CHAPTER

TEN

ANALYSIS OF STRAIN-GAGE DATA

10.1 INTRODUCTION

Electrical-resistance strain gages are normally employed on the free surface of a specimen to establish the stress at a particular point on this surface. In general it is necessary to measure three strains at a point to completely define either the stress or the strain field. In terms of principal strains it is necessary to measure ϵ_1, ϵ_2, and the direction of ϵ_1 relative to the x axis as given by the principal angle ϕ. Conversion of the strains into stresses requires, in addition, a knowledge of the elastic constants E and ν of the specimen material.

In certain special cases the state of stress can be established with a single strain gage. Consider first a uniaxial state of stress where $\sigma_{yy} = \tau_{xy} = 0$ and the direction of σ_{xx} is known. In this case a single-element strain gage is mounted with its axis coincident with the x axis. The stress σ_{xx} is given by

$$\sigma_{xx} = E\epsilon_{xx} \tag{10.1}$$

Next, consider an isotropic state of stress where $\sigma_{xx} = \sigma_{yy} = \sigma_1 = \sigma_2$ and $\tau_{xy} = 0$. In this case a strain gage may be mounted in any direction, and the magnitude of the stresses can be established from

$$\sigma_{xx} = \sigma_{yy} = \sigma_1 = \sigma_2 = \frac{E}{1-\nu}\epsilon_\theta \tag{10.2}$$

where ϵ_θ is the strain measured in any direction in the isotropic stress field.

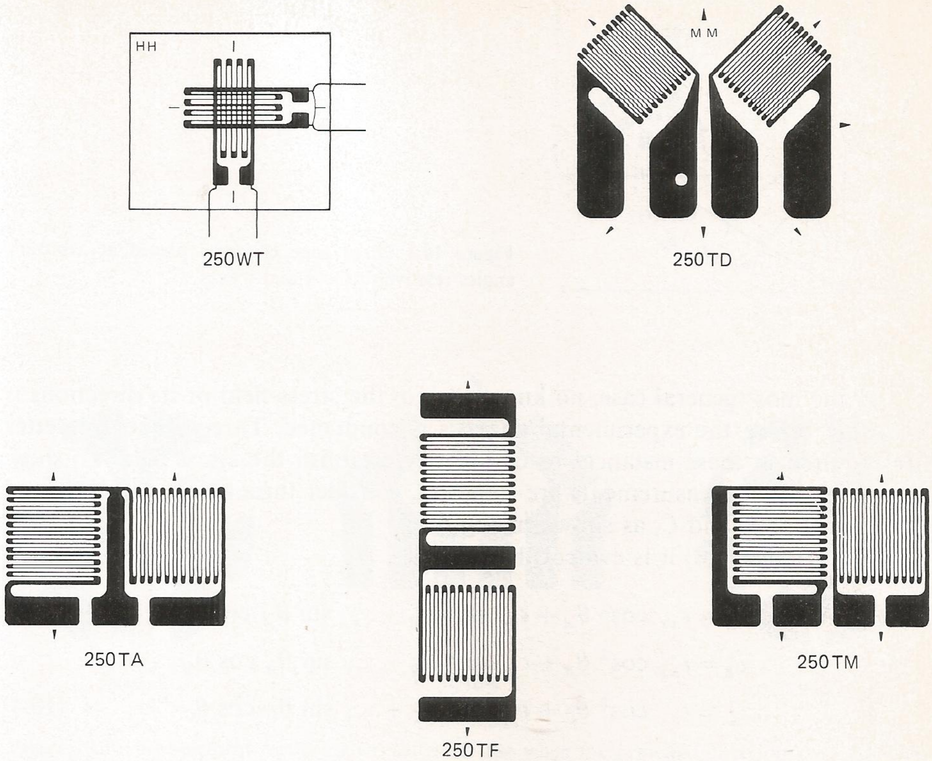


Figure 10.1 Two-element rectangular rosettes for use when the principal directions are known.

When less is known beforehand regarding the state of stress in the specimen, it is necessary to employ multiple-element strain gages to establish the magnitude of the stress field. If the specimen being investigated has an axis of symmetry, or if a brittle-coating analysis has been conducted to establish the principal-stress directions, this knowledge can be used to reduce the number of gage elements required from three to two. A two-element rectangular rosette similar to those illustrated in Fig. 10.1 is mounted on the specimen with its axes coincident with the principal directions. The two principal strains ϵ_1 and ϵ_2 obtained from the gages can be employed to give the principal stresses σ_1 and σ_2:

$$\sigma_1 = \frac{E}{1 - \nu^2}(\epsilon_1 + \nu\epsilon_2) \qquad \sigma_2 = \frac{E}{1 - \nu^2}(\epsilon_2 + \nu\epsilon_1) \tag{10.3}$$

These relations give the complete state of stress since the principal directions are known a priori. The stresses on any plane can be established by employing Eqs. (1.16) with the results obtained from Eq. (10.3).

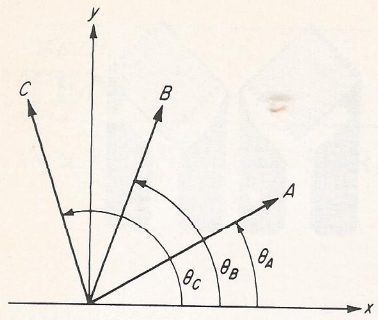


Figure 10.2 Three gage elements placed at arbitrary angles relative to the x and y axes.

In the most general case, no knowledge of the stress field or its directions is available before the experimental analysis is conducted. Three-element rosettes are required in these instances to completely establish the stress field. To show that three strain measurements are sufficient, consider three strain gages aligned along axes A, B, and C, as shown in Fig. 10.2.

From Eqs. (2.18) it is evident that

$$\begin{aligned} \epsilon_A &= \epsilon_{xx} \cos^2 \theta_A + \epsilon_{yy} \sin^2 \theta_A + \gamma_{xy} \sin \theta_A \cos \theta_A \\ \epsilon_B &= \epsilon_{xx} \cos^2 \theta_B + \epsilon_{yy} \sin^2 \theta_B + \gamma_{xy} \sin \theta_B \cos \theta_B \\ \epsilon_C &= \epsilon_{xx} \cos^2 \theta_C + \epsilon_{yy} \sin^2 \theta_C + \gamma_{xy} \sin \theta_C \cos \theta_C \end{aligned} \tag{10.4}$$

The cartesian components of strain ϵ_{xx}, ϵ_{yy}, and γ_{xy} can be determined from a simultaneous solution of Eqs. (10.4). The principal strains and the principal directions can then be established by employing Eqs. (2.7), (1.12), and (1.14). The results are

$$\begin{aligned} \epsilon_1 &= \tfrac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) + \tfrac{1}{2}\sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + \gamma_{xy}^2} \\ \epsilon_2 &= \tfrac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) - \tfrac{1}{2}\sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + \gamma_{xy}^2} \\ \tan 2\phi &= \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} \end{aligned} \tag{10.5}$$

where ϕ is the angle between the principal axis (σ_1) and the x axis. The principal stresses can then be computed from the principal strains by utilizing Eqs. (10.3).

In actual practice, three-element rosettes with fixed angles (that is, θ_A, θ_B, and θ_C fixed at specified values) are employed to provide sufficient data to completely define the stress field. These rosettes are defined by the fixed angles as the rectangular rosette, the delta rosette, and the tee-delta rosette. Examples of commercially available three-element rosettes are presented in Fig. 10.3. Also shown in this figure is a stress gage which can be employed to give the stress in an arbitrary direction. Each of these rosette gages will be discussed in detail in subsequent sections of this chapter.

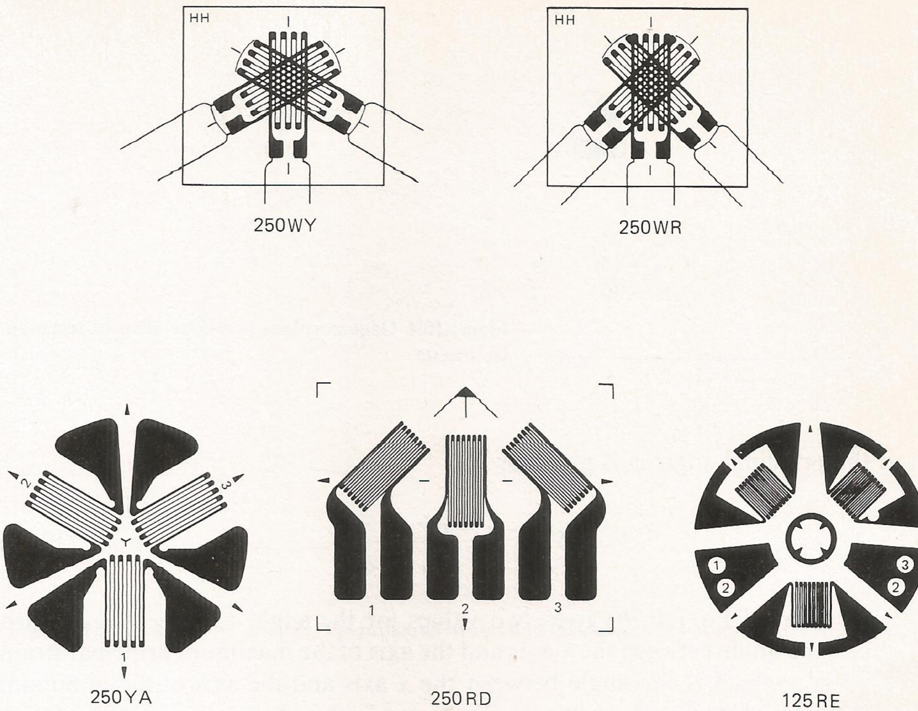


Figure 10.3 Three-element rectangular rosettes for use when the principal directions are unknown.

10.2 THE THREE-ELEMENT RECTANGULAR ROSETTE

The three-element rectangular rosette employs gages placed at the 0, 45, and 90° positions, as indicated in Fig. 10.4.

For this particular rosette it is clear from Eqs. (10.4) that

$$\epsilon_A = \epsilon_{xx} \qquad \epsilon_B = \tfrac{1}{2}(\epsilon_{xx} + \epsilon_{yy} + \gamma_{xy}) \qquad \epsilon_C = \epsilon_{yy} \tag{10.6}$$

and that

$$\gamma_{xy} = 2\epsilon_B - \epsilon_A - \epsilon_C$$

Thus by measuring the strains ϵ_A, ϵ_B, and ϵ_C the cartesian components of strain ϵ_{xx}, ϵ_{yy}, and γ_{xy} can be quickly and simply established through the use of Eqs. (10.6). Next, by utilizing Eqs. (10.5), the principal strains ϵ_1 and ϵ_2 can be established as

$$\epsilon_1 = \tfrac{1}{2}(\epsilon_A + \epsilon_C) + \tfrac{1}{2}\sqrt{(\epsilon_A - \epsilon_C)^2 + (2\epsilon_B - \epsilon_A - \epsilon_C)^2}$$

$$\epsilon_2 = \tfrac{1}{2}(\epsilon_A + \epsilon_C) - \tfrac{1}{2}\sqrt{(\epsilon_A - \epsilon_C)^2 + (2\epsilon_B - \epsilon_A - \epsilon_C)^2} \tag{10.7a}$$

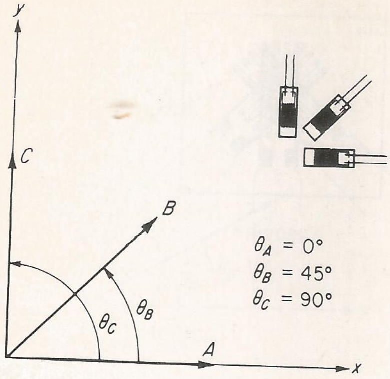


Figure 10.4 Gage positions in a three-element rectangular rosette.

and the principal angle ϕ is given by

$$\tan 2\phi = \frac{2\epsilon_B - \epsilon_A - \epsilon_C}{\epsilon_A - \epsilon_C} \tag{10.7b}$$

The solution of Eq. (10.7*b*) gives two values for the angle ϕ, namely, ϕ_1, which refers to the angle between the x axis and the axis of the maximum principal strain ϵ_1, and ϕ_2, which is the angle between the x axis and the axis of the minimum principal strain ϵ_2. These angles are illustrated in the Mohr's strain circle shown in Fig. 10.5. It is possible to show (see Exercise 10.4) that the principal axes can be identified by applying the following rules:

$$\begin{array}{ll}
0 < \phi_1 < 90^\circ & \text{when } \epsilon_B > \tfrac{1}{2}(\epsilon_A + \epsilon_C) \\
-90^\circ < \phi_1 < 0 & \text{when } \epsilon_B < \tfrac{1}{2}(\epsilon_A + \epsilon_C) \\
\phi_1 = 0 & \text{when } \epsilon_A > \epsilon_C \text{ and } \epsilon_A = \epsilon_1 \\
\phi_1 = \pm 90^\circ & \text{when } \epsilon_A < \epsilon_C \text{ and } \epsilon_A = \epsilon_2
\end{array} \tag{10.8}$$

Finally, the principal stresses occurring in the component can be established by employing Eqs. (10.7) together with Eqs. (10.3) to obtain

$$\begin{aligned}
\sigma_1 &= E\left[\frac{\epsilon_A + \epsilon_C}{2(1-\nu)} + \frac{1}{2(1+\nu)}\sqrt{(\epsilon_A - \epsilon_C)^2 + (2\epsilon_B - \epsilon_A - \epsilon_C)^2}\right] \\
\sigma_2 &= E\left[\frac{\epsilon_A + \epsilon_C}{2(1-\nu)} - \frac{1}{2(1+\nu)}\sqrt{(\epsilon_A - \epsilon_C)^2 + (2\epsilon_B - \epsilon_A - \epsilon_C)^2}\right]
\end{aligned} \tag{10.9}$$

The use of Eqs. (10.6) to (10.9) permits a determination of the cartesian components of strain, the principal strains and their directions, and the principal stresses by a totally analytical approach. However, it is also possible to determine these

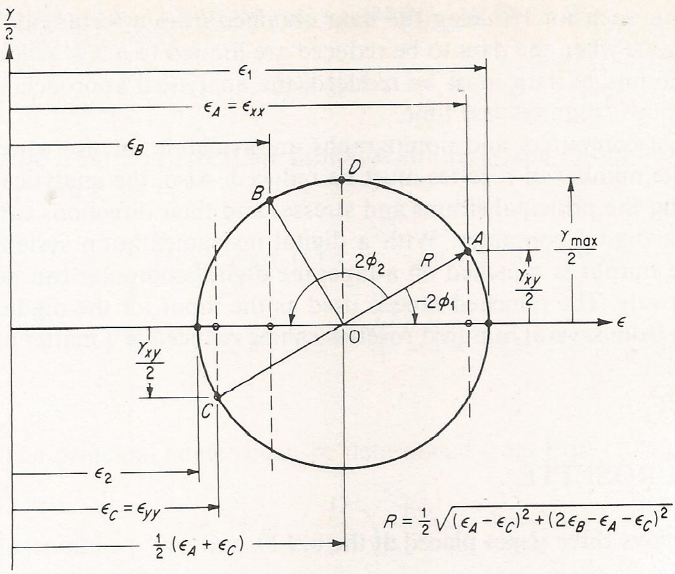


Figure 10.5 Graphical solution for the principal strains and their directions from a rectangular rosette.

quantities with a graphical approach, as illustrated in Fig. 10.5. A Mohr's strain circle is initiated by laying out the ϵ (abscissa) and the $\frac{1}{2}\gamma$ (ordinate) axes. The three strains ϵ_A, ϵ_B, and ϵ_C are then plotted as points on the abscissa. Vertical lines are then drawn through these three points. The shearing strain γ_{xy} is computed from Eqs. (10.6), and $\frac{1}{2}\gamma_{xy}$ is plotted positive downward or negative upward along the vertical line drawn through ϵ_A to establish point A. This shearing strain may also be plotted as positive upward or negative downward along the vertical line through ϵ_C to establish point C. The diameter of the circle is then determined by drawing a line between points A and C which intersects the abscissa and defines the center of the circle at a distance $\frac{1}{2}(\epsilon_A + \epsilon_C)$ from the origin. A circle is then drawn from this center passing through points A and C. The circle will intersect the vertical line drawn through ϵ_B, and this point of intersection is labeled as B. A straight line through the center of the circle and point B should be a perpendicular bisector of the diameter AC. The values of the principal strains ϵ_1 and ϵ_2 are given by the intersections of the circle with the abscissa. The principal angle $2\phi_1$ is given by the angle $AO\epsilon_1$ and is negative if point A lies above the ϵ axis. The principal angle $2\phi_2$ is given by the angle $AO\epsilon_2$ and is positive if point A lies above the ϵ axis. The maximum shearing strain is established by a vertical line drawn through the center of the circle to give point D at the intersection. The projection of point D onto the $\frac{1}{2}\gamma$ axis determines the value of $\frac{1}{2}\gamma_{max}$. The principal stresses can be determined directly from the principal strains by employing Eqs. (10.3).

This graphical approach for reducing the data obtained from a rectangular rosette is quite applicable when the data to be reduced are limited to a few gages. However, if large amounts of data must be reduced, the analytical approach is normally preferred since it requires less time.

Special strain-gage computers and nomographs are available for use when data from a very large number of rosettes must be reduced. Also, the analytical approach for obtaining the principal strains and stresses and their directions can be programmed for a digital computer. With a digital instrumentation system where the strain-gage output is punched on a tape, the digital computer can be employed quite effectively. The punched tape is used as the input for the digital computer, and results from several hundred rosettes can be reduced in a matter of minutes.

10.3 THE DELTA ROSETTE

The delta rosette employs three gages placed at the 0, 120, and 240° positions, as indicated in Fig. 10.6.

For the angular layout of the delta rosette it is clear from Eqs. (10.4) that

$$
\begin{aligned}
\epsilon_A &= \epsilon_{xx} \\
\epsilon_B &= \tfrac{1}{4}(\epsilon_{xx} + 3\epsilon_{yy} - \sqrt{3}\,\gamma_{xy}) \\
\epsilon_C &= \tfrac{1}{4}(\epsilon_{xx} + 3\epsilon_{yy} + \sqrt{3}\,\gamma_{xy})
\end{aligned}
\qquad (10.10)
$$

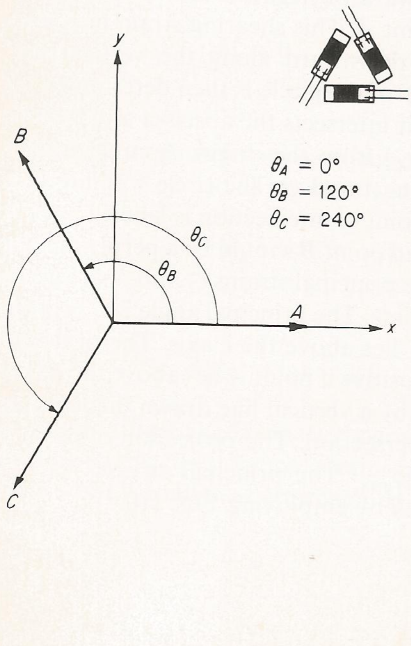


Figure 10.6 Gage positions in a three-element delta rosette.

Solving Eqs. (10.10) for ϵ_{xx}, ϵ_{yy}, and γ_{xy} in terms of ϵ_A, ϵ_B, and ϵ_C gives

$$\epsilon_{xx} = \epsilon_A \qquad \epsilon_{yy} = \tfrac{1}{3}[2(\epsilon_B + \epsilon_C) - \epsilon_A] \qquad \gamma_{xy} = \frac{2\sqrt{3}}{3}(\epsilon_C - \epsilon_B) \tag{10.11}$$

Also from Eqs. (10.5) the principal strains ϵ_1 and ϵ_2 can be written in terms of ϵ_A, ϵ_B, and ϵ_C as

$$\epsilon_1 = \frac{1}{3}(\epsilon_A + \epsilon_B + \epsilon_C) + \frac{\sqrt{2}}{3}\sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2 + (\epsilon_C - \epsilon_A)^2}$$

$$\epsilon_2 = \frac{1}{3}(\epsilon_A + \epsilon_B + \epsilon_C) - \frac{\sqrt{2}}{3}\sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2 + (\epsilon_C - \epsilon_A)^2} \tag{10.12}$$

The principal angle ϕ can be determined from Eqs. (10.5) as

$$\tan 2\phi = \frac{(2/\sqrt{3})(\epsilon_C - \epsilon_B)}{2[\epsilon_A - \frac{1}{3}(\epsilon_A + \epsilon_B + \epsilon_C)]} = \frac{\sqrt{3}(\epsilon_C - \epsilon_B)}{2\epsilon_A - (\epsilon_B + \epsilon_C)} \tag{10.13}$$

The solution of Eq. (10.13) gives two values for the principal angle ϕ, as was the case for the rectangular rosette. The angles ϕ_1 and ϕ_2 are illustrated in the Mohr's strain circle shown in Fig. 10.7. It is possible to show (see Exercise 10.7) that the

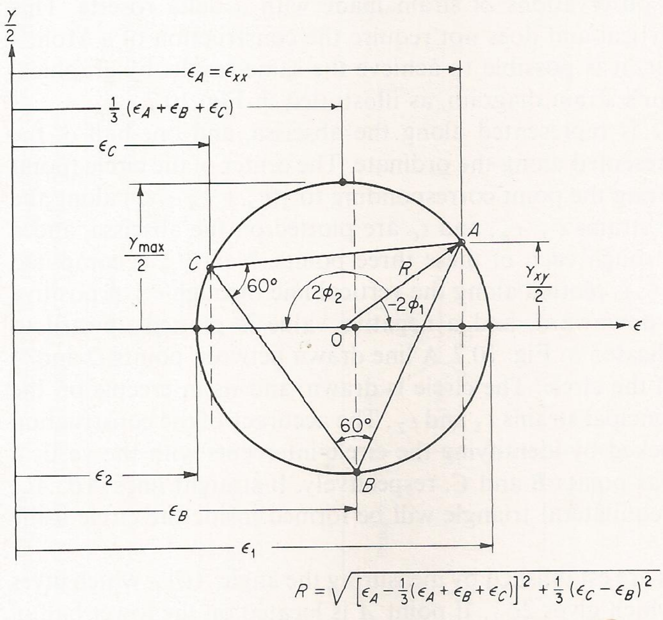


Figure 10.7 Graphical solution for the principal strains and their directions from a delta rosette.

principal angles can be identified by applying the following rules:

$$
\begin{aligned}
&0^\circ < \phi_1 < 90^\circ && \text{when } \epsilon_C > \epsilon_B \\
&-90^\circ < \phi_1 < 0^\circ && \text{when } \epsilon_C < \epsilon_B \\
&\phi_1 = 0 && \text{when } \epsilon_B = \epsilon_C \text{ and } \epsilon_A > \epsilon_B = \epsilon_C \\
&\phi_1 = \pm 90^\circ && \text{when } \epsilon_B = \epsilon_C \text{ and } \epsilon_A < \epsilon_B = \epsilon_C
\end{aligned}
\tag{10.14}
$$

Finally, the principal stresses can be determined from the principal strains by employing Eqs. (10.3) to obtain

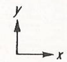
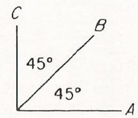
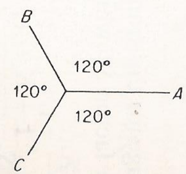
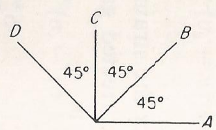
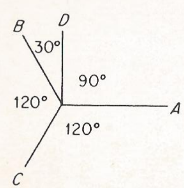
$$
\begin{aligned}
\sigma_1 &= E\left\{\frac{\epsilon_A + \epsilon_B + \epsilon_C}{3(1-\nu)} + \frac{\sqrt{2}}{3(1+\nu)}\sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2 + (\epsilon_C - \epsilon_A)^2}\right\} \\
\sigma_2 &= E\left\{\frac{\epsilon_A + \epsilon_B + \epsilon_C}{3(1-\nu)} - \frac{\sqrt{2}}{3(1+\nu)}\sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2 + (\epsilon_C - \epsilon_A)^2}\right\}
\end{aligned}
\tag{10.15}
$$

By employing Eqs. (10.11) to (10.15), it is possible to determine the cartesian components of strain, the principal strains and their directions, and the principal stresses from the three observations of strain made with a delta rosette. This approach is purely analytical and does not require the construction of a Mohr's strain diagram. However, it is possible to achieve the same results by graphical construction of the Mohr's strain diagram, as illustrated in Fig. 10.7.

The normal strain ϵ is represented along the abscissa, and one-half of the shearing strain $\frac{1}{2}\gamma$ is represented along the ordinate. The center of the circle (point O) is established by plotting the point corresponding to $\frac{1}{3}(\epsilon_A + \epsilon_B + \epsilon_C)$ along the abscissa. Also, the three strains ϵ_A, ϵ_B, and ϵ_C are plotted on the abscissa, and a vertical line is drawn through each of these three points. Next γ_{xy} is computed from Eqs. (10.11) and $\frac{1}{2}\gamma_{xy}$ is plotted along the vertical line through ϵ_A. A positive value of γ_{xy} is plotted downward, and a negative value is plotted upward to establish point A, as indicated in Fig. 10.7. A line drawn between points O and A establishes the radius of the circle. The circle is drawn, and its intercepts on the abscissa establish the principal strains ϵ_1 and ϵ_2. The accuracy of the construction of the circle can be checked by identifying the circle intercepts with the vertical lines through ϵ_B and ϵ_C as points B and C, respectively. If straight lines AB, AC, and BC are drawn, an equilateral triangle will be formed inside the circle if the construction is correct.

The principal angles are established by measuring the angle $AO\epsilon_1$, which gives $2\phi_1$, and angle $AO\epsilon_2$, which gives $2\phi_2$. If point A is located on the lower half of the circle, ϕ_1 will be positive and ϕ_2 negative. If point A is located on the upper half of the circle, ϕ_1 will be negative and ϕ_2 positive, as indicated in Fig. 10.7. The

Table 10.1 A summary of the equations used to determine principal strains, principal stresses, and their directions from four types of rosettes

Type of rosette	Gage arrangement	Principal strain and principal stress	Principal angle	Identification $0 < \phi_1 < 90°$
				
Three-element, rectangular		$\epsilon_{1,2} = \frac{\epsilon_A + \epsilon_C}{2} \pm \frac{1}{2}\sqrt{(\epsilon_A - \epsilon_C)^2 + (2\epsilon_B - \epsilon_A - \epsilon_C)^2}$ $\sigma_{1,2} = \frac{E}{2}\left[\frac{\epsilon_A + \epsilon_C}{1-\nu} \pm \frac{1}{1+\nu}\sqrt{(\epsilon_A - \epsilon_C)^2 + (2\epsilon_B - \epsilon_A - \epsilon_C)^2}\right]$	$\tan 2\phi_1$ $= \frac{2\epsilon_B - \epsilon_A - \epsilon_C}{\epsilon_A - \epsilon_C}$	$\epsilon_B > \frac{\epsilon_A + \epsilon_C}{2}$
Delta		$\epsilon_{1,2} = \frac{\epsilon_A + \epsilon_B + \epsilon_C}{3} \pm \frac{\sqrt{2}}{3}\sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2 + (\epsilon_C - \epsilon_A)^2}$ $\sigma_{1,2} = \frac{E}{3}\left\{\frac{\epsilon_A + \epsilon_B + \epsilon_C}{(1-\nu)} \pm \frac{\sqrt{2}}{(1+\nu)}\sqrt{(\epsilon_A - \epsilon_B)^2 + (\epsilon_B - \epsilon_C)^2 + (\epsilon_C - \epsilon_A)^2}\right\}$	$\tan 2\phi_1$ $= \frac{\sqrt{3}(\epsilon_C - \epsilon_B)}{2\epsilon_A - (\epsilon_B + \epsilon_C)}$	$\epsilon_C > \epsilon_B$
Four-element, rectangular		$\epsilon_{1,2} = \frac{\epsilon_A + \epsilon_B + \epsilon_C + \epsilon_D}{4} \pm \frac{1}{2}\sqrt{(\epsilon_A - \epsilon_C)^2 + (\epsilon_B - \epsilon_D)^2}$ $\sigma_{1,2} = \frac{E}{2}\left[\frac{\epsilon_A + \epsilon_B + \epsilon_C + \epsilon_D}{2(1-\nu)} \pm \frac{1}{1+\nu}\sqrt{(\epsilon_A - \epsilon_C)^2 + (\epsilon_B - \epsilon_D)^2}\right]$	$\tan 2\phi_1 = \frac{\epsilon_B - \epsilon_D}{\epsilon_A - \epsilon_C}$	$\epsilon_B > \epsilon_D$
Tee-delta		$\epsilon_{1,2} = \frac{\epsilon_A + \epsilon_D}{2} \pm \frac{1}{2}\sqrt{(\epsilon_A - \epsilon_D)^2 + \frac{4}{3}(\epsilon_C - \epsilon_B)^2}$ $\sigma_{1,2} = \frac{E}{2}\left[\frac{\epsilon_A + \epsilon_D}{1-\nu} \pm \frac{1}{1+\nu}\sqrt{(\epsilon_A - \epsilon_D)^2 + \frac{4}{3}(\epsilon_C - \epsilon_B)^2}\right]$	$\tan 2\phi_1$ $= \frac{2(\epsilon_C - \epsilon_B)}{\sqrt{3}(\epsilon_A - \epsilon_D)}$	$\epsilon_C > \epsilon_B$

maximum shearing strain is given by the projection of the circle radius onto the ordinate. The principal stresses can be established from the principal strains by employing Eqs. (10.3).

Both the delta and the rectangular rosette can be employed with about equal facility for determining the principal stresses and their directions on the surface of the specimen. The selection of one type of rosette over the other will depend primarily on the nature of the stress field at the point of interest and on the type of rosette commercially available in the size (gage length) required.

A complete summary of the analytical expressions used to reduce the data obtained from four different rosettes is presented in Table 10.1.

10.4 CORRECTIONS FOR TRANSVERSE STRAIN EFFECTS [11–14]

In Sec. 6.5 it was noted that foil-type resistance strain gages exhibit a sensitivity S_t to transverse strains. Reference to Fig. 6.12 shows that in certain instances this transverse sensitivity can lead to large errors, and it is important to correct the data to eliminate this effect. Two different procedures for correcting data have been developed.

The first procedure requires a priori knowledge of the ratio ϵ_t/ϵ_a of the strain field. The correction factor is evident in Eq. (6.8), where

$$\epsilon_a = \epsilon'_a \frac{1 - \nu_0 K_t}{1 + K_t \epsilon_t/\epsilon_a} \tag{6.8}$$

The term ϵ'_a is the apparent strain, and the correction factor CF is given by

$$\text{CF} = \frac{1 - \nu_0 K_t}{1 + K_t \epsilon_t/\epsilon_a} \tag{10.16}$$

It is possible to correct the strain gage for this transverse sensitivity by adjusting its gage factor. The corrected gage factor S_g^* which should be dialed into the measuring instrument is

$$S_g^* = S_g \frac{1 + K_t \epsilon_t/\epsilon_a}{1 - \nu_0 K_t} \tag{10.17}$$

Correction for the cross-sensitivity effect when the strain field is unknown is more involved and requires the experimental determination of strain in both the x and y directions. If ϵ'_{xx} and ϵ'_{yy} are the apparent strains recorded in the x and y directions, respectively, then from Eq. (6.8) it is evident that

$$\epsilon'_{xx} = \frac{1}{1 - \nu_0 K_t}(\epsilon_{xx} + K_t \epsilon_{yy}) \qquad \epsilon'_{yy} = \frac{1}{1 - \nu_0 K_t}(\epsilon_{yy} + K_t \epsilon_{xx}) \tag{10.18}$$

where the unprimed quantities ϵ_{xx} and ϵ_{yy} are the true strains. Solving Eqs. (10.18) for ϵ_{xx} and ϵ_y gives

$$\epsilon_{xx} = \frac{1 - v_0 K_t}{1 - K_t^2}(\epsilon'_{xx} - K_t \epsilon'_{yy}) \qquad \epsilon_{yy} = \frac{1 - v_0 K_t}{1 - K_t^2}(\epsilon'_{yy} - K_t \epsilon'_{xx}) \tag{10.19}$$

Equations (10.19) give the true strains ϵ_{xx} and ϵ_{yy} in terms of the apparent strains ϵ'_{xx} and ϵ'_{yy}. Correction equations for transverse strains in two- and three-element rosettes are given in Ref. 14.

10.5 THE STRESS GAGE [15–18]

The transverse sensitivity which was shown in the previous section to result in errors in strain measurements can be employed to produce a special-purpose transducer known as a *stress gage*. The stress gage looks very much like a strain gage (see Fig. 10.3*c*) except that its grid is designed to give a select value of K_t so that the output $\Delta R/R$ is proportional to the stress along the axis of the gage. The stress gage serves a very useful purpose when a stress determination in a particular direction is the ultimate objective of the analysis, for it can be obtained with a single gage rather than a three-element rosette.

The principle upon which a stress gage is based is exhibited in the following derivation. The output of a gage $\Delta R/R$ as expressed by Eq. (6.4) is

$$\frac{\Delta R}{R} = S_a(\epsilon_a + K_t \epsilon_t)$$

The relationship between stress and strain for a plane state of stress is given by Eqs. (2.19) as

$$\epsilon_a = \frac{1}{E}(\sigma_a - v\sigma_t) \qquad \epsilon_t = \frac{1}{E}(\sigma_t - v\sigma_a)$$

Substituting Eqs. (2.19) into Eq. (6.4) yields

$$\begin{aligned}\frac{\Delta R}{R} &= \frac{S_a}{E}(\sigma_a - v\sigma_t) + \frac{K_t S_a}{E}(\sigma_t - v\sigma_a) \\ &= \frac{\sigma_a S_a}{E}(1 - vK_t) + \frac{\sigma_t S_a}{E}(K_t - v)\end{aligned} \tag{a}$$

Examination of Eq. (*a*) indicates that the output of the gage $\Delta R/R$ will be independent of σ_t if $K_t = v$. It can also be shown that the axial sensitivity S_a of a gage is related to the alloy sensitivity S_A by the expression

$$S_a = \frac{S_A}{1 + K_t} \tag{b}$$

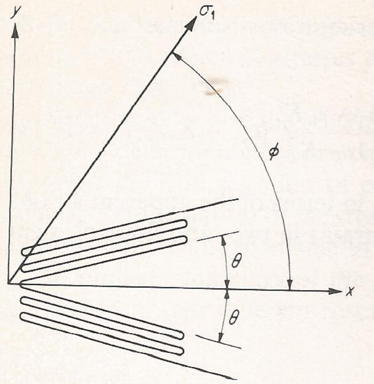


Figure 10.8 The stress gage relative to the x axis and the principal axis corresponding to σ_1.

Substituting Eq. (b) into Eq. (a) and letting $K_t = \nu$ leads to

$$\sigma_a = \frac{E}{S_A(1-\nu)}\frac{\Delta R}{R} \tag{10.20}$$

Since the factor $E/S_A(1-\nu)$ is a constant for a given gage alloy and specimen material, the gage output in terms of $\Delta R/R$ is linearly proportional to stress.

In practice the stress gage is made with a V-type grid configuration, as shown in Fig. 6.5m. Further analysis of the stress gage is necessary to understand its operation in a strain field which is unknown and in which the strain gage is placed in an arbitrary direction. Consider the placement of the gage, as shown in Fig. 10.8, along an arbitrary x axis which is at some unknown angle ϕ with the principal axis corresponding to σ_1. The grid elements are at a known angle θ relative to the x axis.

The strain along the top grid element is given by a modified form of Eq. (10.4) as

$$\epsilon_{\phi-\theta} = \tfrac{1}{2}(\epsilon_1 + \epsilon_2) + \tfrac{1}{2}(\epsilon_1 - \epsilon_2)\cos 2(\phi - \theta) \tag{c}$$

The strain along the lower grid element is

$$\epsilon_{\phi+\theta} = \tfrac{1}{2}(\epsilon_1 + \epsilon_2) + \tfrac{1}{2}(\epsilon_1 - \epsilon_2)\cos 2(\phi + \theta) \tag{d}$$

Summing Eqs. (c) and (d) and expanding the cosine terms yield

$$\epsilon_{\phi-\theta} + \epsilon_{\phi+\theta} = (\epsilon_1 + \epsilon_2) + (\epsilon_1 - \epsilon_2)\cos 2\phi \cos 2\theta \tag{e}$$

Note from the Mohr's strain circles presented earlier in this chapter that

$$\epsilon_{xx} + \epsilon_{yy} = \epsilon_1 + \epsilon_2 \tag{f}$$

$$\epsilon_{xx} - \epsilon_{yy} = (\epsilon_1 - \epsilon_2)\cos 2\phi \tag{g}$$

Substituting Eqs. (f) and (g) into Eq. (e) gives

$$\begin{aligned}\epsilon_{\phi-\theta} + \epsilon_{\phi+\theta} &= (\epsilon_{xx} + \epsilon_{yy}) + (\epsilon_{xx} - \epsilon_{yy}) \cos 2\theta \\ &= 2(\epsilon_{xx} \cos^2 \theta + \epsilon_{yy} \sin^2 \theta) \\ &= 2 \cos^2 \theta(\epsilon_{xx} + \epsilon_{yy} \tan^2 \theta) \qquad (h)\end{aligned}$$

If the gage is manufactured so that θ is equal to arctan $\sqrt{\nu}$, then

$$\tan^2 \theta = \nu \qquad \cos^2 \theta = \frac{1}{1+\nu}$$

and Eq. (h) becomes

$$\epsilon_{\phi-\theta} + \epsilon_{\phi+\theta} = \frac{2}{1+\nu}(\epsilon_{xx} + \nu\epsilon_{yy}) \qquad (i)$$

Substituting Eq. (i) into Eqs. (2.20) results in

$$\sigma_{xx} = \frac{E}{2(1-\nu)}(\epsilon_{\phi+\theta} + \epsilon_{\phi-\theta}) \qquad (10.21)$$

where $\frac{1}{2}(\epsilon_{\phi+\theta} + \epsilon_{\phi-\theta})$ is the average strain indicated by the two elements of the gage and is equal to $(\Delta R/R)/S_g$.

The gage reading will give $\frac{1}{2}(\epsilon_{\phi+\theta} + \epsilon_{\phi-\theta})$, and it is only necessary to multiply this reading by $E/(1-\nu)$ to obtain σ_{xx}. The stress gage will thus give σ_{xx} directly with a single gage. However, it does not give any data regarding σ_{yy} or the principal angle ϕ. Moreover, σ_{xx} may not be the most important stress since it may differ appreciably from σ_1. If the directions of the principal stresses are known, the stress gage may be used more effectively by choosing the x axis to coincide with the principal axis corresponding to σ_1 so that $\sigma_{xx} = \sigma_1$. In fact, when the principal directions are known, a conventional single-element strain gage can be employed as a stress gage.

This adaption is possible if the gage is located along a line which makes an angle θ with respect to the principal axis, as shown in Fig. 10.9. In this case the

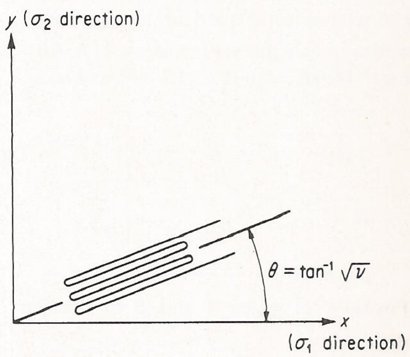


Figure 10.9 A single-element strain gage employed as a stress gage when the principal directions are known.

strains will be symmetrical about the principal axis; hence it is clear that

$$\epsilon_{\phi-\theta} = \epsilon_{\phi+\theta} = \epsilon_\theta$$

and Eq. (10.21) reduces to

$$\sigma_1 = \sigma_{xx} = \frac{E}{1-\nu}\epsilon_\theta \qquad (10.22)$$

The value ϵ_θ is recorded on the strain gage and converted to σ_{xx} or σ_1 directly by multiplying by $E/(1-\nu)$. This procedure reduces the number of gages necessary if only the value of σ_1 is to be determined. The saving of a gage is of particular importance in dynamic work when the instrumentation required becomes complex and the number of available channels of recording equipment is limited.

10.6 PLANE-SHEAR GAGES [19]

Consider two strain gages A and B oriented at angles θ_A and θ_B with respect to the x axis, as shown in Fig. 10.10. The strains along the gage axes are given by a modified form of Eqs. (2.18) as

$$\epsilon_A = \frac{\epsilon_{xx}+\epsilon_{yy}}{2} + \frac{\epsilon_{xx}-\epsilon_{yy}}{2}\cos 2\theta_A + \frac{\gamma_{xy}}{2}\sin 2\theta_A$$

$$\epsilon_B = \frac{\epsilon_{xx}+\epsilon_{yy}}{2} + \frac{\epsilon_{xx}-\epsilon_{yy}}{2}\cos 2\theta_B + \frac{\gamma_{xy}}{2}\sin 2\theta_B \qquad (10.23)$$

From Eqs. (10.23), the shear strain γ_{xy} is

$$\gamma_{xy} = \frac{2(\epsilon_A - \epsilon_B) - (\epsilon_{xx}-\epsilon_{yy})(\cos 2\theta_A - \cos 2\theta_B)}{\sin 2\theta_A - \sin 2\theta_B} \qquad (a)$$

If gages A and B are oriented such that

$$\cos 2\theta_A = \cos 2\theta_B \qquad (b)$$

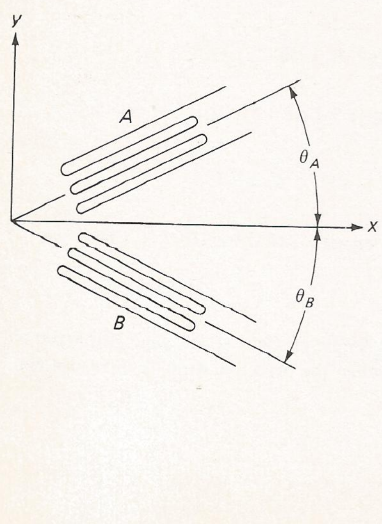


Figure 10.10 Positions of gages A and B for measuring γ_{xy}.

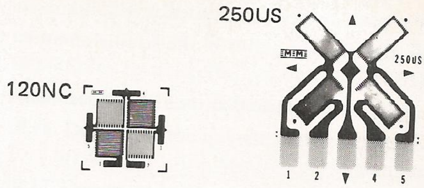


Figure 10.11 Four-element shear-strain gages.

then Eq. (a) reduces to

$$\gamma_{xy} = \frac{(\epsilon_A - \epsilon_B)}{\sin 2\theta_A - \sin 2\theta_B} \tag{10.24}$$

Since the cosine is an even function, $\theta_A = -\theta_B$ satisfies Eq. (b). Thus, the shearing strain γ_{xy} is proportional to the difference between normal strains experienced by gages A and B when they are oriented with respect to the x axis as shown in Fig. 10.10. The angle $\theta_A = -\theta_B$ can be arbitrary; however, for the angle $\theta_A = \pi/4$, Eq. (10.24) reduces simply to

$$\gamma_{xy} = \epsilon_A - \epsilon_B \tag{10.25}$$

Equation (10.25) indicates that the shearing strain γ_{xy} can be measured with a two-element rectangular rosette by orienting the gages at 45 and $-45°$ with respect to the x axis and connecting one gage into arm R_1 and the other into arm R_4 of a Wheatstone bridge. The subtraction $\epsilon_A - \epsilon_B$ will be performed automatically in the bridge, and the output will give γ_{xy} directly.

Four-element shear gages are marketed commercially for this measurement. The four elements provide a complete four-arm bridge with twice the output of the two-element rectangular rosette. Typical shear gages are illustrated in Fig. 10.11.

EXERCISES

10.1 Suppose a state of pure shear stress occurs (say on a circular shaft under pure torsion). Show how a single-element strain gage can be employed to determine the principal stresses σ_1 and σ_2.

10.2 In a thin-walled cylinder loaded with internal pressure ($\sigma_h = 2\sigma_a$), show how a single-element strain gage in the hoop direction can be used to establish the stresses σ_h and σ_a.

10.3 A two-element rectangular rosette was used to determine the two principal stresses at the point shown in Fig. E10.3. If $\sigma_1 = 200$ MPa and $\sigma_2 = 75$ MPa, find σ_{xx}, σ_{yy}, and τ_{xy} when $\phi = 30°$.

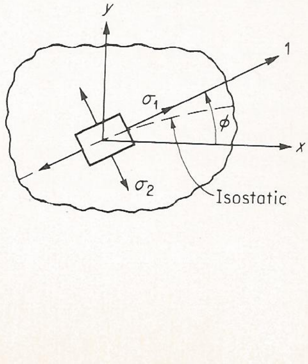


Figure E10.3