CHAPTER
THREE

BASIC EQUATIONS AND PLANE ELASTICITY THEORY

3.1 FORMULATION OF THE PROBLEM

In the general three-dimensional elasticity problem there are 15 unknown quantities which must be determined at every point in the body, namely, the 6 cartesian components of stress, the 6 cartesian components of strain, and the 3 components of displacement. Attempts can be made to obtain a solution to a given problem after the following quantities have been adequately defined:

1. The geometry of the body
2. The boundary conditions
3. The body-force field as a function of position
4. The elastic constants

In order to solve for the above-mentioned 15 unknown quantities, 15 independent equations are required. Three are provided by the stress equations of equilibrium [Eqs. (1.3)], six are provided by the strain-displacement relations [Eqs. (2.4)], and the remaining six can be obtained from the stress-strain expressions [Eqs. (2.16)].

A solution to an elasticity problem, in addition to satisfying these 15 equations, must also satisfy the boundary conditions. In other words, the stresses acting over the surface of the body must produce tractions which are equivalent to the loads being applied to the body. Boundary conditions are often classified to define the four different types of boundary-value problem listed below:

Type 1. If the displacements are prescribed over the entire boundary, the problem is classified as a type 1 boundary-value problem. As an example, consider a long, slender rod which is given an axial displacement, say, u and transverse displacements v and w. In this instance displacements are prescribed over the entire boundary of the rod.

Type 2. The most frequently encountered boundary-value problem is the type where normal and shearing forces are given over the entire surface. For instance, a sphere subjected to a uniform hydrostatic pressure has zero shearing stress and a normal stress equal to $-p$ on the surface and hence is a type 2 boundary-value problem.

Type 3. This is a mixed boundary-value problem where the normal and shearing forces are given over a portion of the boundary and the displacements are given over the remainder of the body. To illustrate this type of problem, consider the shrinking of a sleeve over a shaft. In the shrinking process a radial displacement is given to the sleeve at the interface between the shaft and the sleeve. On all other surfaces of the sleeve, both the normal and the shearing components of stress are zero.

Type 4. This type of boundary-value problem is the most general of the four considered. Over a portion of the surface, displacements are prescribed. Over a second portion of the surface, normal and shearing stresses are prescribed. Over a third portion of the surface, the normal component of displacement and the shearing component of stress are prescribed. Over a fourth portion of the surface, the shearing component of displacement and the normal component of stress are prescribed. Obviously, the first three types of problem can be regarded as special cases of this general fourth type.

One of the most difficult problems encountered in any experimental study is the design and construction of the loading fixture for applying the required displacements or tractions to the model being studied. The classifications given previously should be kept in mind when one designs the fixture. In general, it has been found that tractions cannot be adequately simulated by applying a displacement field to the model and vice versa. Type 1 and type 2 boundary-value problems are usually the easiest to approach experimentally. In general, type 3 and type 4 problems offer more difficulties in properly loading the model.

3.2 FIELD EQUATIONS

Thus far in the development four sets of field equations have been discussed, namely, the stress equations of equilibrium, the strain-displacement relations, the stress-strain expressions, and the equations of compatibility. Quite often two or more of these sets of equations can be combined to give a new set which may be more applicable to a specific problem. As an example, consider the six stress-displacement equations which can be obtained from the six stress-strain relations and the six strain-displacement equations by substituting Eqs. (2.4) into Eqs. (2.16).

$$\frac{\partial u}{\partial x} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$

$$\frac{\partial v}{\partial y} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]$$

$$\frac{\partial w}{\partial z} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{\mu}\tau_{xy} \qquad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{\mu}\tau_{yz} \qquad \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{1}{\mu}\tau_{zx} \tag{3.1}$$

It is interesting to note that the set of equations consisting of the stress equations of equilibrium [Eqs. (1.3)] and the stress-displacement relations [Eqs. (3.1)] are expressed as nine equations in terms of nine unknowns. The reduction in the number of unknowns from 15 to 9 was made possible by eliminating the strains.

The problem can be reduced further (from nine to three unknowns) if the stress equations of equilibrium [Eqs. (1.3)] are combined with the stress-displacement equations (3.1). The displacement equations of equilibrium obtained can be written as follows:

$$\nabla^2 u + \frac{1}{1-2\nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{1}{\mu}F_x = 0$$

$$\nabla^2 v + \frac{1}{1-2\nu}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{1}{\mu}F_y = 0 \tag{3.2}$$

$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{1}{\mu}F_z = 0$$

where ∇^2 is the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

It is clear that a solution of the displacement equations of equilibrium will yield the three displacements u, v, w. Once the displacements are known, the six strains and the six stresses can easily be obtained by using Eqs. (2.4) to obtain the strains and Eqs. (2.16) to obtain the stresses.

Analytical solutions for three-dimensional elasticity problems are quite difficult to obtain, and the number of problems which have been solved in an exact fashion to date is surprisingly small. The most successful approach to date has been through the use of the Boussinesq-Popkovich stress functions, which are defined so as to satisfy Eq. (3.2). The development of this approach is somewhat involved and is therefore beyond the scope and objectives of this elementary treatment of the theory of elasticity. The interested student should consult the selected references at the end of the chapter for a detailed development of the Boussinesq-Popkovich stress-function approach.

Before this section is completed, the stress equations of compatibility will be developed since they are the basis for an important theorem regarding the dependence of stresses on the elastic constants. If the stress-strain relations [Eqs. (2.16)], the stress equations of equilibrium [Eqs. (1.3)], and the strain compatibility equations [Eqs. (2.10)] are combined, the six stress equations of compatibility are obtained as follows:

$$
\begin{aligned}
\nabla^2\sigma_{xx} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x^2}I_1 &= -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_x}{\partial x}\\
\nabla^2\sigma_{yy} + \frac{1}{1+\nu}\frac{\partial^2}{\partial y^2}I_1 &= -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_y}{\partial y}\\
\nabla^2\sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2}{\partial z^2}I_1 &= -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_z}{\partial z}\\
\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x\,\partial y}I_1 &= -\left(\frac{\partial F_x}{\partial y}+\frac{\partial F_y}{\partial x}\right)\\
\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2}{\partial y\,\partial z}I_1 &= -\left(\frac{\partial F_y}{\partial z}+\frac{\partial F_z}{\partial y}\right)\\
\nabla^2\tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2}{\partial z\,\partial x}I_1 &= -\left(\frac{\partial F_x}{\partial z}+\frac{\partial F_z}{\partial x}\right)
\end{aligned}
\tag{3.3}
$$

where I_1 is the first invariant of stress $\sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ and F_x, F_y, F_z are the body-force intensities in the x, y, z directions, respectively.

If this system of six equations is solved for the six cartesian stress components, and if the boundary conditions are satisfied, the problem can be considered solved. Of great importance to the experimentalist is the appearance of elastic constants in Eqs. (3.3). Recall that equations of stress equilibrium did not contain elastic constants. Since only Poisson's ratio ν appears in Eqs. (3.3), it follows that the stresses are independent of the modulus of elasticity E of the model material and can at most depend upon Poisson's ratio alone. Of course, this is true only for a simply connected body since the strain compatibility equations are valid only for this condition.

This independence of the stresses on the elastic modulus is very important in three-dimensional photoelasticity, where a low-modulus plastic model is used to simulate a metal prototype. Only the difference in Poisson's ratio between the model and the prototype is a source of error. The very large difference between the moduli of elasticity of the model and the prototype does not produce any significant errors in the determination of stresses using a three-dimensional photoelastic approach, provided the strains induced in the photoelastic model remain sufficiently small.

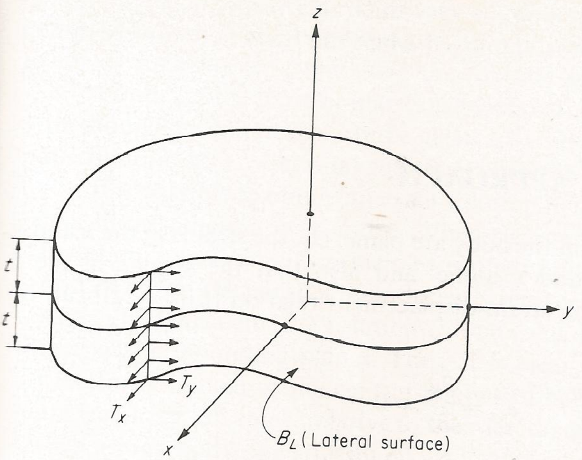

Figure 3.1 A body which may be considered for the plane-elasticity approach is bounded on the top and bottom by two parallel planes and is bounded laterally by any surface which is normal to the top and bottom planes.

3.3 THE PLANE ELASTIC PROBLEM

In the theory of elasticity there exists a special class of problems, known as *plane problems*, which can be solved more readily than the general three-dimensional problem since certain simplifying assumptions can be made in their treatment. The geometry of the body and the nature of the loading on the boundaries which permit a problem to be classified as a plane problem are as follows:

By definition a plane body consists of a region of uniform thickness bounded by two parallel planes and by any closed lateral surface B_L, as indicated by Fig. 3.1. Although the thickness of the body must be uniform, it need not be limited. It may be very thick or very thin; in fact, these two extremes represent the most desirable cases for this approach, as will be pointed out later.
In addition to the restrictions on the geometry of the body, the following restrictions are imposed on the loads applied to the plane body.

1. Body forces, if they exist, cannot vary through the thickness of the region, that is, $F_x = F_x(x, y)$ and $F_y = F_y(x, y)$. Furthermore, the body force in the z direction must equal zero.
2. The surface tractions or loads on the lateral boundary B_L must be in the plane of the model and must be uniformly distributed across the thickness, i.e., constant in the z direction. Hence, $T_x = T_x(x, y)$, $T_y = T_y(x, y)$, and $T_z = 0$.
3. No loads can be applied on the parallel planes bounding the top and bottom surfaces, that is, $\mathbf{T}_n = 0$ on $z = \pm t$.

Once the geometry and loading have been defined, stresses can be determined by using either the plane-strain or the plane-stress approach. Usually the plane-strain approach is used when the body is very thick relative to its lateral dimensions. The plane-stress approach is employed when the body is relatively thin in relation to its lateral dimensions.

3.4 THE PLANE-STRAIN APPROACH

If it is assumed that the strains in the body are plane, i.e., the strains in the x and y directions are functions of x and y alone, and also that the strains in the z directions are equal to zero, the strain-displacement relation [Eqs. (2.4)] can be simplified as follows:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \tag{3.4}$$

Similarly, if Eqs. (3.4) are substituted into Eqs. (2.13), a reduced form of the stress-strain relations for the case of plane strain is obtained:

$$\sigma_{xx} = \lambda J_1 + 2\mu\epsilon_{xx} \qquad \sigma_{yy} = \lambda J_1 + 2\mu\epsilon_{yy} \qquad \sigma_{zz} = \lambda J_1$$

$$\tau_{xy} = \mu\gamma_{xy} \qquad \tau_{yz} = \tau_{zx} = 0 \tag{3.5}$$

where $J_1 = \epsilon_{xx} + \epsilon_{yy}$. In addition, the stress equations of equilibrium [Eqs. (1.3)] reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \tag{3.6}$$

Any solution for a plane-strain problem must satisfy Eqs. (3.4) to (3.6) in addition to the boundary conditions on the lateral boundary B_L and the bounding planes. The boundary conditions on B_L can be expressed in terms of the stresses by referring to Eqs. (1.2), which give the x, y, and z components of the resultant-stress vector in terms of the cartesian components of stress. Thus, on B_L the following relations must be satisfied:

$$\begin{aligned} T_{nx} &= \sigma_{xx} \cos(n, x) + \tau_{xy} \cos(n, y) \\ T_{ny} &= \tau_{xy} \cos(n, x) + \sigma_{yy} \cos(n, y) \\ T_{nz} &= 0 \end{aligned} \tag{3.7}$$

where T_{nx}, T_{ny}, T_{nz} are the x, y, z components of the stresses applied to the body on surface B_L. Finally, on the two parallel bounding planes,

$$\mathbf{T}_n = 0 \tag{3.8}$$

i.e., no tractions are applied to these surfaces; hence, τ_{yz}, τ_{zx}, σ_{zz} must be zero on these surfaces.

It is clear from Eqs. (3.5) that σ_{zz} will be equal to zero, as demanded by Eq. (3.8), only when the dilatation J_1 is equal to zero. In most problems J_1 will not be equal to zero; therefore, the solution will not be exact since the boundary conditions on the parallel planes are violated. In many problems this violation of the boundary conditions can be cleared by superimposing an equal and opposite distribution of σ_{zz} (residual solution) onto the original solution.

It is possible to obtain an exact solution to the residual problem only when σ_{zz} is a linear function of x and y. When σ_{zz} is nonlinear, an approximate solution based on Saint-Venant's principle† is often utilized. When the nonlinear distribution of σ_{zz} on the parallel boundaries is replaced by a linear distribution which is statically equivalent, the solution will be valid only in regions well removed from the parallel bounding planes. Thus, it is clear that the plane-strain approach is necessarily limited to the central regions of bodies such as shafts or dams which are very long, i.e., thick, relative to their lateral dimensions. In the central region of such a long body, the stresses σ_{xx}, σ_{yy}, and τ_{xy} can be found from the solution of the original problem since the superposition of the residual solution onto the original problem does not influence these stresses but only serves to make σ_{zz} vanish.

In this section the plane-strain approach has been discussed without indicating a method for solving for σ_{xx}, σ_{yy}, and τ_{xy}. This problem will be treated later in this chapter when the Airy's-stress-function approach is discussed. In this plane-strain section it is important for the student to understand the plane-strain assumption, why it usually leads to a violation of the boundary conditions on the two parallel planes, and finally how these undesired stresses can be removed from the planes by superimposing a statically equivalent linear stress system. Also quite important is Saint-Venant's principle, since an experimentalist in simulating loads often relies on this principle to permit simplification in the design of the loading fixtures.

3.5 PLANE STRESS

In the preceding section it was noted that the plane-strain method is limited to very long or thick bodies. In those cases where the body thickness is small relative to its lateral dimensions, it is advantageous to assume that

$$\sigma_{zz} = \tau_{yz} = \tau_{zx} = 0 \tag{3.9}$$

throughout the thickness of the plate. With this assumption the stress equations of equilibrium again reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \tag{3.10}$$

† Saint-Venant's principle states that a system of forces acting over a small region of the boundary can be replaced by a statically equivalent system of forces without introducing appreciable changes in the distribution of stresses in regions well removed from the area of load application.

and the stress-strain relations [Eqs. (2.13)] become

$$\sigma_{xx} = \lambda J_1 + 2\mu\epsilon_{xx} \qquad \sigma_{yy} = \lambda J_1 + 2\mu\epsilon_{yy} \qquad \sigma_{zz} = \lambda J_1 + 2\mu\epsilon_{zz} = 0$$

$$\tau_{xy} = \mu\gamma_{xy} \qquad \tau_{yz} = \mu\gamma_{yz} = 0 \qquad \tau_{zx} = \mu\gamma_{zx} = 0 \tag{3.11}$$

From the third of Eqs. (3.11) the following relationship can be obtained:

$$\epsilon_{zz} = -\frac{\lambda}{\lambda + 2\mu}(\epsilon_{xx} + \epsilon_{yy}) \tag{a}$$

With this value of ϵ_{zz} the first strain invariant J_1 becomes

$$J_1 = \frac{2\mu}{\lambda + 2\mu}(\epsilon_{xx} + \epsilon_{yy}) \tag{b}$$

Substituting the value for J_1 given in Eq. (*b*) into Eqs. (3.11) yields

$$\sigma_{xx} = \frac{2\lambda\mu}{\lambda + 2\mu}(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{xx}$$

$$\sigma_{yy} = \frac{2\lambda\mu}{\lambda + 2\mu}(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{yy} \tag{3.12}$$

$$\tau_{xy} = \mu\gamma_{xy} \qquad \sigma_{zz} = \tau_{yz} = \tau_{zx} = 0$$

Unfortunately, in the general case σ_{xx}, σ_{yy}, and τ_{xy} are not independent of z, and thus the boundary conditions imposed on the boundary B_L cannot be rigorously satisfied. To overcome this difficulty, average stresses and displacements over the thickness are commonly used. If the body is relatively thin, these averages closely approximate the true boundary conditions on B_L. Average values for the stresses and displacements over the thickness of the body are obtained as follows:

$$\tilde{\sigma}_{xx} = \frac{1}{2t}\int_{-t}^{t} \sigma_{xx}\,dz \qquad \tilde{\sigma}_{yy} = \frac{1}{2t}\int_{-t}^{t} \sigma_{yy}\,dz \qquad \tilde{\tau}_{xy} = \frac{1}{2t}\int_{-t}^{t} \tau_{xy}\,dz$$

$$\tilde{u} = \frac{1}{2t}\int_{-t}^{t} u\,dz \qquad \tilde{v} = \frac{1}{2t}\int_{-t}^{t} v\,dz \tag{3.13}$$

The symbol $\sim$ over the stresses and displacements indicates average values. Substituting the average values of the stresses into Eqs. (1.2) gives the boundary conditions which must be satisfied on B_L:

$$T_{nx} = \tilde{\sigma}_{xx}\cos(n, x) + \tilde{\tau}_{xy}\cos(n, y)$$

$$T_{ny} = \tilde{\tau}_{xy}\cos(n, x) + \tilde{\sigma}_{yy}\cos(n, y) \tag{3.14}$$

If the equations which the plane-strain and the plane-stress solutions must satisfy are compared, it can be observed that they are identical except for the comparison between Eqs. (3.5) and (3.11). An examination of a typical equation from each of these sets,

$$\sigma_{xx} = \begin{cases} \lambda(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{xx} & \text{plane strain} \\ \dfrac{2\lambda\mu}{\lambda + 2\mu}(\epsilon_{xx} + \epsilon_{yy}) + 2\mu\epsilon_{xx} & \text{plane stress} \end{cases}$$

indicates that they are identical except for the coefficients of the $\epsilon_{xx} + \epsilon_{yy}$ term. Since all other equations for the plane-stress and plane-strain solutions are identical, results from plane strain can be transformed into plane stress by letting

$$\lambda \rightarrow \frac{2\lambda\mu}{\lambda + 2\mu}$$

which is equivalent to letting

$$\frac{\nu}{1 - \nu} \rightarrow \nu \tag{3.15}$$

In a similar manner a plane-stress solution can be transformed into a plane-strain solution by letting

$$\frac{2\lambda\mu}{\lambda + 2\mu} \rightarrow \lambda$$

or

$$\nu \rightarrow \frac{\nu}{1 - \nu} \tag{3.16}$$

In the plane-stress approach it is generally assumed that

$$\sigma_{zz} = \tau_{yz} = \tau_{zx} = 0$$

and the unknown stresses σ_{xx}, σ_{yy}, and τ_{xy} will have a z dependence. As a result of this z dependence, the boundary conditions on B_L are violated. This difficulty can be eliminated and an approximate solution to the problem can be obtained by using average values for the stresses and displacements. Finally, it was shown that plane-stress and plane-strain solutions can be transformed from one case into the other by a simple replacement involving Poisson's ratio, as indicated in Eqs. (3.15) and (3.16).

3.6 AIRY'S STRESS FUNCTION

In the plane problem three unknowns σ_{xx}, σ_{yy}, and τ_{xy} must be determined which will satisfy the required field equations and boundary conditions. The most convenient sets of field equations to use in this determination are the two equations of equilibrium and one stress equation of compatibility.

The equilibrium equations in two dimensions are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \tag{3.17a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \tag{3.17b}$$

The stress compatibility equation for the case of plane strain is

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\frac{2(\lambda + \mu)}{\lambda + 2\mu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \tag{3.17c}$$

Suppose the body-force field is defined by $\Omega(x, y)$ so that the body-force intensities are given by

$$F_x = -\frac{\partial \Omega}{\partial x} \qquad F_y = -\frac{\partial \Omega}{\partial y} \tag{3.18}$$

Then by substituting Eqs. (3.18) into Eqs. (3.17) and noting that $2(\lambda + \mu)/(\lambda + 2\mu) = 1/(1 - \nu)$, it is apparent that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \frac{\partial \Omega}{\partial x} \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial \Omega}{\partial y} \tag{3.19}$$

$$\nabla^2\left(\sigma_{xx} + \sigma_{yy} - \frac{\Omega}{1 - \nu}\right) = 0$$

Equations (3.19) represent the three field equations which σ_{xx}, σ_{yy}, and τ_{xy} must satisfy.

Assume that the stresses can be represented by a stress function ϕ such that

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + \Omega \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + \Omega \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y} \tag{3.20}$$

If Eqs. (3.20) are substituted into Eqs. (3.19), it can be seen that the two equations of equilibrium are exactly satisfied, and the last of Eqs. (3.19) gives

$$\nabla^4 \phi = -\frac{1 - 2\nu}{1 - \nu} \nabla^2 \Omega \tag{3.21}$$

Thus, equilibrium and compatibility are immediately satisfied if ϕ satisfies Eq. (3.21). The expression ϕ is known as Airy's stress function. If Eq. (3.21) is solved for ϕ, an expression containing x, y, and a number of constants will be obtained. The constants are evaluated from the boundary conditions given in Eqs. (3.17), and the stresses are computed from ϕ according to Eqs. (3.20). Of course, evaluation of ϕ from Eq. (3.21) produces stresses for the plane-strain case.

Stresses for the plane-stress case can be obtained by letting $\nu/(1-\nu) \rightarrow \nu$, as indicated in Eq. (3.15). This substitution leads to

$$\nabla^4 \phi = -(1-\nu)\,\nabla^2 \Omega \tag{3.22}$$

which is valid for plane-stress problems.

It is important to note that if the body-force intensities are zero or constant, such as those encountered in a gravitational field, then

$$\nabla^2 \Omega = 0$$

and Eqs. (3.21) and (3.22) both become

$$\nabla^4 \phi = 0 \tag{3.23a}$$

This is a biharmonic equation, which can also be written in the form

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{3.23b}$$

Examination of this equation shows that ϕ and thus σ_{xx}, σ_{yy}, and τ_{xy} are independent of the elastic constants. This consideration is very important in two-dimensional photoelasticity since it indicates that the stresses obtained from a plastic model are identical to those in a metal prototype if the model is simply connected and subjected to a zero or a uniform body-force field. Differences in the values of the modulus of elasticity and Poisson's ratio between model and prototype do not influence the results for the stresses. There are exceptions to the simply connected restriction, however, which will be covered in a later chapter on photoelasticity.

3.7 AIRY'S STRESS FUNCTION IN CARTESIAN COORDINATES

Any Airy's stress function used in the solution of a plane problem must satisfy Eqs. (3.23*a*) and (3.23*b*) and provide stresses via Eqs. (3.20) which satisfy the defined boundary conditions. Some Airy's stress functions commonly used are polynomials in x and y. In this section, polynomials from the first to the fifth degree will be considered.

A. Airy's Stress Function in Terms of a First-Degree Polynomial

$$\phi_1 = a_1 x + b_1 y$$

It is clear from Eqs. (3.20) that

$$\sigma_{xx} = \sigma_{yy} = \tau_{xy} = 0 \tag{3.24}$$

and that Eqs. (3.23*a*) and (3.23*b*) are satisfied. This function is suitable only for indicating a stress-free field and therefore is of little use in the solution of any problem.

B. Airy's Stress Function in Terms of a Second-Degree Polynomial

$$\phi_2 = a_2 x^2 + b_2 xy + c_2 y^2$$

From Eqs. (3.20) the stresses are

$$\sigma_{xx} = 2c_2 \qquad \sigma_{yy} = 2a_2 \qquad \tau_{xy} = -b_2 \tag{3.25}$$

Note that Eqs. (3.23*a*) and (3.23*b*) are satisfied and that the stress function ϕ_2 gives a uniform stress field over the entire body which is independent of x and y.

C. Airy's Stress Function in Terms of a Third-Degree Polynomial

$$\phi_3 = a_3 x^3 + b_3 x^2 y + c_3 xy^2 + d_3 y^3$$

Again, by use of Eqs. (3.20) the stresses are given by

$$\sigma_{xx} = 2c_3 x + 6d_3 y \qquad \sigma_{yy} = 6a_3 x + 2b_3 y \qquad \tau_{xy} = -2b_3 x - 2c_3 y \tag{3.26}$$

Equations (3.23*a*) and (3.23*b*) are satisfied unconditionally, and the stress function ϕ_3 provides a linearly varying stress field over the body.

D. Airy's Stress Function in Terms of a Fourth-Degree Polynomial

$$\phi_4 = a_4 x^4 + b_4 x^3 y + c_4 x^2 y^2 + d_4 xy^3 + e_4 y^4$$

From Eqs. (3.20) it is apparent that

$$\begin{aligned} \sigma_{xx} &= 2c_4 x^2 + 6d_4 xy + 12e_4 y^2 \\ \sigma_{yy} &= 12a_4 x^2 + 6b_4 xy + 2c_4 y^2 \\ \tau_{xy} &= -3b_4 x^2 - 4c_4 xy - 3d_4 y^2 \end{aligned} \tag{3.27}$$

When ϕ_4 is substituted into Eq. (3.23*b*), it should be noted that it is not unconditionally satisfied. In order for $\nabla^4 \phi = 0$ it is necessary that

$$e_4 = -\left(a_4 + \frac{c_4}{3}\right)$$

Substituting this equation into the relations for the stresses gives

$$\sigma_{xx} = 2c_4 x^2 + 6d_4 xy - 12a_4 y^2 - 4c_4 y^2$$

and σ_{yy} and τ_{xy} are unchanged. Thus, ϕ_4 yields a stress field which is a second-degree polynomial in x and y.

E. Airy's Stress Function in Terms of a Fifth-Degree Polynomial

$$\phi_5 = a_5 x^5 + b_5 x^4 y + c_5 x^3 y^2 + d_5 x^2 y^3 + e_5 xy^4 + f_5 y^5$$

Employing Eqs. (3.20) to solve for the stresses gives

$$\sigma_{xx} = 2c_5 x^3 + 6d_5 x^2 y + 12e_5 xy^2 + 20f_5 y^3$$
$$\sigma_{yy} = 20a_5 x^3 + 12b_5 x^2 y + 6c_5 xy^2 + 2d_5 y^3$$
$$\tau_{xy} = -4b_5 x^3 - 6c_5 x^2 y - 6d_5 xy^2 - 4e_5 y^3$$

Again, note that ϕ_5 must be subjected to certain conditions involving the constants e_5 and f_5. For Eqs. (3.23*a*) and (3.23*b*) to be satisfied, these conditions are

$$e_5 = -(5a_5 + c_5) \qquad f_5 = -\tfrac{1}{5}(b_5 + d_5)$$

Subject to the restrictive conditions listed above, the cartesian stress components become

$$\begin{aligned} \sigma_{xx} &= 2c_5 x^3 + 6d_5 x^2 y - 12(5a_5 + c_5)xy^2 - 4(b_5 + d_5)y^3 \\ \sigma_{yy} &= 20a_5 x^3 + 12b_5 x^2 y + 6c_5 xy^2 + 2d_5 y^3 \\ \tau_{xy} &= -4b_5 x^3 - 6c_5 x^2 y - 6d_5 xy^2 + 4(5a_5 + c_5)y^3 \end{aligned} \tag{3.28}$$

Thus, it is clear that ϕ_5 yields a stress field which is a third-degree polynomial in x and y. It is possible to continue this procedure to ϕ_6, ϕ_7, etc., as long as Eqs. (3.23*a*) and (3.23*b*) are satisfied. It is also possible to add together two or more stress functions to form another, for example, $\phi^* = \phi_2 + \phi_3$. Thus, by simply adding terms or by eliminating terms from the stress function it is theoretically possible to build up any stress field that can be expressed as a function of x and y.

3.8 EXAMPLE PROBLEM

Airy's stress function expressed in cartesian coordinates can be employed to solve a particular class of two-dimensional problems where the boundaries of the body can be adequately represented by the cartesian reference frame. As an example, consider the simply supported beam with uniform loads shown in Fig. 3.2. An examination of the loading conditions indicates that

$$\sigma_{yy} = \begin{cases} \tau_{xy} = 0 & \text{at } y = \dfrac{-h}{2} \\ -q & \text{at } y = \dfrac{+h}{2} \end{cases} \tag{a}$$

$$\tau_{xy} = 0 \qquad \text{at } y = \frac{+h}{2} \tag{b}$$

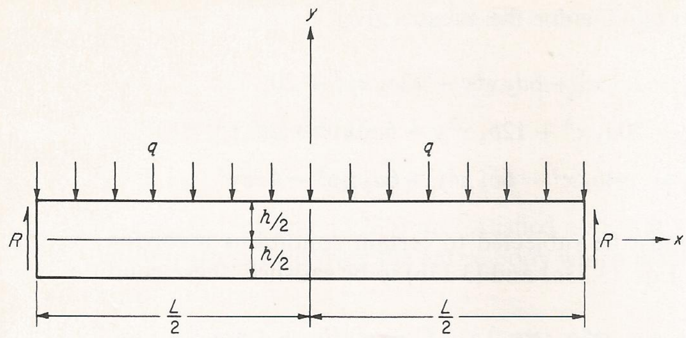

Figure 3.2 Simply supported beam of length L, height h, and unit depth subjected to a uniformly distributed load.

Also at $x = \pm L/2$

$$\int_{-h/2}^{h/2} \tau_{xy}\, dy = R = \frac{qL}{2} \tag{c}$$

$$\int_{-h/2}^{h/2} \sigma_{xx}\, dy = 0 \tag{d}$$

$$\int_{-h/2}^{h/2} \sigma_{xx} y\, dy = 0 \tag{e}$$

Note that the bending moment (and consequently σ_{xx}) is a maximum at position $x = 0$ and decreases with a change in x in either the positive or the negative direction. This is possible only if the stress function contains even functions of x. Note also that σ_{yy} varies from zero at $y = -h/2$ to a maximum value of $-q$ at $y = +h/2$; thus the stress function must contain odd functions of y. From the stress functions listed in Sec. 3.7, the following even and odd functions can be selected to form a new stress function ϕ which satisfies the previously listed conditions.

$$\phi = a_2 x^2 + b_3 x^2 y + d_3 y^3 + a_4 x^4 + b_5 x^4 y + d_5 x^2 y^3 + f_5 y^5 \tag{f}$$

This stress function ϕ must satisfy the equation $\nabla^4 \phi = 0$; hence

$$a_4 = 0 \qquad f_5 = -\tfrac{1}{5}(b_5 + d_5) \tag{g}$$

From Eqs. (3.20) the cartesian stress components are

$$\begin{aligned} \sigma_{xx} &= 6d_3 y + 6d_5 x^2 y - 4(b_5 + d_5) y^3 \\ \sigma_{yy} &= 2a_2 + 2b_3 y + 12b_5 x^2 y + 2d_5 y^3 \\ \tau_{xy} &= -2b_3 x - 4b_5 x^3 - 6d_5 x y^2 \end{aligned} \tag{h}$$

Examination of the boundary conditions shown in Eq. (*a*) indicates that σ_{yy} must be independent of x; hence the coefficient $b_5 = 0$. Consequently, Eqs. (*h*) reduce to

$$\begin{aligned} \sigma_{xx} &= 6d_3 y + 6d_5 x^2 y - 4d_5 y^3 \\ \sigma_{yy} &= 2a_2 + 2b_3 y + 2d_5 y^3 \\ \tau_{xy} &= -2b_3 x - 6d_5 xy^2 \end{aligned} \tag{i}$$

The problem can be solved if the coefficients a_2, b_3, d_3, and d_5 can be selected so that the boundary conditions given in Eqs. (*a*) to (*e*) are satisfied. From Eqs. (*a*)

$$\sigma_{yy} = 0 = 2a_2 + 2b_3\left(-\frac{h}{2}\right) + 2d_5\left(-\frac{h}{2}\right)^3 \qquad a_2 - \frac{b_3 h}{2} - \frac{d_5 h^3}{8} = 0 \tag{j}$$

and from Eqs. (*b*)

$$\sigma_{yy} = -q = 2a_2 + 2b_3\frac{h}{2} + 2d_5\left(\frac{h}{2}\right)^3 \qquad a_2 + \frac{b_3 h}{2} + \frac{d_5 h^3}{8} = -\frac{q}{2} \tag{k}$$

Adding Eqs. (*j*) and (*k*) gives

$$a_2 = -\frac{q}{4} \tag{l}$$

From Eqs. (*a*) and (*b*)

$$\tau_{xy} = 0 = -2x\left[b_3 + 3d_5\left(\pm\frac{h}{2}\right)^2\right] \qquad b_3 = -\frac{3}{4}h^2 d_5 \tag{m}$$

Substituting Eqs. (*m*) into Eqs. (*j*),

$$\frac{d_5 h^3}{8} - \frac{3h^3 d_5}{8} = -\frac{q}{4} \qquad d_5 = \frac{q}{h^3} \tag{n}$$

and

$$b_3 = -\frac{3}{4}\frac{q}{h} \tag{o}$$

With the values of a_2, b_3, and d_5 given by Eqs. (*l*), (*o*), and (*n*), respectively, Eqs. (*c*) and (*d*) are identically satisfied. Equation (*e*) can be used to solve for the remaining unknown d_3.

$$\int_{-h/2}^{h/2}\left(6d_3 y^2 + \frac{3}{2}\frac{q}{h^3}L^2 y^2 - 4\frac{qy^4}{L^3}\right) dy = 0$$

$$\left[2d_3 y^3 + \frac{qL^2 y^3}{2h^3} - \frac{4qy^5}{5h^3}\right]_{-h/2}^{+h/2} = 0 \tag{p}$$

Solving Eq. (*p*) for d_3 gives

$$d_3 = \frac{q}{240I}(2h^2 - 5L^2) \tag{q}$$

where $I = h^3/12$ is the moment of inertia of the unit-width beam. Substituting Eqs. (*q*), (*o*), (*n*), and (*l*) into Eqs. (*i*) gives the final equations for the cartesian components of stress:

$$\sigma_{xx} = \frac{q}{8I}(4x^2 - L^2)y + \frac{q}{60I}(3h^2y - 20y^3)$$

$$\sigma_{yy} = \frac{q}{24I}(4y^3 - 3h^2y - h^3) \qquad (r)$$

$$\tau_{xy} = \frac{qx}{8I}(h^2 - 4y^2)$$

The conventional strength-of-materials solution for this problem, namely, that $\sigma_{xx} = My/I$, gives

$$\sigma_{xx} = \frac{q}{8I}(4x^2 - L^2)y \qquad (s)$$

which is identical with the first term of the relation given for σ_{xx} in Eqs. (*r*). The second term, $(q/60I)(3h^2y - 20y^3)$, is a correction term for the strength-of-materials solution. In the strength-of-materials approach, recall that it is assumed that plane sections remain plane after bending. This is not exactly true, and as a consequence the solution obtained lacks the correction term shown above. It is clear that the correction term is small when $L \gg h$, and the strength-of-materials solution will be sufficiently accurate.

This simple example illustrates how elementary elasticity theory can be employed to extend the student's understanding of the distribution of stresses in simple two-dimensional problems. Other examples are included in the exercises at the end of this chapter.

3.9 TWO-DIMENSIONAL PROBLEMS IN POLAR COORDINATES

In Sec. 3.6 the Airy's-stress-function approach to the solution of two-dimensional elasticity problems in cartesian coordinates was developed. This method was then applied to solve an elementary problem which was well suited to the cartesian reference frame. In many problems, however, the geometry of the body does not lend itself to the use of a cartesian coordinate system, and it is more expeditious to work with a different system. A large class of problems (such as circular rings, curved beams, and half-planes) can be solved by employing a commonly used system, the polar coordinate system. In any elasticity problem the proper choice of the coordinate system is extremely important since this choice establishes the complexity of the mathematical expressions employed to satisfy the field equations and the boundary conditions.

In order to solve two-dimensional elasticity problems by employing a polar-coordinate reference frame, the equations of equilibrium, the definition of Airy's stress function, and one of the stress equations of compatibility must be reestablished in terms of polar coordinates. On the following pages the equations of equilibrium will be derived by considering a polar element instead of a cartesian element. The equations for the polar components of stress in terms of Airy's stress function as well as the stress equation of compatibility will be transformed from cartesian to polar coordinates. Finally, a set of stress functions is developed which satisfies the stress equation of compatibility.

The stress equations of equilibrium in polar coordinates can be derived from the free-body diagram of the polar element shown in Fig. 3.3. The element is assumed to be very small. The average values of the normal and shearing stresses which act on surface 1 are denoted by σ_{rr} and $\tau_{r\theta}$, respectively. Since the stresses may vary as a function of r, values of the normal and shearing stresses on surface 3 are given by $\sigma_{rr} + (\partial\sigma_{rr}/\partial r)\, dr$ and $\tau_{r\theta} + (\partial\tau_{r\theta}/\partial r)\, dr$. Similarly, the average values of the normal and shearing stresses which act on surface 2 are given by $\sigma_{\theta\theta}$ and $\tau_{r\theta}$. Since the stresses may also vary as a function of θ, values of the normal and shearing stresses on surface 4 are $\sigma_{\theta\theta} + (\partial\sigma_{\theta\theta}/\partial\theta)\, d\theta$ and $\tau_{r\theta} + (\partial\tau_{r\theta}/\partial\theta)\, d\theta$.

For a polar element of unit thickness to be in a state of equilibrium the sum of all forces in the radial r and tangential θ directions must equal zero. Summing forces first in the radial direction and considering the body-force intensity F_r gives the equation of equilibrium

$$\left(\sigma_{rr} + \frac{\partial\sigma_{rr}}{\partial r}\, dr\right)(r + dr)\, d\theta - \sigma_{rr} r\, d\theta - \left[\sigma_{\theta\theta}\, dr + \left(\sigma_{\theta\theta} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta}\, d\theta\right) dr\right]\frac{d\theta}{2}$$
$$+ \left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta}\, d\theta - \tau_{r\theta}\right) dr + F_r r\, d\theta\, dr = 0 \qquad (a)$$

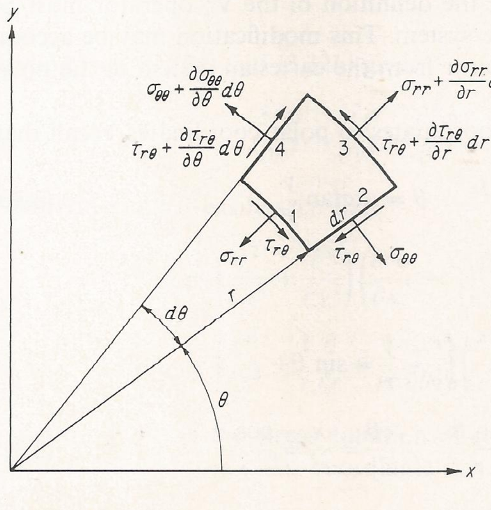

Figure 3.3 Polar element of unit depth showing the stresses acting on the four faces.

Dividing Eq. (*a*) by $dr\,d\theta$ and simplifying gives

$$\frac{\partial \sigma_{rr}}{\partial r}dr - \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\frac{d\theta}{2} + \sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r}r - \sigma_{\theta\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} + F_r r = 0 \tag{b}$$

If the element is made infinitely small by permitting dr and $d\theta$ each to approach zero, the first two terms in Eq. (*b*) also approach zero and the expression can be rewritten as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + F_r = 0 \tag{3.29a}$$

The equation of equilibrium in the tangential direction can be derived in the same manner if the forces acting in the θ direction on the polar element are summed and set equal to zero. Hence

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0 \tag{3.29b}$$

Equations (3.29*a*) and (3.29*b*) represent the equations of equilibrium in polar coordinates. They are analogous to the equations of equilibrium in cartesian coordinates presented in Eqs. (3.17*a*) and (3.17*b*). Any solution to an elasticity problem must satisfy these field equations.

3.10 TRANSFORMATION OF THE EQUATION $\nabla^4\phi = 0$ INTO POLAR COORDINATES

In the coverage of Airy's stress function given in Sec. 3.6 it was shown that the stress function ϕ had to satisfy the biharmonic equation $\nabla^4\phi = 0$, provided the body forces are zero or constants. In polar coordinates the stress function must satisfy this same equation; however, the definition of the ∇^4 operator must be modified to suit the polar-coordinate system. This modification may be accomplished by transforming the ∇^4 operator from the cartesian system to the polar system.

In transforming from cartesian coordinates to polar coordinates, recall that

$$r^2 = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x} \tag{3.30}$$

where r and θ are defined in Fig. 3.3.

Differentiating Eqs. (3.30) gives

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin\theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos\theta}{r} \end{aligned} \tag{3.31}$$

The form of the ∇^4 operator in cartesian coordinates is

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$$

Individual elements of this expression can be transformed by employing Eqs. (3.30) and (3.31) as follows. If it is assumed that ϕ is a function of r and θ,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x} \tag{a}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial r}\frac{\partial^2 r}{\partial x^2} + \left(\frac{\partial r}{\partial x}\right)^2\frac{\partial^2\phi}{\partial r^2} + 2\frac{\partial^2\phi}{\partial r\,\partial\theta}\frac{\partial r}{\partial x}\frac{\partial\theta}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial^2\theta}{\partial x^2} + \left(\frac{\partial\theta}{\partial x}\right)^2\frac{\partial^2\phi}{\partial\theta^2} \tag{b}$$

Substituting the equalities given in Eqs. (3.31) into Eq. (*b*) yields

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\sin^2\theta}{r}\frac{\partial\phi}{\partial r} + \cos^2\theta\frac{\partial^2\phi}{\partial r^2} - \frac{\sin 2\theta}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta} + \frac{\sin 2\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{3.32a}$$

Following the same procedure makes it clear that

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\cos^2\theta}{r}\frac{\partial\phi}{\partial r} + \sin^2\theta\frac{\partial^2\phi}{\partial r^2} + \frac{\sin 2\theta}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta} - \frac{\sin 2\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{3.32b}$$

$$\frac{\partial^2\phi}{\partial x\,\partial y} = -\frac{\sin\theta\cos\theta}{r}\frac{\partial\phi}{\partial r} + \sin\theta\cos\theta\frac{\partial^2\phi}{\partial r^2} + \frac{\cos 2\theta}{r}\frac{\partial^2\phi}{\partial r\,\partial\theta} - \frac{\cos 2\theta}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{3.32c}$$

Adding Eqs. (3.32*a*) and (3.32*b*) gives

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{3.33}$$

Furthermore, it is easily seen that

$$\begin{aligned}\nabla^4\phi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0\end{aligned} \tag{3.34}$$

Equation (3.34) is the stress equation of compatibility in terms of Airy's stress function referred to a polar coordinate system.

3.11 POLAR COMPONENTS OF STRESS IN TERMS OF AIRY'S STRESS FUNCTION

By referring to the two-dimensional equations of stress transformation [Eqs. (1.11)], expressions can be obtained which relate the polar stress components σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ to the cartesian stress components σ_{xx}, σ_{yy}, and τ_{xy} as follows:

$$\begin{aligned}\sigma_{rr} &= \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + \tau_{xy} \sin 2\theta \\ \sigma_{\theta\theta} &= \sigma_{yy} \cos^2 \theta + \sigma_{xx} \sin^2 \theta - \tau_{xy} \sin 2\theta \\ \tau_{r\theta} &= (\sigma_{yy} - \sigma_{xx}) \sin \theta \cos \theta + \tau_{xy} \cos 2\theta\end{aligned} \tag{3.35}$$

If Eqs. (3.20) are substituted into Eqs. (3.35) and Ω set equal to zero (which is equivalent to setting both F_x and F_y equal to zero), then

$$\begin{aligned}\sigma_{rr} &= \frac{\partial^2 \phi}{\partial y^2} \cos^2 \theta + \frac{\partial^2 \phi}{\partial x^2} \sin^2 \theta - \frac{\partial^2 \phi}{\partial x\, \partial y} \sin 2\theta \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial x^2} \cos^2 \theta + \frac{\partial^2 \phi}{\partial y^2} \sin^2 \theta + \frac{\partial^2 \phi}{\partial x\, \partial y} \sin 2\theta \\ \tau_{r\theta} &= \left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}\right) \sin \theta \cos \theta - \frac{\partial^2 \phi}{\partial x\, \partial y} \cos 2\theta\end{aligned} \tag{3.36}$$

If the results from Eqs. (3.32*a*) to (3.32*c*) are substituted into Eqs. (3.36), the polar components of stress in terms of Airy's stress function are obtained:

$$\begin{aligned}\sigma_{rr} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\, \partial \theta}\end{aligned} \tag{3.37}$$

When Airy's stress function ϕ in polar coordinates has been established, these relations can be employed to determine the stress field as a function of r and θ.

3.12 FORMS OF AIRY'S STRESS FUNCTION IN POLAR COORDINATES

The equation $\nabla^4 \phi = 0$ is a fourth-order biharmonic partial differential equation which can be reduced to an ordinary fourth-order differential equation by using a separation-of-variables technique, where

$$\phi^{(n)} = R_n(r) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}$$

The resulting differential equation is an Euler type which yields four different stress functions upon solution. These stress functions are tabulated, together with the stress and displacement distributions which they provide, on the following pages.

One of the stress functions obtained can be expressed in the following form:

$$\phi^{(0)}(r) = a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r \tag{3.38a}$$

By using Eqs. (3.37), the stresses associated with this particular stress function can be expressed as

$$\sigma_{rr} = \frac{b_0}{r^2} + 2c_0 + d_0(1 + 2 \ln r)$$

$$\sigma_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + d_0(3 + 2 \ln r) \qquad \tau_{r\theta} = 0 \tag{3.38b}$$

The displacements associated with this function can be determined by integrating the stress displacement relations, giving

$$u_r = \frac{1}{E}\left[-(1+\nu)\frac{b_0}{r} + 2(1-\nu)c_0 r + 2(1-\nu)d_0 r \ln r - (1+\nu)d_0 r\right]$$
$$+ \alpha_2 \cos\theta + \alpha_3 \sin\theta$$

$$u_\theta = \frac{1}{E}(4d_0 r\theta) - \alpha_1 r - \alpha_2 \sin\theta + \alpha_3 \cos\theta \tag{3.38c}$$

where u_r and u_θ are the displacements in the radial and circumferential directions, respectively. The terms containing α_1, α_2, and α_3 are associated with rigid-body displacements.

It should be noted that the stresses in this solution are independent of θ; hence, the stress function $\phi^{(0)}$ should be employed to solve problems which have rotational symmetry.

One of the other stress functions and the stresses and displacements associated with it can be expressed as follows:

$$\phi^{(1)} = \left(a_1 r + \frac{b_1}{r} + c_1 r^3 + d_1 r \ln r\right)\begin{Bmatrix}\sin\theta \\ \cos\theta\end{Bmatrix}$$

$$\sigma_{rr} = \left(-\frac{2b_1}{r^3} + 2c_1 r + \frac{d_1}{r}\right)\begin{Bmatrix}\sin\theta \\ \cos\theta\end{Bmatrix}$$

$$\sigma_{\theta\theta} = \left(\frac{2b_1}{r^3} + 6c_1 r + \frac{d_1}{r}\right)\begin{Bmatrix}\sin\theta \\ \cos\theta\end{Bmatrix}$$

$$\tau_{r\theta} = \left(-\frac{2b_1}{r^3} + 2c_1 r + \frac{d_1}{r}\right)\begin{Bmatrix}-\cos\theta \\ \sin\theta\end{Bmatrix}$$

$$u_r = \frac{1}{E}\left\{\left[(1+\nu)\frac{b_1}{r^2} + (1-3\nu)c_1 r^2 - (1+\nu)d_1\right.\right. \tag{3.39}$$

$$\left.\left. + (1-\nu)d_1 \ln r\right]\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix} - (2d_1\theta)\begin{Bmatrix}\cos\theta\\ -\sin\theta\end{Bmatrix}\right\} + \alpha_2\cos\theta + \alpha_3\sin\theta$$

$$u_\theta = \frac{1}{E}\left\{\left[-(1+\nu)\frac{b_1}{r^2} - (5+\nu)c_1 r^2 + (1-\nu)d_1 \ln r\right]\begin{Bmatrix}\cos\theta\\ -\sin\theta\end{Bmatrix}\right.$$

$$\left. + (2d_1\theta)\begin{Bmatrix}\sin\theta\\ \cos\theta\end{Bmatrix}\right\} - \alpha_1 r - \alpha_2\sin\theta + \alpha_3\cos\theta$$

The third stress function of interest and its associated stresses and displacements are as follows:

$$\phi^{(n)} = (a_n r^n + b_n r^{-n} + c_n r^{2+n} + d_n r^{2-n})\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}$$

$$\sigma_{rr} = [a_n(n-n^2)r^{n-2} - b_n(n+n^2)r^{-n-2} + c_n(2+n-n^2)r^n$$

$$+ d_n(2-n-n^2)r^{-n}]\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}$$

$$\sigma_{\theta\theta} = [a_n(n^2-n)r^{n-2} + b_n(n^2+n)r^{-n-2} + c_n(2+3n+n^2)r^n$$

$$+ d_n(2-3n+n^2)r^{-n}]\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}$$

$$\tau_{r\theta} = [a_n(n^2-n)r^{n-2} - b_n(n+n^2)r^{-n-2} + c_n(n+n^2)r^n$$

$$+ d_n(n-n^2)r^{-n}]\begin{Bmatrix}-\cos n\theta\\ \sin n\theta\end{Bmatrix} \tag{3.40}$$

$$u_r = \frac{1}{E}\{-a_n(1+\nu)nr^{n-1} + b_n(1+\nu)nr^{-n-1}$$

$$+ c_n[4-(1+\nu)(2+n)]r^{n+1}$$

$$+ d_n[4-(1+\nu)(2-n)]r^{-n+1}\}\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix} + \alpha_2\cos\theta + \alpha_3\sin\theta$$

$$u_\theta = \frac{1}{E}\{-a_n(1+\nu)nr^{n-1} - b_n(1+\nu)nr^{-n-1}$$

$$- c_n[4+(1+\nu)n]r^{n+1} + d_n[4-(1+\nu)n]r^{-n+1}\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix}$$

$$- \alpha_1 r - \alpha_2\sin\theta + \alpha_3\cos\theta$$

For the stress function $\phi^{(n)}$ the value of n can be greater than or equal to 2 (that is, $n \geq 2$).

The fourth stress function of interest and the associated stresses and displacements are expressed as follows:

$$
\begin{aligned}
\phi^{(*)} &= a_*\theta + b_* r^2\theta + c_* r\theta \sin\theta + d_* r\theta\cos\theta \\
\sigma_{rr} &= 2b_*\theta + 2c_*\frac{\cos\theta}{r} - 2d_*\frac{\sin\theta}{r} \\
\sigma_{\theta\theta} &= 2b_*\theta \\
\tau_{r\theta} &= \frac{a_*}{r^2} - b_* \qquad (3.41)\\
u_r &= \frac{1}{E}[2(1-\nu)b_* r\theta + (1-\nu)c_*\theta\sin\theta + 2c_* \ln r\cos\theta \\
&\quad + (1-\nu)d_*\theta\cos\theta - 2d_* \ln r\sin\theta] + \alpha_2\cos\theta + \alpha_3\sin\theta \\
u_\theta &= \frac{1}{E}\Big[-(1+\nu)\frac{a_*}{r} + (3-\nu)b_* r - 4b_* r\ln r \\
&\quad - (1+\nu)c_*\sin\theta - 2c_*\ln r\sin\theta + (1-\nu)c_*\theta\cos\theta \\
&\quad - (1+\nu)d_*\cos\theta - 2d_*\ln r\cos\theta - (1-\nu)d_*\theta\sin\theta\Big] \\
&\quad - \alpha_1 r - \alpha_2\sin\theta + \alpha_3\cos\theta
\end{aligned}
$$

In the example problems which follow, the stress functions previously listed will be employed to determine the stresses and displacements for problems which lend themselves to polar coordinates. As the selection of the stress function is often the most difficult phase of the problem, particular emphasis will be placed on the reasoning behind the selection.

3.13 STRESSES AND DISPLACEMENTS IN A CIRCULAR CYLINDER SUBJECTED TO INTERNAL AND EXTERNAL PRESSURE

Consider the long hollow cylinder shown in Fig. 3.4, which is subjected to an internal pressure p_i and an external pressure p_o. The inside and outside radii of the cylinder are denoted as a and b, respectively.

As stated previously, the first step in the solution of an elasticity problem after the geometry of the body has been defined is to establish the boundary conditions. For the problem under consideration these conditions can be listed as follows:

$$
\begin{aligned}
\sigma_{rr} &= -p_i \qquad \tau_{r\theta} = 0 \qquad \text{at } r = a \\
\sigma_{rr} &= -p_o \qquad \tau_{r\theta} = 0 \qquad \text{at } r = b \qquad (a)
\end{aligned}
$$

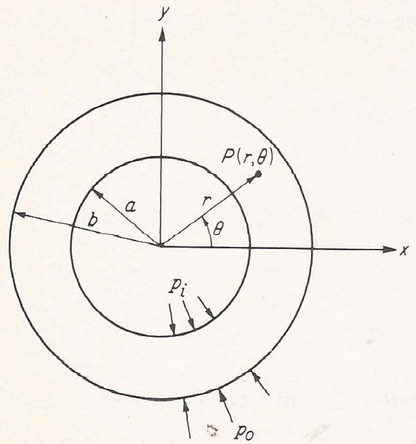

Figure 3.4 Circular cylinder subjected to internal and external pressures.

An examination of the boundary conditions indicates that they are independent of θ; hence the four stress functions $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(n)}$, and $\phi^{(*)}$ should be inspected to determine which will provide a stress field independent of θ. The stress function $\phi^{(0)}$ given in Eq. (3.38*a*) yields stresses which satisfy this requirement, as shown below:

$$\sigma_{rr} = \frac{b_0}{r^2} + 2c_0 + d_0(1 + 2\ln r)$$

$$\sigma_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + d_0(3 + 2\ln r) \tag{b}$$

$$\tau_{r\theta} = 0$$

Equations (*b*) will provide the desired solution to the problem if the constants b_0, c_0, and d_0 can be determined so that the boundary conditions given in Eqs. (*a*) are satisfied.

An examination of Eqs. (*b*) indicates that the condition $\tau_{r\theta} = 0$ throughout the body satisfies part of the boundary conditions. From symmetry considerations it is also obvious that both u_r and u_θ must be independent of θ. This condition can be satisfied only if $d_0 = 0$ in Eqs. (3.38*c*). The two remaining constants b_0 and c_0 can be evaluated by using the remaining boundary conditions in Eqs. (*a*):

$$\sigma_{rr} = -p_i = \frac{b_0}{a^2} + 2c_0 \qquad \sigma_{rr} = -p_o = \frac{b_0}{b^2} + 2c_0 \tag{c}$$

Solving Eqs. (*c*) for b_0 and c_0 yields

$$b_0 = \frac{a^2b^2(p_o - p_i)}{b^2 - a^2} \qquad c_0 = \frac{a^2p_i - b^2p_o}{2(b^2 - a^2)}$$

These values when substituted into Eqs. (3.38) provide the required solution.

$$\sigma_{rr} = \frac{a^2b^2(p_o - p_i)}{(b^2 - a^2)r^2} + \frac{a^2p_i - b^2p_o}{b^2 - a^2}$$

$$\sigma_{\theta\theta} = -\frac{a^2b^2(p_o - p_i)}{(b^2 - a^2)r^2} + \frac{a^2p_i - b^2p_o}{b^2 - a^2}$$

$$\tau_{r\theta} = 0 \tag{3.42}$$

$$u_r = \frac{1}{E}\left[-(1+\nu)\frac{a^2b^2(p_o - p_i)}{(b^2 - a^2)r} + (1-\nu)\frac{a^2p_i - b^2p_o}{b^2 - a^2}r\right]$$

$$u_\theta = 0$$

Equations (3.42) give the stresses and displacements at a point $P(r, \theta)$ in the cylinder if the two pressures, the radii, and the elastic constants are known. Three special cases of this problem are of interest.

Case 1: External pressure equals zero Setting $p_o = 0$ in Eqs. (3.42) leads to

$$\sigma_{rr} = \frac{a^2p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \qquad \sigma_{\theta\theta} = \frac{a^2p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \qquad \tau_{r\theta} = 0$$

$$u_r = \frac{a^2p_i}{Er(b^2 - a^2)}[(1+\nu)b^2 + (1-\nu)r^2] \qquad u_\theta = 0 \tag{3.43}$$

This special case is often encountered when dealing with stresses in piping systems or pressure vessels.

Case 2: Internal pressure equals zero Setting $p_i = 0$ in Eqs. (3.42) leads to

$$\sigma_{rr} = \frac{b^2p_o}{b^2 - a^2}\left(\frac{a^2}{r^2} - 1\right) \qquad \sigma_{\theta\theta} = -\frac{b^2p_o}{b^2 - a^2}\left(\frac{a^2}{r^2} + 1\right) \qquad \tau_{r\theta} = 0$$

$$u_r = -\frac{b^2p_o}{Er(b^2 - a^2)}[(1+\nu)a^2 + (1-\nu)r^2] \qquad u_\theta = 0 \tag{3.44}$$

When external pressure is applied to a cylindrical shell, the problem of buckling should also be considered.

Case 3: External pressure on a solid circular cylinder When one sets $a = 0$ in Eqs. (3.44), the hole in the cylinder vanishes and the stresses become

$$\sigma_{rr} = \sigma_{\theta\theta} = -p_o \qquad \tau_{r\theta} = 0$$

$$u_r = -\frac{1-\nu}{E}p_o r \qquad u_\theta = 0 \tag{3.45}$$

3.14 STRESS DISTRIBUTION IN A THIN, INFINITE PLATE WITH A CIRCULAR HOLE SUBJECTED TO UNIAXIAL TENSILE LOADS

A thin plate of infinite length and width with a circular hole is shown in Fig. 3.5. The plate is subjected to a uniform tensile-type load which produces a uniform stress σ_0 in the y direction at $r = \infty$. The distribution of the stresses about the hole, along the x axis, and along the y axis can be determined by using the Airy's-stress-function approach.

The boundary conditions which must be satisfied are

$$\sigma_{rr} = \tau_{r\theta} = 0 \qquad \text{at } r = a \qquad (a)$$

$$\sigma_{yy} = \sigma_0 \qquad \text{at } r \to \infty$$

$$\sigma_{xx} = \tau_{xy} = 0 \qquad \text{at } r \to \infty \qquad (b)$$

Selection of a stress function for this particular problem is difficult since none of the four functions previously tabulated is satisfactory. In order to overcome this difficulty, a method of superposition is commonly used which employs two different stress functions. The first function is selected such that the stresses associated with it satisfy the boundary conditions at $r \to \infty$ but in general violate the conditions on the boundary of the hole. The second stress function must have associated stresses which cancel the stresses on the boundary of the hole without influencing the stresses at $r \to \infty$. An illustration of this superposition process is presented in Fig. 3.6.

The boundary conditions at $r \to \infty$ can be satisfied by the uniform stress field associated with the stress function ϕ_2 in Eqs. (3.25). For the case of uniaxial

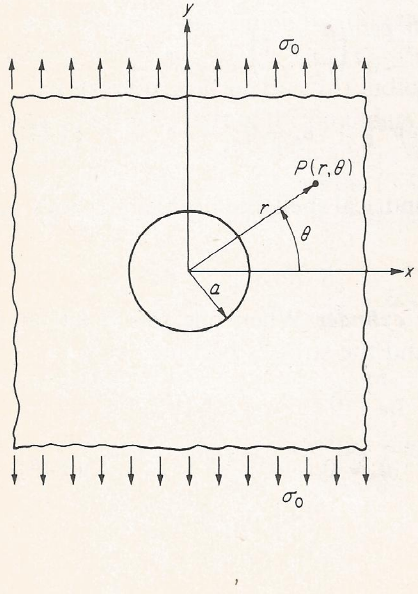

Figure 3.5 Thin, infinite plate with a circular hole subjected to a uniaxial tensile stress σ_0.

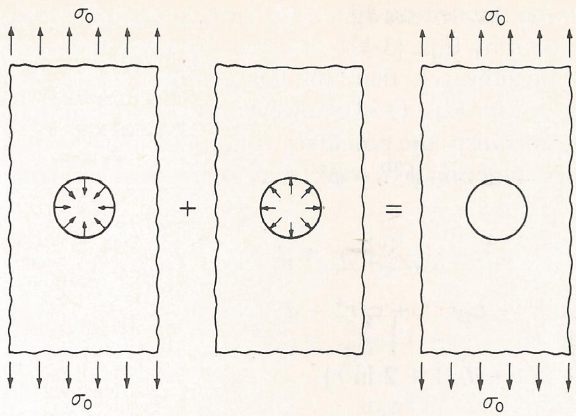

Figure 3.6 The method of superposition.

tension in the y direction, ϕ_2 reduces to

$$\phi_2 = a_2 x^2 = \frac{\sigma_0 x^2}{2} \tag{c}$$

The stresses throughout the plate for a plate without a hole are

$$\sigma_{yy} = \sigma_0 \qquad \sigma_{xx} = \tau_{xy} = 0 \tag{d}$$

If an imaginary hole of radius a is cut into the plate, the stresses σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ on the boundary of the imaginary hole can be computed from Eqs. (3.31) as follows:

$$\begin{aligned}
\sigma_{rr}^I &= \sigma_0 \sin^2\theta = \frac{\sigma_0}{2}(1 - \cos 2\theta) \\
\sigma_{\theta\theta}^I &= \sigma_0 \cos^2\theta = \frac{\sigma_0}{2}(1 + \cos 2\theta) \\
\tau_{r\theta}^I &= \sigma_0 \sin\theta \cos\theta = \frac{\sigma_0}{2} \sin 2\theta
\end{aligned} \tag{e}$$

In the original problem the boundary conditions at $r = a$ were

$$\sigma_{rr} = \tau_{r\theta} = 0$$

The boundary conditions to be satisfied by the stresses associated with the second stress function are therefore

$$\begin{aligned}
\sigma_{rr} &= -\sigma_0 \sin^2\theta = -\frac{\sigma_0}{2}(1 - \cos 2\theta) && \text{at } r = a \\
\sigma_{rr} &= \tau_{r\theta} = \sigma_{\theta\theta} = 0 && \text{at } r \to \infty \\
\tau_{r\theta} &= -\sigma_0 \sin\theta \cos\theta = -\frac{\sigma_0}{2} \sin 2\theta && \text{at } r = a
\end{aligned} \tag{f}$$

From Eqs. (f) it is apparent that the stresses σ'_{rr} and $\tau_{r\theta}$ are functions of sin 2θ and cos 2θ, which suggests $\phi^{(2)}$ given by Eqs. (3.40) as a possible stress function. Inspection of Eqs. (3.40) indicates, however, that this function can satisfy the boundary conditions only for $\tau_{r\theta}$. From Eqs. (3.38), however, it can be seen that the stresses associated with $\phi^{(0)}$ can satisfy the boundary conditions for σ_{rr} without influencing $\tau_{r\theta}$. Thus, the stress function $\phi^{(0)} + \phi^{(2)}$ may be applicable. From Eqs. (3.38) and (3.40),

$$\phi^{(0)} + \phi^{(2)} = a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r + (a_2 r^2 + b_2 r^{-2} + c_2 r^4 + d_2) \cos 2\theta \tag{g}$$

$$\sigma_{rr} = \frac{b_0}{r^2} + 2c_0 + d_0(1 + 2 \ln r) - \left(2a_2 + \frac{6b_2}{r^4} + \frac{4d_2}{r^2}\right) \cos 2\theta \tag{h}$$

$$\sigma_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + d_0(3 + 2 \ln r) + \left(2a_2 + \frac{6b_2}{r^4} + 10c_2 r^2\right) \cos 2\theta \tag{i}$$

$$\tau_{r\theta} = \left(2a_2 - \frac{6b_2}{r^4} + 6c_2 r^2 - \frac{2d_2}{r^2}\right) \sin 2\theta \tag{j}$$

Equations (h) to (j) contain seven unknowns: $b_0, c_0, d_0, a_2, b_2, c_2$, and d_2. Since $\sigma_{\theta\theta} = \sigma_{rr} = \tau_{r\theta} = 0$ as $r \to \infty$,

$$c_0 = d_0 = a_2 = c_2 = 0 \tag{k}$$

and Eqs. (h) to (j) reduce to

$$\sigma_{rr} = \frac{1}{r^2}\left[b_0 - \left(\frac{6b_2}{r^2} + 4d_2\right) \cos 2\theta\right] \tag{l}$$

$$\sigma_{\theta\theta} = \frac{1}{r^2}\left(-b_0 + \frac{6b_2}{r^2} \cos 2\theta\right) \tag{m}$$

$$\tau_{r\theta} = -\frac{1}{r^2}\left[\left(\frac{6b_2}{r^2} + 2d_2\right) \sin 2\theta\right] \tag{n}$$

From the boundary conditions at $r = a$,

$$\tau_{r\theta} = -\frac{1}{a^2}\left[\left(\frac{6b_2}{a^2} + 2d_2\right) \sin 2\theta\right] = -\frac{\sigma_0}{2} \sin 2\theta \tag{o}$$

$$\sigma_{rr} = \frac{1}{a^2}\left[b_0 - \left(\frac{6b_2}{a^2} + 4d_2\right) \cos 2\theta\right] = -\frac{\sigma_0}{2}(1 - \cos 2\theta) \tag{p}$$

Solving Eqs. (*o*) and (*p*) for the coefficients gives

$$b_0 = -\frac{\sigma_0 a^2}{2} \qquad b_2 = \frac{\sigma_0 a^4}{4} \qquad d_2 = -\frac{\sigma_0 a^2}{2} \tag{q}$$

Substituting Eqs. (*q*) into Eqs. (*l*) to (*n*) gives

$$\sigma_{rr}^{\mathrm{II}} = -\frac{\sigma_0 a^2}{2r^2}\left[1 + \left(\frac{3a^2}{r^2} - 4\right)\cos 2\theta\right]$$

$$\sigma_{\theta\theta}^{\mathrm{II}} = \frac{\sigma_0 a^2}{2r^2}\left(1 + \frac{3a^2}{r^2}\cos 2\theta\right)$$

$$\tau_{r\theta}^{\mathrm{II}} = -\frac{\sigma_0 a^2}{2r^2}\left[\left(\frac{3a^2}{r^2} - 2\right)\sin 2\theta\right]$$

The required solution for the original problem is obtained by superposition as follows:

$$\begin{aligned}
\sigma_{rr} &= \sigma_{rr}^{\mathrm{I}} + \sigma_{rr}^{\mathrm{II}} = \frac{\sigma_0}{2}\left\{\left(1 - \frac{a^2}{r^2}\right)\left[1 + \left(\frac{3a^2}{r^2} - 1\right)\cos 2\theta\right]\right\} \\
\sigma_{\theta\theta} &= \sigma_{\theta\theta}^{\mathrm{I}} + \sigma_{\theta\theta}^{\mathrm{II}} = \frac{\sigma_0}{2}\left[\left(1 + \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\right] \\
\tau_{r\theta} &= \tau_{r\theta}^{\mathrm{I}} + \tau_{r\theta}^{\mathrm{II}} = \frac{\sigma_0}{2}\left[\left(1 + \frac{3a^2}{r^2}\right)\left(1 - \frac{a^2}{r^2}\right)\sin 2\theta\right]
\end{aligned} \tag{3.46}$$

Equations (3.46) give the polar components of stress at any point in the body defined by r, θ. Through the use of Eqs. (3.46) the stresses along the x axis, along the y axis, and about the boundary of the hole can easily be computed.

The stresses along the x axis can be obtained by setting $\theta = 0$ and $r = x$ in Eqs. (3.46):

$$\begin{aligned}
\sigma_{rr} &= \sigma_{xx} = \frac{\sigma_0}{2}\left(1 - \frac{a^2}{x^2}\right)\frac{3a^2}{x^2} \\
\sigma_{\theta\theta} &= \sigma_{yy} = \frac{\sigma_0}{2}\left(2 + \frac{a^2}{x^2} + \frac{3a^4}{x^4}\right) \\
\tau_{r\theta} &= \tau_{xy} = 0
\end{aligned} \tag{3.47}$$

The distribution of σ_{xx}/σ_0 and σ_{yy}/σ_0 is plotted as a function of position along the x axis in Fig. 3.7. An examination of this figure clearly indicates that the presence of the hole in the infinite plate under uniaxial tension increases the σ_{yy} stress by a factor of 3. This factor is often called a *stress concentration factor*. In a later chapter it will be shown how photoelasticity can be effectively employed to determine stress concentration factors.

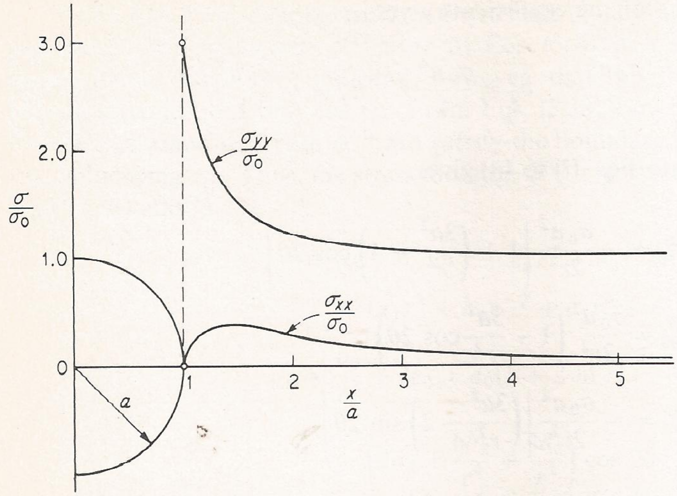

Figure 3.7 Distribution of σ_{xx}/σ_0 and σ_{yy}/σ_0 along the x axis.

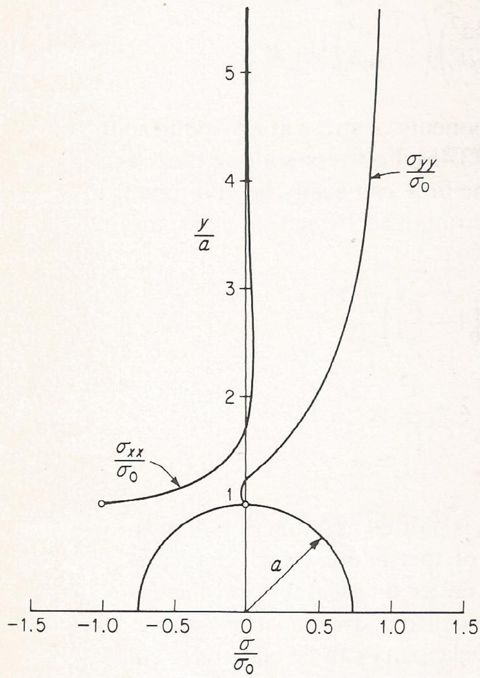

Figure 3.8 Distribution of σ_{xx}/σ_0 and σ_{yy}/σ_0 along the y axis.

In a similar manner the stresses along the y axis can be obtained by setting $\theta = \pi/2$ and $r = y$ in Eqs. (3.46):

$$\sigma_{rr} = \sigma_{yy} = \frac{\sigma_0}{2}\left(2 - \frac{5a^2}{y^2} + \frac{3a^4}{y^4}\right)$$

$$\sigma_{\theta\theta} = \sigma_{xx} = \frac{\sigma_0}{2}\left(\frac{a^2}{y^2} - \frac{3a^4}{y^4}\right) \tag{3.48}$$

$$\tau_{r\theta} = \tau_{xy} = 0$$

A distribution of σ_{xx}/σ_0 and σ_{yy}/σ_0 is plotted as a function of position along the y axis in Fig. 3.8. In this figure it can be noted that $\sigma_{xx}/\sigma_0 = -1$ at the boundary of the hole; thus the influence of the hole not only produces a concentration of the stresses but in this case also produces a change in the sign of the stresses.

The distribution of $\sigma_{\theta\theta}$ about the boundary of the hole is obtained by setting $r = a$ into Eqs. (3.46):

$$\sigma_{rr} = \tau_{r\theta} = 0 \qquad \sigma_{\theta\theta} = \sigma_0(1 + 2\cos 2\theta) \tag{3.49}$$

The distribution of $\sigma_{\theta\theta}/\sigma_0$ about the boundary of the hole is shown in Fig. 3.9. The maximum $\sigma_{\theta\theta}/\sigma_0$ occurs at the x axis ($\sigma_{\theta\theta}/\sigma_0 = 3$), and the minimum occurs at the y axis ($\sigma_{\theta\theta}/\sigma_0 = -1$). At the point defined by $\theta = 60°$ on the boundary of the hole, all stresses are zero. This type of point is commonly referred to as a singular point.

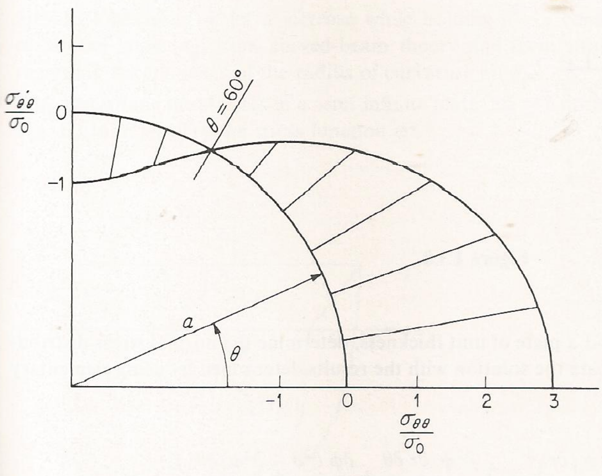

Figure 3.9 Distribution of $\sigma_{\theta\theta}/\sigma_0$ about the boundary of the hole.