axis. Compressive stresses are plotted to the left. Shear-stress components which tend to produce a clockwise rotation of a small element surrounding the point are plotted above the σ axis. Those tending to produce a counterclockwise

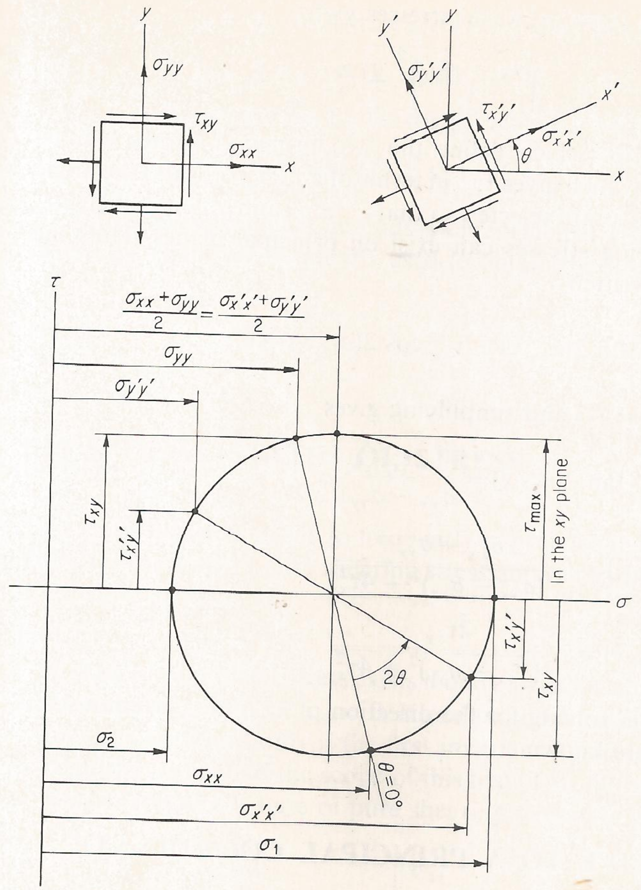

Figure 1.12 Mohr's circle of stress.

rotation are plotted below. When plotted in this manner, the stress components associated with each plane through the point are represented by a point on the circle. The diagram thus gives an excellent visual picture of the state of stress at a point. Mohr's circle and Eqs. (1.11) are often used in experimental stress-analysis work when stress components are transformed from one coordinate system to another. These relationships will be used frequently in later sections of this text, where strain gages and photoelasticity methods of analysis are discussed. Since two-dimensional stress systems are often considered in subsequent chapters, it will be useful to consider the principal stresses which occur in a two-dimensional stress system. If a coordinate system is chosen so that $\sigma_{zz} = \tau_{zx} = \tau_{yz} = 0$, then a state of plane stress exists and Eq. (1.7) reduces to

$$\sigma_n[\sigma_n^2 - (\sigma_{xx} + \sigma_{yy})\sigma_n + (\sigma_{xx}\sigma_{yy} - \tau_{xy}^2)] = 0 \qquad (a)$$

Solving this equation for the three principal stresses yields

$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_3 = 0 \tag{1.12}$$

The two direction cosines which define the two principal planes can be determined from Eq. (1.11*c*), which gives $\tau_{x'y'}$ in terms of σ_{xx}, σ_{yy}, τ_{xy} and the angle θ between x and x'. If x' and y' are selected so that $x' = n_1$ and $y' = n_2$, then $\tau_{x'y'}$ must vanish since no shearing stresses can exist on principal planes. Thus the following equation can be written:

$$\frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2(n_1, x) + \tau_{xy} \cos 2(n_1, x) = 0 \tag{1.13}$$

Dividing through by $\cos 2(n_1, x)$ and simplifying gives

$$\tan 2(n_1, x) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{1.14a}$$

and hence

$$\cos 2(n_1, x) = \frac{\sigma_{xx} - \sigma_{yy}}{\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\tau_{xy}^2}} \tag{1.14b}$$

$$\sin 2(n_1, x) = \frac{2\tau_{xy}}{\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\tau_{xy}^2}} \tag{1.14c}$$

Equations (1.14) are used in solving for the direction of n_1 if the cartesian stress components τ_{xy}, σ_{xx}, σ_{yy} are known.

1.9 STRESSES RELATIVE TO A PRINCIPAL COORDINATE SYSTEM

If the coordinate system *Oxyz* is selected to coincide with the three principal directions n_1, n_2, n_3, then $\sigma_1 = \sigma_{xx}$, $\sigma_2 = \sigma_{yy}$, $\sigma_3 = \sigma_{zz}$, and $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$. This reduces the six components of stress to three, which permits a considerable simplification in some of the previous results. Equations (1.2) become

$$T_{nx} = \sigma_1 \cos(n, x) \qquad T_{ny} = \sigma_2 \cos(n, y) \qquad T_{nz} = \sigma_3 \cos(n, z) \tag{1.15}$$

and Equations (1.6) reduce to

$$\begin{aligned}
\sigma_{x'x'} &= \sigma_1 \cos^2(x', x) + \sigma_2 \cos^2(x', y) + \sigma_3 \cos^2(x', z) \\
\sigma_{y'y'} &= \sigma_1 \cos^2(y', x) + \sigma_2 \cos^2(y', y) + \sigma_3 \cos^2(y', z) \\
\sigma_{z'z'} &= \sigma_1 \cos^2(z', x) + \sigma_2 \cos^2(z', y) + \sigma_3 \cos^2(z', z) \\
\tau_{x'y'} &= \sigma_1 \cos(x', x)\cos(y', x) + \sigma_2 \cos(x', y)\cos(y', y) \\
&\quad + \sigma_3 \cos(x', z)\cos(y', z)
\end{aligned} \tag{1.16}$$

$$\tau_{y'z'} = \sigma_1 \cos(y', x) \cos(z', x) + \sigma_2 \cos(y', y) \cos(z', y) + \sigma_3 \cos(y', z) \cos(z', z)$$

$$\tau_{z'x'} = \sigma_1 \cos(z', x) \cos(x', x) + \sigma_2 \cos(z', y) \cos(x', y) + \sigma_3 \cos(z', z) \cos(x', z)$$

Often experimental methods yield principal stresses directly, and in these cases Eqs. (1.16) are frequently used to obtain the stresses acting on other planes.

1.10 SPECIAL STATES OF STRESS

Two states of stress occur so frequently in practice that they have been classified. They are the state of pure shearing stress and the hydrostatic state of stress. Both are defined below.

1. A state of pure shear stress exists if one particular set of axes $Oxyz$ can be found such that $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$. It can be shown that this particular set of axes $Oxyz$ exists if and only if the first invariant of stress $I_1 = 0$. The proof of this condition is beyond the scope of this text. Two of the infinite number of arrays which represent a state of pure shearing stress are given below.

$$\begin{Vmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{Vmatrix} \quad \text{or} \quad \begin{Vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} = -\sigma_{xx} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{Vmatrix}$$

Pure shear

Can be converted to the form shown on the left by a suitable rotation of the coordinate system

2. A state of stress is said to be hydrostatic if $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ and all the shearing stresses vanish. In photoelastic work a hydrostatic state of stress is often called an isotropic state of stress. The stress array for this case is

$$\begin{Vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{Vmatrix}$$

One particularly important property of these two states of stress is that they can be combined to form a general state of stress. Of more importance, however, is the fact that any state of stress can be separated into a state of pure shear plus a hydrostatic state of stress. This is easily seen from the three arrays shown below:

$$\begin{Vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{Vmatrix} = \begin{Vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{Vmatrix} + \begin{Vmatrix} \sigma_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy}+p & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}+p \end{Vmatrix} \tag{1.17}$$

General state of stress = hydrostatic state of stress
+ state of pure shearing stress

It is immediately clear that the array on the left represents a general state of stress and that the center array represents a hydrostatic state of stress; however, the right-hand array represents a state of pure shear if and only if its first stress invariant is zero. This fact implies that

$$(\sigma_{xx} + p) + (\sigma_{yy} + p) + (\sigma_{zz} + p) = 0$$

Hence

$$p = -\tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{1.18}$$

If the p represented in the hydrostatic state of stress satisfies Eq. (1.18), then the separation of the state of stress given in Eq. (1.17) is valid. In the study of plasticity, the effect of the hydrostatic stresses is usually neglected; consequently, the principle illustrated above is quite important.

EXERCISES

1.1 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 60$ MPa, $\sigma_{yy} = -30$ MPa, $\sigma_{zz} = 30$ MPa, $\tau_{xy} = 40$ MPa, $\tau_{yz} = \tau_{zx} = 0$. Determine the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, x) = \tfrac{6}{11} \qquad \cos(n, y) = \tfrac{6}{11} \qquad \cos(n, z) = \tfrac{7}{11}$$

1.2 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 70$ MPa, $\sigma_{yy} = 60$ MPa, $\sigma_{zz} = 50$ MPa, $\tau_{xy} = 20$ MPa, $\tau_{yz} = -20$ MPa, $\tau_{zx} = 0$. Determine the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, x) = \tfrac{12}{25} \qquad \cos(n, y) = \tfrac{15}{25} \qquad \cos(n, z) = \tfrac{16}{25}$$

1.3 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 40$ MPa, $\sigma_{yy} = 60$ MPa, $\sigma_{zz} = 40$ MPa, $\tau_{xy} = 80$ MPa, $\tau_{yz} = 50$ MPa, $\tau_{zx} = 60$ MPa. Determine (*a*) the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, x) = \tfrac{4}{9} \qquad \cos(n, y) = \tfrac{4}{9} \qquad \cos(n, z) = \tfrac{7}{9}$$

and (*b*) the angle between T_n and the outer normal n.

1.4 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 60$ MPa, $\sigma_{yy} = 40$ MPa, $\sigma_{zz} = 20$ MPa, $\tau_{xy} = 40$ MPa, $\tau_{yz} = 20$ MPa, $\tau_{zx} = 30$ MPa. Determine (*a*) the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, x) = \tfrac{1}{3} \qquad \cos(n, y) = \tfrac{2}{3} \qquad \cos(n, z) = \tfrac{2}{3}$$

and (*b*) the angle between T_n and the outer normal n.

1.5 Determine the normal and shear stresses on a plane whose outer normal makes equal angles with the x, y, and z axes if the cartesian components of stress at the point are

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \qquad \tau_{xy} = 75 \text{ MPa} \qquad \tau_{yz} = 0 \qquad \tau_{zx} = 100 \text{ MPa}$$

1.6 The following stress distribution has been determined for a machine component:

$$\sigma_{xx} = 3x^2 - 3y^2 - z \qquad \sigma_{yy} = 3y^2 \qquad \sigma_{zz} = 3x + y - z + \tfrac{5}{4}$$

$$\tau_{xy} = z - 6xy - \tfrac{3}{4} \qquad \tau_{yz} = 0 \qquad \tau_{zx} = x + y - \tfrac{3}{2}$$

Is equilibrium satisfied in the absence of body forces?

1.7 If the state of stress at any point in a body is given by the equations

$$\sigma_{xx} = ax + by + cz \qquad \sigma_{yy} = dx^2 + ey^2 + fz^2 \qquad \sigma_{zz} = gx^3 + hy^3 + iz^3$$

$$\tau_{xy} = k \qquad \tau_{yz} = ly + mz \qquad \tau_{zx} = nx^2 + pz^2$$

what equations must the body-force intensities F_x, F_y, F_z satisfy?

1.8 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 90$ MPa, $\sigma_{yy} = 60$ MPa, $\sigma_{zz} = 30$ MPa, $\tau_{xy} = 30$ MPa, $\tau_{yz} = 30$ MPa, $\tau_{zx} = 60$ MPa. Transform this set of cartesian stress components into a new set of cartesian stress components relative to an $Ox'y'z'$ set of coordinates where the $Ox'y'z'$ axes are defined as:

θ	Case 1	Case 2	Case 3	Case 4
$x - x'$	$\pi/4$	$\pi/2$	0	$\pi/2$
$y - y'$	$\pi/4$	$\pi/2$	$\pi/2$	0
$z - z'$	0	0	$\pi/2$	$\pi/2$

1.9 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 70$ MPa, $\sigma_{yy} = 60$ MPa, $\sigma_{zz} = 50$ MPa, $\tau_{xy} = 20$ MPa, $\tau_{yz} = -20$ MPa, $\tau_{zx} = 0$. Transform this set of cartesian stress components into a new set of cartesian stress components relative to an $Ox'y'z'$ set of coordinates where the $Ox'y'z'$ axes are defined by the following direction cosines:

	x	y	z
x'	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$
y'	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$
z'	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$

1.10 For the state of stress at the point of Exercise 1.1, determine the principal stresses and the maximum shear stress at the point.

1.11 For the state of stress at the point of Exercise 1.2, determine the principal stresses and the maximum shear stress at the point.

1.12 For the state of stress at the point of Exercise 1.3, determine the principal stresses and the maximum shear stress at the point.

1.13 For the state of stress at the point of Exercise 1.4, determine the principal stresses and the maximum shear stress at the point.

1.14 Determine the principal stresses and the maximum shear stress at the point $x = \frac{1}{2}$, $y = 1$, $z = \frac{3}{4}$ for the stress distribution given in Exercise 1.6.

1.15 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = 50$ MPa, $\sigma_{yy} = 50$ MPa, $\sigma_{zz} = 50$ MPa, $\tau_{xy} = 100$ MPa, $\tau_{yz} = 0$, $\tau_{zx} = 50$ MPa. Determine (*a*) the principal stresses and the maximum shear stress at the point and (*b*) the orientation of the plane on which the maximum tensile stress acts.

1.16 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\tau_{xy} = 75$ MPa, $\tau_{yz} = 0$, $\tau_{zx} = 100$ MPa. Determine (*a*) the principal stresses and the maximum shear stress at the point and (*b*) the orientation of the plane on which the maximum tensile stress acts.

1.17 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 25$ MPa, $\tau_{xy} = 100$ MPa, $\tau_{yz} = 0$, $\tau_{zx} = 75$ MPa. Determine the principal stresses and the associated principal directions. Check on the invariance of I_1, I_2, and I_3.

1.18 At a point in a stressed body, the cartesian components of stress are $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$, $\tau_{xy} = \tau_{yz} = \tau_{zx} = 100$ MPa. Determine the principal stresses and the associated principal directions. Check on the invariance of I_1, I_2, and I_3.

1.19 A machine component is subjected to loads which produce the following stress field in a region where an oilhole must be drilled: $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = -50$ MPa, $\sigma_{zz} = 50$ MPa, $\tau_{xy} = 50$ MPa, $\tau_{yz} = \tau_{zx} = 0$. To minimize the effects of stress concentrations, the hole must be drilled along a line parallel to the direction of the maximum tensile stress in the region. Determine the direction cosines associated with the centerline of the hole with respect to the reference *Oxyz* coordinate system.

1.20 A two-dimensional state of stress ($\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$) exists at a point on the free surface of a machine component. The remaining cartesian components of stress are $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = -80$ MPa, $\tau_{xy} = -40$ MPa. Determine (*a*) the principal stresses and their associated directions at the point and (*b*) the maximum shear stress at the point.

1.21 A two-dimensional state of stress ($\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$) exists at a point on the surface of a loaded member. Determine the principal stresses and the maximum shear stress at the point if the remaining cartesian components of stress are $\sigma_{xx} = 90$ MPa, $\sigma_{yy} = 60$ MPa, $\tau_{xy} = 40$ MPa.

1.22 A two-dimensional state of stress ($\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$) exists at a point on the surface of a loaded member. The remaining cartesian components of stress are $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = 70$ MPa, $\tau_{xy} = 20$ MPa. Determine the principal stresses and the maximum shear stress at the point.

1.23 A two-dimensional state of stress ($\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$) exists at a point on the surface of a loaded member. The remaining cartesian components of stress are $\sigma_{xx} = 90$ MPa, $\sigma_{yy} = 40$ MPa, $\tau_{xy} = 60$ MPa. Determine the principal stresses and the maximum shear stress at the point.

1.24 Solve Exercise 1.22 by means of Mohr's circle.

1.25 Solve Exercise 1.23 by means of Mohr's circle.

1.26 At the point of Exercise 1.22, determine the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, x) = \tfrac{3}{5} \qquad \cos(n, y) = \tfrac{4}{5} \qquad \cos(n, z) = 0$$

1.27 At the point of Exercise 1.23, determine the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, x) = \tfrac{1}{3} \qquad \cos(n, y) = \tfrac{2}{3} \qquad \cos(n, z) = \tfrac{2}{3}$$

1.28 There is a crack in a plate of steel which makes the material in that area weak in tension and shear. The plate must be used for a member which will be loaded to produce the following state of stress in the plane of the plate: $\sigma_{xx} = 100$ MPa, $\sigma_{yy} = -60$ MPa, $\tau_{xy} = 20$ MPa. How should the x and y axes be oriented with respect to the crack in order to minimize the effect of the crack?

1.29 At a point in a metal machine part the principal stresses are $\sigma_1 = 150$ MPa, $\sigma_2 = 100$ MPa, $\sigma_3 = 50$ MPa. Determine the normal and shear stresses on a plane whose outer normal has the direction cosines

$$\cos(n, n_1) = \frac{\sqrt{3}}{2} \qquad \cos(n, n_2) = 0 \qquad \cos(n, n_3) = \frac{1}{2}$$

1.30 If the three principal stresses relative to the $Oxyz$ reference system are $\sigma_1 = \sigma_{xx} = 100$ MPa, $\sigma_2 = \sigma_{yy} = 80$ MPa, $\sigma_3 = \sigma_{zz} = -20$ MPa, determine the six cartesian components of stress relative to the $Ox'y'z'$ reference system where $Ox'y'z'$ is defined as:

θ	Case 1	Case 2	Case 3	Case 4
$x - x'$	$\pi/4$	$\pi/2$	0	$\pi/4$
$y - y'$	$\pi/4$	$\pi/2$	$\pi/4$	0
$z - z'$	0	0	$\pi/4$	$\pi/4$

1.31 Resolve the general state of stress given in Exercise 1.1 into a hydrostatic state of stress and a state of pure shearing stress.

1.32 Resolve the general state of stress given in Exercise 1.2 into a hydrostatic state of stress and a state of pure shearing stress.

1.33 Resolve the general state of stress given in Exercise 1.4 into a hydrostatic state of stress and a state of pure shearing stress.

1.34 Resolve the two-dimensional state of stress given in Exercise 1.21 into a hydrostatic state of stress and a state of pure shearing stress.

1.35 Determine the octahedral normal and shearing stresses associated with the principal stresses σ_1, σ_2, and σ_3. Octahedral normal and shearing stresses occur on planes whose outer normal makes equal angles with the principal directions n_1, n_2, and n_3.

REFERENCES

1. Boresi, A. P., and P. P. Lynn: "Elasticity in Engineering Mechanics," chap. 3, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.
2. Chou, P. C., and N. J. Pagano: "Elasticity," chap. 1, D. Van Nostrand Company, Inc., Princeton, N.J., 1967.
3. Durelli, A. J., E. A. Phillips, and C. H. Tsao: "Introduction to the Theoretical and Experimental Analysis of Stress and Strain," chap. 1, McGraw-Hill Book Company, New York, 1958.
4. Love, A. E. H.: "A Treatise on the Mathematical Theory of Elasticity," chap. 2, Dover Publications, Inc., New York, 1944.
5. Sechler, E. E.: "Elasticity in Engineering," chaps. 2 and 3, John Wiley & Sons, Inc., New York, 1952.
6. Sokolnikoff, I. S.: "Mathematical Theory of Elasticity," 2d ed., chap. 2, McGraw-Hill Book Company, New York, 1956.
7. Southwell, R. V.: "An Introduction to the Theory of Elasticity," chap. 8, Oxford University Press, Fair Lawn, N.J., 1953.
8. Timoshenko, S. P., and J. N. Goodier: "Theory of Elasticity," 2d ed., chaps. 1, 8, and 9, McGraw-Hill Book Company, New York, 1951.