CHAPTER

TWO

STRAIN AND THE STRESS-STRAIN RELATIONS

2.1 INTRODUCTION

In the preceding chapter the state of stress which develops at an arbitrary point within a body as a result of surface- or body-force loadings was discussed. The relationships obtained were based on the conditions of equilibrium, and since no assumptions were made regarding body deformations or physical properties of the material of which the body was composed, the results are valid for any material and for any amount of body deformation. In this chapter the subject of body deformation and associated strain will be discussed. Since strain is a pure geometric quantity, no restrictions on body material will be required. However, in order to obtain linear equations relating displacement to strain, restrictions must be placed on the allowable deformations. In a later section, when the stress-strain relations are developed, the elastic constants of the body material must be considered.

2.2 DEFINITIONS OF DISPLACEMENT AND STRAIN

If a given body is subjected to a system of forces, individual points of the body will, in general, move. This movement of an arbitrary point is a vector quantity known as a *displacement*. If the various points in the body undergo different movements, each can be represented by its own unique displacement vector. Each vector can be resolved into components parallel to a set of cartesian coordinate axes such that u, v, and w are the displacement components in the x, y, and z directions, respectively.

Motion of the body may be considered as the sum of two parts:

1. A translation and/or rotation of the body as a whole
2. The movement of the points of the body relative to each other

The translation or rotation of the body as a whole is known as *rigid-body motion.* This type of motion is applicable to either the idealized rigid body or the real deformable body. The movement of the points of the body relative to each other is known as a *deformation* and is obviously a property of real bodies only. Rigid-body motions can be large or small. Deformations, in general, are small except when rubberlike materials or specialized structures such as long, slender beams are involved.

Strain is a geometric quantity which depends on the relative movements of two or three points in the body and therefore is related only to the deformation displacements. Since rigid-body displacements do not produce strains, they will be neglected in all further developments in this chapter. In the preceding chapter two types of stress were discussed: normal stress and shear stress. This same classification will be used for strains. A normal strain is defined as the change in length of a line segment between two points divided by the original length of the line segment. A shearing strain is defined as the angular change between two line segments which were originally perpendicular. The relationships between strains and displacements can be determined by considering the deformation of an arbitrary cube in a body as a system of loads is applied. This deformation is illustrated in Fig. 2.1, in which a general point P is moved through a distance u in the x direction, v in the y direction, and w in the z direction. The other corners of the cube are also displaced and, in general, they will be displaced by amounts which differ from those at point P. For example the displacements u^*, v^*, and w^* associated with point Q can be expressed in terms of the displacements u, v, and w at point P by means of a Taylor-series expansion. Thus

$$
\begin{aligned}
u^* &= u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \cdots \\
v^* &= v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z + \cdots \\
w^* &= w + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \cdots
\end{aligned}
\tag{2.1}
$$

The terms shown in the above expressions are the only significant terms if it is assumed that the cube is sufficiently small for higher-order terms such as $(\Delta x)^2$, $(\Delta y)^2$, $(\Delta z)^2$, ... to be neglected. Under these conditions, planes will remain plane and straight lines will remain straight lines in the deformed cube, as shown in Fig. 2.1.

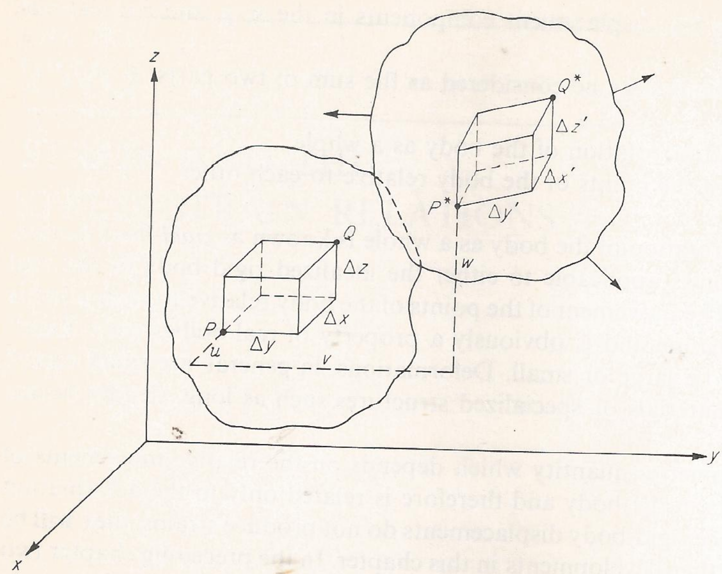

Figure 2.1 The distortion of an arbitrary cube in a body due to the application of a system of forces.

The average normal strain along an arbitrary line segment was previously defined as the change in length of the line segment divided by its original length. This normal strain can be expressed in terms of the displacements experienced by points at the ends of the segment. For example, consider the line PQ originally oriented parallel to the x axis, as shown in Fig. 2.2. Since y and z are constant along PQ, Eqs. (2.1) yield the following displacements for point Q if the displacements for point P are u, v, and w:

$$u^* = u + \frac{\partial u}{\partial x}\Delta x \qquad v^* = v + \frac{\partial v}{\partial x}\Delta x \qquad w^* = w + \frac{\partial w}{\partial x}\Delta x$$

From the definition of normal strain,

$$\epsilon_{xx} = \frac{\Delta x' - \Delta x}{\Delta x} \tag{a}$$

which is equivalent to

$$\Delta x' = (1 + \epsilon_{xx})\,\Delta x \tag{b}$$

As shown in Fig. 2.2, the deformed length $\Delta x'$ can be expressed in terms of the displacement gradients as

$$(\Delta x')^2 = \left[\left(1 + \frac{\partial u}{\partial x}\right)\Delta x\right]^2 + \left(\frac{\partial v}{\partial x}\Delta x\right)^2 + \left(\frac{\partial w}{\partial x}\Delta x\right)^2 \tag{c}$$

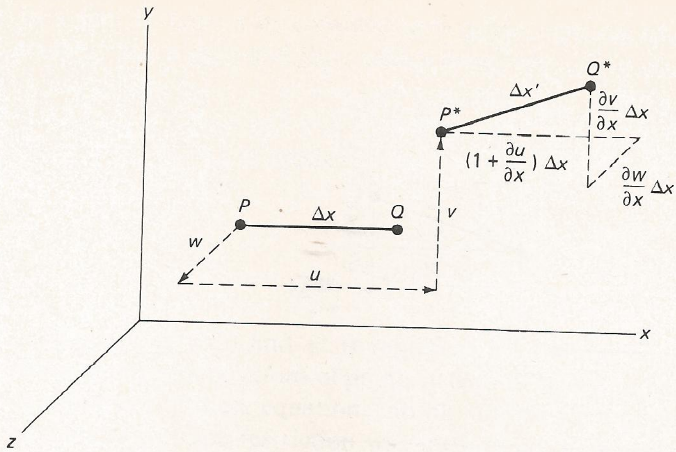

Figure 2.2 Displacement gradients associated with the normal strain ϵ_{xx}.

Squaring Eq. (*b*) and substituting Eq. (*c*) yields

$$(1 + \epsilon_{xx})^2(\Delta x)^2 = \left[1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right](\Delta x)^2$$

or

$$\epsilon_{xx} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \tag{2.2a}$$

In a similar manner considering line segments originally oriented parallel to the y and z axes leads to

$$\epsilon_{yy} = \sqrt{1 + 2\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} - 1 \tag{2.2b}$$

$$\epsilon_{zz} = \sqrt{1 + 2\frac{\partial w}{\partial z} + \left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2} - 1 \tag{2.2c}$$

The shear-strain components can also be related to the displacements by considering the changes in right angle experienced by the edges of the cube during deformation. For example, consider lines PQ and PR, as shown in Fig. 2.3. The angle θ^* between P^*Q^* and P^*R^* in the deformed state can be expressed in terms of the displacement gradients since the cosine of the angle between any two intersecting lines in space is the sum of the pairwise products of the direction cosines of the lines with respect to the same set of reference axes. Thus

$$\cos\theta^* = \left[\left(1 + \frac{\partial u}{\partial x}\right)\frac{\Delta x}{\Delta x'}\right]\left(\frac{\partial u}{\partial y}\frac{\Delta y}{\Delta y'}\right) + \left(\frac{\partial v}{\partial x}\frac{\Delta x}{\Delta x'}\right)\left[\left(1 + \frac{\partial v}{\partial y}\right)\frac{\Delta y}{\Delta y'}\right] + \left(\frac{\partial w}{\partial x}\frac{\Delta x}{\Delta x'}\right)\left(\frac{\partial w}{\partial y}\frac{\Delta y}{\Delta y'}\right) \tag{d}$$

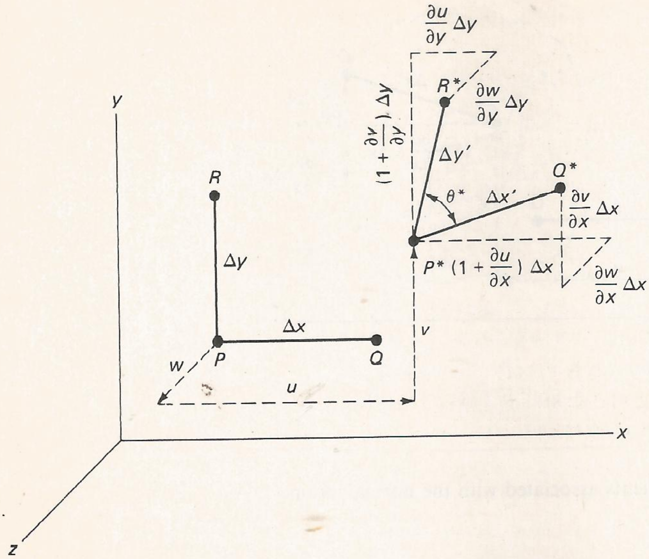

Figure 2.3 Displacement gradients associated with the shear strain γ_{xy}.

From the definition of shear strain

$$\gamma_{xy} = \left(\frac{\pi}{2} - \theta^*\right) \tag{e}$$

therefore

$$\sin \gamma_{xy} = \sin\left(\frac{\pi}{2} - \theta^*\right) = \cos \theta^* \tag{f}$$

Substituting Eq. (d) into Eq. (f) and simplifying yields

$$\sin \gamma_{xy} = \left[\left(1 + \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + \left(1 + \frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\left(\frac{\Delta x}{\Delta x'}\frac{\Delta y}{\Delta y'}\right)$$

From Eq. (b)

$$\Delta x' = (1 + \epsilon_{xx})\,\Delta x \qquad \text{and} \qquad \Delta y' = (1 + \epsilon_{yy})\,\Delta y$$

therefore

$$\gamma_{xy} = \arcsin \frac{\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}}{(1 + \epsilon_{xx})(1 + \epsilon_{yy})} \tag{2.3a}$$

In a similar manner by considering two line segments originally oriented parallel to the y and z axes and the z and x axes

$$\gamma_{yz} = \arcsin \frac{\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial y}\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\dfrac{\partial w}{\partial z} + \dfrac{\partial u}{\partial y}\dfrac{\partial u}{\partial z}}{(1 + \epsilon_{yy})(1 + \epsilon_{zz})} \tag{2.3b}$$

$$\gamma_{zx} = \arcsin \frac{\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial z}\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial z}\dfrac{\partial v}{\partial x}}{(1 + \epsilon_{zz})(1 + \epsilon_{xx})} \tag{2.3c}$$

Equations (2.2) and (2.3) represent a common engineering description of strain in terms of positions of points in a body before and after deformation. In the development of these equations, no limitations were imposed on the magnitudes of the strains. One restriction was introduced, however, when the higher-order terms in the Taylor-series expansion for displacement were neglected. This restriction has the effect of limiting the length of the line segment (gage length) used for strain determinations unless displacement gradients ($\partial u/\partial x$, $\partial u/\partial y$, ...) in the region of interest are essentially constant. If displacement gradients change rapidly with position in the region of interest, very short gage lengths will be required for accurate strain measurements.

In a wide variety of engineering problems, the displacements and strains produced by the applied loads are very small. Under these conditions, it can be assumed that products and squares of displacement gradients will be small with respect to the displacement gradients and therefore can be neglected. With this assumption Eqs. (2.2) and (2.3) reduce to the strain-displacement equations frequently encountered in the theory of elasticity. The reduced form of the equations is

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \tag{2.4a}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \tag{2.4b}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} \tag{2.4c}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2.4d}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{2.4e}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2.4f}$$

Equations (2.4) indicate that it is a simple matter to convert a displacement field into a strain field. However, as will be emphasized later, an entire displacement field is rarely determined experimentally. Usually, strains are determined at a number of small areas on the surface of the body through the use of strain gages. In certain problems, however, the displacement field can be computed analytically, and in these instances Eqs. (2.4) become very important.

2.3 STRAIN EQUATIONS OF TRANSFORMATION

Now that the normal and shearing strains in the x, y, z directions have been determined, consider the normal strain in an arbitrary direction. Refer to Fig. 2.4 and consider the elongation of the diagonal PQ. By definition the strain along PQ is

$$\epsilon_{PQ} = \frac{P^*Q^* - PQ}{PQ} \tag{a}$$

From geometric considerations, as illustrated in Fig. 2.4,

$$(PQ)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{b}$$

$$(P^*Q^*)^2 = (\Delta x^*)^2 + (\Delta y^*)^2 + (\Delta z^*)^2 \tag{c}$$

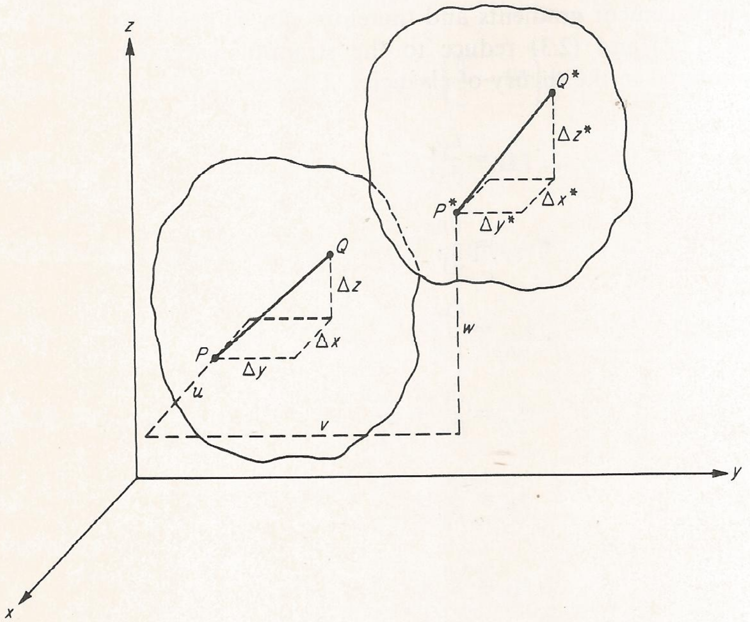

Figure 2.4 Displacements of points P and Q in a body which result from the application of a system of loads.

In general, the component Δx^* will have a different length than the component Δx because of the deformation of the body in the x direction. From Fig. 2.4 it can easily be seen that

$$\begin{aligned}
\Delta x^* &= \left(1 + \frac{\partial u}{\partial x}\right) \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z \\
\Delta y^* &= \frac{\partial v}{\partial x} \Delta x + \left(1 + \frac{\partial v}{\partial y}\right) \Delta y + \frac{\partial v}{\partial z} \Delta z \qquad (d) \\
\Delta z^* &= \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \left(1 + \frac{\partial w}{\partial z}\right) \Delta z
\end{aligned}$$

If Eqs. (d) are substituted into Eq. (c), the length of the deformed line segment P^*Q^* can be computed. In the substitution, since the deformations are extremely small, the products and squares of derivatives can be neglected. Thus

$$\begin{aligned}
(P^*Q^*)^2 = {} & \left(1 + 2\frac{\partial u}{\partial x}\right)(\Delta x)^2 + \left(1 + 2\frac{\partial v}{\partial y}\right)(\Delta y)^2 \\
& + \left(1 + 2\frac{\partial w}{\partial z}\right)(\Delta z)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \Delta x \, \Delta y \\
& + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \Delta y \, \Delta z + 2\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \Delta z \, \Delta x \qquad (e)
\end{aligned}$$

Equation (a) can be rearranged in the following form:

$$\epsilon_{PQ} = \frac{P^*Q^*}{PQ} - 1$$

or

$$(\epsilon_{PQ} + 1)^2 = \left(\frac{P^*Q^*}{PQ}\right)^2$$

If Eqs. (b) and (e) are substituted into this rearranged form of Eq. (a), the following equation can be obtained after some rearrangement of terms:

$$\begin{aligned}
(\epsilon_{PQ} + 1)^2 = {} & \cos^2(x, PQ) + \cos^2(y, PQ) + \cos^2(z, PQ) \\
& + 2\frac{\partial u}{\partial x}\cos^2(x, PQ) + 2\frac{\partial v}{\partial y}\cos^2(y, PQ) + 2\frac{\partial w}{\partial z}\cos^2(z, PQ) \\
& + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(x, PQ)\cos(y, PQ) \\
& + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\cos(y, PQ)\cos(z, PQ) \\
& + 2\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\cos(z, PQ)\cos(x, PQ) \qquad (f)
\end{aligned}$$

If the left side of Eq. (f) is expanded, the ϵ_{PQ}^2 term can be neglected since it is of the same order of magnitude as the products and squares of displacement derivatives which were neglected in a previous step of this development. Recall also that

$$\cos^2(x, PQ) + \cos^2(y, PQ) + \cos^2(z, PQ) = 1$$

Thus the basic equation for the strain along an arbitrary line segment is

$$\begin{aligned}\epsilon_{PQ} = {} & \frac{\partial u}{\partial x}\cos^2(x, PQ) + \frac{\partial v}{\partial y}\cos^2(y, PQ) + \frac{\partial w}{\partial z}\cos^2(z, PQ) \\ & + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(x, PQ)\cos(y, PQ) \\ & + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\cos(y, PQ)\cos(z, PQ) \\ & + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\cos(z, PQ)\cos(x, PQ) \qquad (2.5a)\end{aligned}$$

Equations (2.4) and (2.5a) can be used to determine $\epsilon_{x'x'}$ by choosing the direction of PQ parallel to the x' axis. Then

$$\begin{aligned}\epsilon_{x'x'} = {} & \epsilon_{xx}\cos^2(x, x') + \epsilon_{yy}\cos^2(y, x') \\ & + \epsilon_{zz}\cos^2(z, x') + \gamma_{xy}\cos(x, x')\cos(y, x') \\ & + \gamma_{yz}\cos(y, x')\cos(z, x') + \gamma_{zx}\cos(z, x')\cos(x, x') \qquad (2.6a)\end{aligned}$$

In a similar manner $\epsilon_{y'y'}$ and $\epsilon_{z'z'}$ can be determined by choosing the direction of PQ parallel to the y' and z' axes, respectively:

$$\begin{aligned}\epsilon_{y'y'} = {} & \epsilon_{yy}\cos^2(y, y') + \epsilon_{zz}\cos^2(z, y') \\ & + \epsilon_{xx}\cos^2(x, y') + \gamma_{yz}\cos(y, y')\cos(z, y') \\ & + \gamma_{zx}\cos(z, y')\cos(x, y') + \gamma_{xy}\cos(x, y')\cos(y, y') \qquad (2.6b)\\ \epsilon_{z'z'} = {} & \epsilon_{zz}\cos^2(z, z') + \epsilon_{xx}\cos^2(x, z') \\ & + \epsilon_{yy}\cos^2(y, z') + \gamma_{zx}\cos(z, z')\cos(x, z') \\ & + \gamma_{xy}\cos(x, z')\cos(y, z') + \gamma_{yz}\cos(y, z')\cos(z, z') \qquad (2.6c)\end{aligned}$$

A similar but somewhat more involved derivation can be used to establish the shearing strains. Consider the angular change in an arbitrary right angle formed by two line segments PQ_1 and PQ_2. The shearing strain γ_{PQ_1, PQ_2} can be shown

[3, p. 44]† to be given by

$$\begin{aligned}\gamma_{PQ_1,PQ_2} = 2\epsilon_{xx}\cos(x, PQ_1)\cos(x, PQ_2)\\ + 2\epsilon_{yy}\cos(y, PQ_1)\cos(y, PQ_2)\\ + 2\epsilon_{zz}\cos(z, PQ_1)\cos(z, PQ_2)\\ + \gamma_{xy}[\cos(x, PQ_1)\cos(y, PQ_2) + \cos(x, PQ_2)\cos(y, PQ_1)]\\ + \gamma_{yz}[\cos(y, PQ_1)\cos(z, PQ_2) + \cos(y, PQ_2)\cos(z, PQ_1)]\\ + \gamma_{zx}[\cos(z, PQ_1)\cos(x, PQ_2) + \cos(z, PQ_2)\cos(x, PQ_1)]\end{aligned} \tag{2.5b}$$

By choosing PQ_1 parallel to x' and PQ_2 parallel to y', an expression for $\gamma_{x'y'}$ is obtained as follows:

$$\begin{aligned}\gamma_{x'y'} = 2\epsilon_{xx}\cos(x, x')\cos(x, y')\\ + 2\epsilon_{yy}\cos(y, x')\cos(y, y')\\ + 2\epsilon_{zz}\cos(z, x')\cos(z, y')\\ + \gamma_{xy}[\cos(x, x')\cos(y, y') + \cos(x, y')\cos(y, x')]\\ + \gamma_{yz}[\cos(y, x')\cos(z, y') + \cos(y, y')\cos(z, x')]\\ + \gamma_{zx}[\cos(z, x')\cos(x, y') + \cos(z, y')\cos(x, x')]\end{aligned} \tag{2.6d}$$

Similarly

$$\begin{aligned}\gamma_{y'z'} = 2\epsilon_{yy}\cos(y, y')\cos(y, z')\\ + 2\epsilon_{zz}\cos(z, y')\cos(z, z')\\ + 2\epsilon_{xx}\cos(x, y')\cos(x, z')\\ + \gamma_{yz}[\cos(y, y')\cos(z, z') + \cos(y, z')\cos(z, y')]\\ + \gamma_{zx}[\cos(z, y')\cos(x, z') + \cos(z, z')\cos(x, y')]\\ + \gamma_{xy}[\cos(x, y')\cos(y, z') + \cos(x, z')\cos(y, y')]\end{aligned} \tag{2.6e}$$

$$\begin{aligned}\gamma_{z'x'} = 2\epsilon_{zz}\cos(z, z')\cos(z, x')\\ + 2\epsilon_{xx}\cos(x, z')\cos(x, x')\\ + 2\epsilon_{yy}\cos(y, z')\cos(y, x')\\ + \gamma_{zx}[\cos(z, z')\cos(x, x') + \cos(z, x')\cos(x, z')]\\ + \gamma_{xy}[\cos(x, z')\cos(y, x') + \cos(x, x')\cos(y, z')]\\ + \gamma_{yz}[\cos(y, z')\cos(z, x') + \cos(y, x')\cos(z, z')]\end{aligned} \tag{2.6f}$$

† Numbers in brackets refer to numbered references at the end of the chapter.

Equations (2.6*a*) to (2.6*f*) are the strain equations of transformation and can be used to transform the six cartesian components of strain $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ relative to the *Oxyz* reference system to six other cartesian components of strain relative to the *Ox'y'z'* reference system.

A comparison of Eqs. (2.6) with the stress equations of transformation [Eq. (1.6)] shows remarkable similarities:

$$\begin{array}{ll} \sigma_{xx} \leftrightarrow \epsilon_{xx} & 2\tau_{xy} \leftrightarrow \gamma_{xy} \\ \sigma_{yy} \leftrightarrow \epsilon_{yy} & 2\tau_{yz} \leftrightarrow \gamma_{yz} \\ \sigma_{zz} \leftrightarrow \epsilon_{zz} & 2\tau_{zx} \leftrightarrow \gamma_{zx} \end{array} \tag{2.7}$$

Here the symbol $\leftrightarrow$ indicates an interchange. This interchange is important since many of the derivations given in the preceding chapter for stresses can be converted directly into strains. Some of these conversions are indicated in the next section.

2.4 PRINCIPAL STRAINS

From the similarity between the laws of stress and strain transformation it can be concluded that there exist at most three distinct principal strains with their three associated principal directions. By substituting the conversions indicated by Eqs. (2.7) into Eq. (1.7), the cubic equation whose roots give the principal strains is obtained:

$$\begin{aligned} &\epsilon_n^3 - (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})\epsilon_n^2 \\ &+ \left(\epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} - \frac{\gamma_{xy}^2}{4} - \frac{\gamma_{yz}^2}{4} - \frac{\gamma_{zx}^2}{4}\right)\epsilon_n \\ &- \left(\epsilon_{xx}\epsilon_{yy}\epsilon_{zz} - \epsilon_{xx}\frac{\gamma_{yz}^2}{4} - \epsilon_{yy}\frac{\gamma_{zx}^2}{4} - \epsilon_{zz}\frac{\gamma_{xy}^2}{4} + \frac{\gamma_{xy}\gamma_{yz}\gamma_{zx}}{4}\right) = 0 \end{aligned} \tag{2.8}$$

As with principal stresses, three situations exist:

$$\epsilon_1 \neq \epsilon_2 \neq \epsilon_3 \qquad \epsilon_1 = \epsilon_2 \neq \epsilon_3 \qquad \epsilon_1 = \epsilon_2 = \epsilon_3$$

The significance of these three cases is determined from the discussion in Sec. 1.6, page 14.

Similarly, there are three strain invariants which are analogous to the three stress invariants. By substituting Eqs. (2.7) into Eqs. (1.8), the following expressions are obtained for the strain invariants:

$$\begin{aligned} J_1 &= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\ J_2 &= \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} - \frac{\gamma_{xy}^2}{4} - \frac{\gamma_{yz}^2}{4} - \frac{\gamma_{zx}^2}{4} \\ J_3 &= \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} - \frac{\epsilon_{xx}\gamma_{yz}^2}{4} - \frac{\epsilon_{yy}\gamma_{zx}^2}{4} - \frac{\epsilon_{zz}\gamma_{xy}^2}{4} + \frac{\gamma_{xy}\gamma_{yz}\gamma_{zx}}{4} \end{aligned} \tag{2.9}$$

It is clear that other equations derived in Chap. 1 for stresses could easily be converted into equations in terms of strains. A few more will be covered in the exercises at the end of the chapter, and others will be converted as the need arises.

2.5 COMPATIBILITY

From a given displacement field, i.e., three equations expressing u, v, and w as functions of x, y, and z, a unique strain field can be determined by using Eqs. (2.4). However, an arbitrary strain field may yield an impossible displacement field, i.e., one in which the body might contain voids after deformation. A valid displacement field can be ensured only if the body under consideration is simply connected and if the strain field satisfies a set of equations known as the *compatibility relations*. The six equations of compatibility which must be satisfied are

$$\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} \tag{2.10a}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y\,\partial z} = \frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} \tag{2.10b}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial z\,\partial x} = \frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} \tag{2.10c}$$

$$2\frac{\partial^2 \epsilon_{xx}}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{2.10d}$$

$$2\frac{\partial^2 \epsilon_{yy}}{\partial z\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{2.10e}$$

$$2\frac{\partial^2 \epsilon_{zz}}{\partial x\,\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{2.10f}$$

In order to derive Eq. (2.10*a*), begin by recalling

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{a}$$

Differentiating γ_{xy} once with respect to x and then again with respect to y gives

$$\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \frac{\partial^3 u}{\partial x\,\partial y^2} + \frac{\partial^3 v}{\partial x^2\,\partial y} \tag{b}$$

Note that

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x\,\partial y^2} \quad \text{and} \quad \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2\,\partial y} \tag{c}$$

Substituting Eqs. (*c*) into Eq. (*b*) gives

$$\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} \tag{d}$$

which establishes Eq. (2.10*a*), and by the same methods Eqs. (2.10*b*) and (2.10*c*) could be verified. The proof of Eq. (2.10*d*) is obtained by considering four identities:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y\, \partial z} = \frac{\partial^3 u}{\partial x\, \partial y\, \partial z} \tag{e}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial z} = \frac{\partial^3 u}{\partial x\, \partial y\, \partial z} + \frac{\partial^3 v}{\partial x^2\, \partial z} \tag{f}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial x\, \partial y} = \frac{\partial^3 w}{\partial x^2\, \partial y} + \frac{\partial^3 u}{\partial x\, \partial y\, \partial z} \tag{g}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial x^2} = \frac{\partial^3 w}{\partial x^2\, \partial y} + \frac{\partial^3 v}{\partial x^2\, \partial z} \tag{h}$$

Now by forming

$$2(e) = (f) + (g) - (h) \tag{i}$$

thus obtaining

$$2\frac{\partial^2 \epsilon_{xx}}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x}\right) \tag{j}$$

Eq. (2.10*d*) is verified. The remaining two compatibility relations can be established in an identical manner.

In order to gain a better physical understanding of the compatibility relations, consider a two-dimensional body made up of a large number of small, square elements. When the body is loaded, the elements deform. By measuring angle changes and length changes, the strains which develop in each element can be determined. This procedure is accomplished theoretically by differentiating the displacement field. Consider now the inverse problem. Suppose a large number of small, deformed elements are given which must be fitted together to form a body free of voids and discontinuities. If and only if each element is properly strained can the body be reassembled without voids. The deformed elements correspond to the case of the prescribed strain field. The check to determine whether the elements are all properly strained and hence compatible with each other represents the compatibility relations. If these relations are satisfied, the elements will fit together properly, thus guaranteeing a satisfactory displacement field.

2.6 EXAMPLE OF A DISPLACEMENT FIELD COMPUTED FROM A STRAIN FIELD

If a circular shaft of radius a is loaded in torsion, the following strain field is produced:

$$\gamma_{zx} = -ay \qquad \gamma_{yz} = ax \qquad \epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \gamma_{xy} = 0 \tag{a}$$

where the z axis of the reference system is coincident with the centerline of the shaft. The first step in solving for the displacement field is to check the compatibility conditions:

1. The body must be simply connected, a condition which is obviously satisfied in this case.
2. The strain relations given in Eqs. (a) must satisfy all the compatibility relations given in Eqs. (2.10). It is clear that this linear system of strains does satisfy this requirement.

Next substitute Eqs. (a) into Eqs. (2.4) and integrate:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 0 \qquad u = f(y, z)$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0 \qquad v = g(x, z)$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0 \qquad w = h(x, y) \tag{b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 = \frac{\partial f(y, z)}{\partial y} + \frac{\partial g(x, z)}{\partial x}$$

The last of Eqs. (b) can be satisfied only if both right-hand terms are functions of z alone; hence

$$\frac{\partial f(y, z)}{\partial y} = -\frac{\partial g(x, z)}{\partial x} = F(z) \tag{c}$$

Integrating Eq. (c) gives

$$f(z) = yF(z) + C_1 = u \qquad g(z) = -xF(z) + C_2 = v \tag{d}$$

Recall the value of γ_{yz} from Eqs. (a), the definition of γ_{yz} from Eq. (2.4e), and the functional relation for w from the third of Eqs. (b) and form

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = ax = \frac{\partial h(x, y)}{\partial y} - x\frac{dF(z)}{dz} \tag{e}$$

Equation (e) can be satisfied if and only if both the right-hand terms are functions of x alone; hence

$$\frac{\partial h(x)}{\partial y} = H(x) \qquad \frac{dF(z)}{dz} = C_3 \tag{f}$$

Substituting Eqs. (f) into Eq. (e) gives

$$H(x) - C_3 x = ax$$

$$H(x) = (a + C_3)x \tag{g}$$

Integrating the second of Eqs. (f) yields

$$F(z) = C_3 z + C_4 \tag{h}$$

Substituting Eq. (h) into Eqs. (d) gives

$$u = y(C_3 z + C_4) + C_1 \qquad v = -x(C_3 z + C_4) + C_2 \tag{i}$$

By Eqs. (b), (f), and (g),

$$w = h(x, y) = \int H(x)\, \partial y = \int (a + C_3)x\, \partial y = (a + C_3)xy + C_5 \tag{j}$$

Thus far five constants of integration have been introduced, and five of the six cartesian components of strain which were given have been used. By employing the last strain relation, the arbitrary constant C_3 can be evaluated. Recall from Eqs. (a):

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = (a + C_3)y + C_3 y = -ay \qquad C_3 = -a \tag{k}$$

Substituting Eqs. (k) into Eqs. (i) and (j) gives

$$u = -ayz + C_4 y + C_1 \qquad v = axz - C_4 x + C_2 \qquad w = C_5 \tag{l}$$

The constants C_1, C_2, and C_5 indicate rigid-body translation of the shaft. The constant C_4 indicates rigid-body rotation of the shaft. It is clear upon differentiation that C_1, C_2, C_3, and C_4 do not enter into the strains produced in the shaft and hence are not a part of the displacement due to deformation. The deformation displacements are given by

$$u = -ayz \qquad v = axz \qquad w = 0 \tag{m}$$

As indicated by this simple example, the process by which the displacement field is calculated from a given strain field is quite lengthy. On the other hand, it is a very simple matter to go from a complex displacement field to a strain field.

2.7 VOLUME DILATATION

Consider a small, rectangular element in a deformed body which has its edges oriented along the principal axes. The length of each side of the block may have changed; however, the element will not be distorted since there are no shearing strains acting on the faces. The change in volume of such an element divided by the initial volume is, by definition, the volume dilatation D, that is,

$$D = \frac{V^* - V}{V}$$

where V is the initial volume, equal to the product of the three sides of the element, a_1, a_2, a_3, before deformation and V^* is the final volume after straining, equal to the product of the three sides a_1^*, a_2^*, a_3^*, after deformation. Since

$$a_1^* = a_1(1 + \epsilon_1) \qquad a_2^* = a_2(1 + \epsilon_2) \qquad a_3^* = a_3(1 + \epsilon_3)$$

it follows that

$$D = \frac{a_1 a_2 a_3(1 + \epsilon_1)(1 + \epsilon_2)(1 + \epsilon_3) - a_1 a_2 a_3}{a_1 a_2 a_3}$$

If the higher-order strain terms are neglected,

$$D = \epsilon_1 + \epsilon_2 + \epsilon_3 = J_1 \tag{2.11}$$

Equation (2.11) indicates that the volume dilatation D is equal to the first invariant of strain. Since the first invariant of strain is independent of the coordinate system being used, the volume dilatation of an element is independent of the reference frame forming its sides. Volume dilatation is thus a coordinate-independent concept.

2.8 STRESS-STRAIN RELATIONS

Thus far stress and strain have been discussed individually, and no assumptions have been required regarding the behavior of the material except that it was a continuous medium.† In this section, stress will be related to strain; therefore, certain restrictive assumptions regarding the body material must be introduced. The first of these assumptions regards linearity of the stress versus strain in the body. With a linear stress-strain relationship it is possible to write the general stress-strain expressions as follows:

$$\begin{aligned}
\sigma_{xx} &= K_{11}\epsilon_{xx} + K_{12}\epsilon_{yy} + K_{13}\epsilon_{zz} + K_{14}\gamma_{xy} + K_{15}\gamma_{yz} + K_{16}\gamma_{zx} \\
\sigma_{yy} &= K_{21}\epsilon_{xx} + K_{22}\epsilon_{yy} + K_{23}\epsilon_{zz} + K_{24}\gamma_{xy} + K_{25}\gamma_{yz} + K_{26}\gamma_{zx} \\
\sigma_{zz} &= K_{31}\epsilon_{xx} + K_{32}\epsilon_{yy} + K_{33}\epsilon_{zz} + K_{34}\gamma_{xy} + K_{35}\gamma_{yz} + K_{36}\gamma_{zx} \\
\tau_{xy} &= K_{41}\epsilon_{xx} + K_{42}\epsilon_{yy} + K_{43}\epsilon_{zz} + K_{44}\gamma_{zy} + K_{45}\gamma_{yz} + K_{46}\gamma_{zx} \\
\tau_{yz} &= K_{51}\epsilon_{xx} + K_{52}\epsilon_{yy} + K_{53}\epsilon_{zz} + K_{54}\gamma_{xy} + K_{55}\gamma_{yz} + K_{56}\gamma_{zx} \\
\tau_{zx} &= K_{61}\epsilon_{xx} + K_{62}\epsilon_{yy} + K_{63}\epsilon_{zz} + K_{64}\gamma_{xy} + K_{65}\gamma_{yz} + K_{66}\gamma_{zx}
\end{aligned} \tag{2.12}$$

where K_{11} to K_{66} are the coefficients of elasticity of the material and are independent of the magnitudes of both the stress and the strain, provided the elastic limit of the material is not exceeded. If the elastic limit is exceeded, the linear relationship between stress and strain no longer holds, and Eqs. (2.12) are not valid.

There are 36 coefficients of elasticity in Eqs. (2.12); however, they are not all independent. By strain energy considerations, which are beyond the scope of this

† Actually most metals are not strictly continuous since they are composed of a large number of rather small grains. However, the grains are in almost all cases small enough in comparison with the size of the body for the body to behave as if it were a continuous medium.

book, the number of independent coefficients of elasticity can be reduced to 21. This reduction is quite significant; however, even with 21 constants, Eqs. (2.12) may be considered rather long and involved. By assuming that the material is isotropic, i.e., that the elastic constants are the same in all directions and hence independent of the choice of a coordinate system, the 21 coefficients of elasticity reduce to two constants. The stress-strain relationships then reduce to

$$\sigma_{xx} = \lambda J_1 + 2\mu\epsilon_{xx} \qquad \sigma_{yy} = \lambda J_1 + 2\mu\epsilon_{yy} \qquad \sigma_{zz} = \lambda J_1 + 2\mu\epsilon_{zz}$$

$$\tau_{xy} = \mu\gamma_{xy} \qquad \tau_{yz} = \mu\gamma_{yz} \qquad \tau_{zx} = \mu\gamma_{zx} \tag{2.13}$$

where J_1 = first invariant of strain $(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$
λ = Lamé's constant
μ = shear modulus

Equations (2.13) can be solved to give the strains as a function of stress:

$$\epsilon_{xx} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_{xx} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(\sigma_{yy} + \sigma_{zz})$$

$$\epsilon_{yy} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_{yy} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(\sigma_{xx} + \sigma_{zz})$$

$$\epsilon_{zz} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_{zz} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}(\sigma_{yy} + \sigma_{xx})$$

$$\gamma_{xy} = \frac{1}{\mu}\tau_{xy} \qquad \gamma_{yz} = \frac{1}{\mu}\tau_{yz} \qquad \gamma_{zx} = \frac{1}{\mu}\tau_{zx} \tag{2.14}$$

The elastic coefficients μ and λ shown in Eqs. (2.13) and (2.14) arise from a mathematical treatment of the general linear stress-strain relations. In experimental work, Lamé's constant λ is rarely used since it has no physical significance; however, as will be shown later, the shear modulus has physical significance and can easily be measured.

Consider a two-dimensional case of pure shear where

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{zx} = \tau_{yz} = 0 \qquad \tau_{xy} = \text{applied shearing stress}$$

From Eqs. (2.14),

$$\mu = \frac{\tau_{xy}}{\gamma_{xy}} \tag{2.15a}$$

Hence, the shear modulus μ is the ratio of the shearing stress to the shearing strain in a two-dimensional state of pure shear.

In a conventional tension test which is often used to determine the mechanical properties of materials, a long, slender bar is subjected to a state of uniaxial stress in, say, the x direction. In this instance

$$\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \qquad \sigma_{xx} = \text{applied normal stress}$$

From Eqs. (2.14),

$$\epsilon_{xx} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\sigma_{xx} \tag{a}$$

$$\epsilon_{yy} = \epsilon_{zz} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{xx} \tag{b}$$

In elementary strength-of-materials texts, the stress-strain relations for the case of uniaxial stress are often written

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} \tag{c}$$

$$\epsilon_{yy} = \epsilon_{zz} = -\frac{v}{E}\sigma_{xx} \tag{d}$$

By equating the coefficients in Eqs. (*a*) and (*b*) to those in Eqs. (*c*) and (*d*),

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{2.15b}$$

$$v = \frac{\lambda}{2(\lambda + \mu)} \tag{2.15c}$$

where E is the modulus of elasticity and v is Poisson's ratio, defined as

$$v = -\frac{\epsilon_{yy}}{\epsilon_{xx}} \tag{2.15d}$$

Equations (2.15*b*) and (2.15*c*) indicate the conversion from Lamé's constant λ and the shear modulus μ to the more commonly used modulus of elasticity E and Poisson's ratio v.

To establish the definition and physical significance of a fifth elastic constant, consider a state of hydrostatic stress where

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p \qquad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

where p is the uniform pressure acting on the body.

Adding together the first three of Eqs. (2.13) gives

$$-3p = (3\lambda + 2\mu)J_1$$

or

$$p = -\frac{3\lambda + 2\mu}{3}J_1 = -KJ_1 = -KD$$

Thus

$$K = \frac{3\lambda + 2\mu}{3} = -\frac{p}{D} \tag{2.15e}$$

The constant K is known as the bulk modulus and is the ratio of the applied hydrostatic pressure to the volume dilatation.

Table 2.1 Relationships between the elastic constants

	λ equals	μ equals	E equals	ν equals	K equals
λ, μ			$\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$	$\frac{\lambda}{2(\lambda + \mu)}$	$\frac{3\lambda + 2\mu}{3}$
λ, E		$\frac{A\dagger + (E - 3\lambda)}{4}$		$\frac{A\dagger - (E + \lambda)}{4\lambda}$	$\frac{A\dagger + (3\lambda + E)}{6}$
λ, ν		$\frac{\lambda(1 - 2\nu)}{2\nu}$	$\frac{\lambda(1 + \nu)(1 - 2\nu)}{\nu}$		$\frac{\lambda(1 + \nu)}{3\nu}$
λ, K		$\frac{3(K - \lambda)}{2}$	$\frac{9K(K - \lambda)}{3K - \lambda}$	$\frac{\lambda}{3K - \lambda}$	
μ, E	$\frac{\mu(2\mu - E)}{E - 3\mu}$			$\frac{E - 2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu - E)}$
μ, ν	$\frac{2\mu\nu}{1 - 2\nu}$		$2\mu(1 + \nu)$		$\frac{2\mu(1 + \nu)}{3(1 - 2\nu)}$
μ, K	$\frac{3K - 2\mu}{3}$		$\frac{9K\mu}{3K + \mu}$	$\frac{3K - 2\mu}{2(3K + \mu)}$	
E, ν	$\frac{\nu E}{(1 + \nu)(1 - 2\nu)}$	$\frac{E}{2(1 + \nu)}$			$\frac{E}{3(1 - 2\nu)}$
K, E	$\frac{3K(3K - E)}{9K - E}$	$\frac{3EK}{9K - E}$		$\frac{3K - E}{6K}$	
ν, K	$\frac{3K\nu}{1 + \nu}$	$\frac{3K(1 - 2\nu)}{2(1 + \nu)}$	$3K(1 - 2\nu)$		

† $A = \sqrt{E^2 + 2\lambda E + 9\lambda^2}$.

Five elastic constants λ, μ, E, ν, and K have been discussed. The constant λ has no physical significance and is employed because it simplifies, mathematically speaking, the stress-strain relations. The constant μ has both mathematical and physical significance. It is used extensively in torsional problems. The constants E and ν are the most widely recognized of the five constants considered and are used in almost all areas of stress analysis. The rather specialized bulk modulus K is used primarily for computing volume changes in a given body subjected to hydrostatic pressure. As indicated previously, there are two and only two independent elastic constants. The five constants discussed are related to each other as shown in Table 2.1.

Since the constants E and ν will be used almost exclusively throughout the remainder of this text, Eqs. (2.15*b*) and (2.15*c*) have been substituted into Eqs. (2.13) and (2.14) to obtain expressions for strain in terms of stress and the constants

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$

$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})]$$

$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{yy} + \sigma_{xx})]$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \qquad \gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz} \qquad \gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx} \tag{2.16}$$

and for stress in terms of strain and the constants

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{xx} + \nu(\epsilon_{yy} + \epsilon_{zz})]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{yy} + \nu(\epsilon_{xx} + \epsilon_{zz})]$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{zz} + \nu(\epsilon_{xx} + \epsilon_{yy})]$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \qquad \tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \qquad \tau_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx} \tag{2.17}$$

2.9 STRAIN-TRANSFORMATION EQUATIONS AND STRESS-STRAIN RELATIONS FOR A TWO-DIMENSIONAL STATE OF STRESS

Simplified forms of the strain-transformation equations and the stress-strain relations, which will be extremely useful in later chapters when brittle coating and electrical-resistance strain-gage analyses are discussed, are the equations applicable to the strain field associated with a two-dimensional state of stress ($\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$).

The strain-transformation equations can be obtained from Eqs. (2.6) by selecting z' coincident with z and noting from Eqs. (2.16) that $\gamma_{zx} = \gamma_{yz} = 0$. The notation can also be simplified by denoting the angle between x' and x as θ. The equations obtained are

$$\epsilon_{x'x'} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\epsilon_{y'y'} = \epsilon_{yy}\cos^2\theta + \epsilon_{xx}\sin^2\theta - \gamma_{xy}\sin\theta\cos\theta$$

$$\gamma_{x'y'} = 2(\epsilon_{yy} - \epsilon_{xx})\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \tag{2.18}$$

$$\epsilon_{z'z'} = \epsilon_{zz} \qquad \gamma_{y'z'} = \gamma_{z'x'} = 0$$

The stress-strain relations for a two-dimensional state of stress are obtained by substituting $\sigma_{zz} = \tau_{zx} = \tau_{yz} = 0$ into Eqs. (2.16). Thus

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \qquad \epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \qquad \epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \qquad \gamma_{yz} = \gamma_{zx} = 0 \tag{2.19}$$

In a similar manner the equations for stress in terms of strain for the two-dimensional state of stress are obtained from Eqs. (2.17). Thus

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}) \qquad \sigma_{yy} = \frac{E}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx})$$

$$\sigma_{zz} = \tau_{zx} = \tau_{yz} = 0 \qquad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \tag{2.20}$$

One additional relationship which relates the strain ϵ_{zz} to the measured strains ϵ_{xx} and ϵ_{yy} in experimental analyses is obtained from Eq. (2.17) by substituting $\sigma_{zz} = 0$. Thus

$$\epsilon_{zz} = -\frac{\nu}{1-\nu}(\epsilon_{xx} + \epsilon_{yy}) \tag{2.21}$$

This equation can be used to establish the magnitude of the third principal strain associated with a two-dimensional state of stress. This information is useful for maximum shear-strain determinations.

EXERCISES

2.1 Given the displacement field

$$u = (3x^4 + 2x^2y^2 + x + y + z^3 + 3)(10^{-3})$$
$$v = (3xy + y^3 + y^2z + z^2 + 1)(10^{-3})$$
$$w = x^2 + xy + yz + zx + y^2 + z^2 + 2)(10^{-3})$$

Compute the associated strains at point (1, 1, 1). Compare the results obtained by using Eqs. (2.2) and (2.3) with those obtained by using Eqs. (2.4).

2.2 Given the displacement field

$$u = (x^2 + y^4 + 2y^2z + yz)(10^{-3})$$
$$v = (xy + xz + 3x^2z)(10^{-3})$$
$$w = (y^4 + 4y^3 + 2z^2)(10^{-3})$$

Compute the associated strains at point (2, 2, 2). Compare the results obtained by using Eqs. (2.2) and (2.3) with those obtained by using Eqs. (2.4).