$$V_f = \frac{A_f}{A_c} \qquad V_m = \frac{A_m}{A_c} \tag{3.12}$$

Thus

$$\sigma_c = \sigma_f V_f + \sigma_m V_m \tag{3.13}$$

Now Eq. (3.13) can be differentiated with respect to strain, which is the same for the composite, the fibers, and the matrix. The differentiation yields

$$\frac{d\sigma_c}{d\epsilon} = \frac{d\sigma_f}{d\epsilon} V_f + \frac{d\sigma_m}{d\epsilon} V_m \tag{3.14}$$

where $(d\sigma/d\epsilon)$ represents the slope of the corresponding stress–strain diagrams at the given strain. If the stress–strain curves of the materials are linear, the slopes $(d\sigma/d\epsilon)$ are constants and can be replaced by the corresponding elastic modulus in Eq. (3.14). Thus

$$E_c = E_f V_f + E_m V_m \tag{3.15}$$

Equations (3.13)–(3.15) indicate that the contributions of the fibers and the matrix to the average composite properties are proportional to their volume fractions. Such a relationship is called the *rule of mixtures*. Equations (3.13) and (3.15) can be generalized as

$$\sigma_c = \sum_{i=1}^{n} \sigma_i V_i \tag{3.16}$$

$$E_c = \sum_{i=1}^{n} E_i V_i \tag{3.17}$$

The following numerical example illustrates the influence of elastic modulus and volume fraction of the fibers on the longitudinal modulus of the composite.

Example 3-1: Calculate the ratios of longitudinal modulus of the composite to the matrix modulus for glass–epoxy and carbon–epoxy composites with 10% and 50% fibers by volume. Elastic moduli of glass fibers, carbon fibers, and epoxy resin are 70, 350, and 3.5 GPa, respectively.

Equation (3.15) can be written as

$$\frac{E_c}{E_m} = \left(\frac{E_f}{E_m} - 1\right) V_f + 1$$

Calculations will give the following results:

System (E_f/E_m)	E_c/E_m V_f = 10%	V_f = 50%
Glass–epoxy (20)	2.9	10.5
Carbon–epoxy (100)	10.9	50.5

It may be observed that as the fiber volume fraction increases by a fact of 5, the ratio of E_c/E_m also increases by a similar factor (3.62 for glass epoxy and 4.63 for carbon–epoxy). Further, as the fiber modulus increase by a factor 5, the ratio of E_c/E_m again increases by a similar factor (3.7 at V_f = 10% and 4.81 at V_f = 50%). These calculations show that fibe are very effective in increasing the composite modulus in the longitudin direction. Further, the elastic modulus of fibers has a significant influen on the composite modulus. This behavior will be compared with the infl ence of these factors on the composite transverse modulus in a later sectio

The predictions of Eq. (3.13) can be explained by considering the stress strain diagrams for the fibers and the matrix. Let us consider two composite The fibers in both composites have linear stress–strain curves up to the fracture. The matrix material of one of the composites also has a linear stress strain curve, but that of the other has a nonlinear stress–strain curve (Fig. 4*a*,*b*). The stress in the composite at a given strain can be calculated accordir to Eq. (3.13) by first finding the matrix stress and the fiber stress at the give strain from the corresponding stress–strain diagrams and then adding the proportional to their volume fractions. This process can be repeated for number of strain values up to the fiber fracture strain. Thus a complete stress strain diagram for the composite may be obtained. It may be noted that th procedure is applicable to both the composites being considered here becau

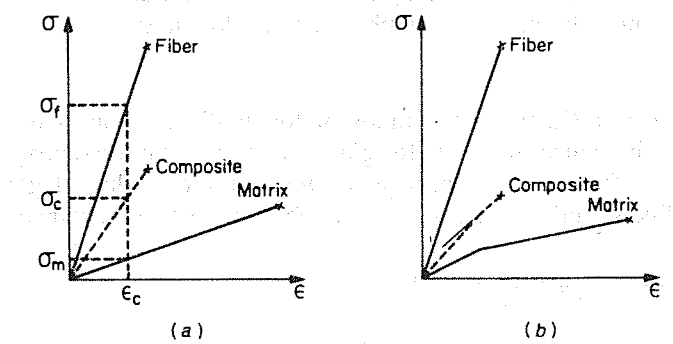

Figure 3-4. Longitudinal stress–strain diagrams for a composite with (*a*) linear and (*b*) nonlin matrix material.

above 80% the composite properties usually begin to decrease because of the inability of the matrix to wet and infiltrate the bundles of fibers. This results in poorly bonded fibers and voids in the composite.

The excellent strengths and strength–weight ratios achieved by glass-fiber-reinforced plastics are a result of the high strength of the glass fibers and the ability of the composite to use this strength because the ratio E_f/E_m is approximately 20. Even at 10% by volume of glass, the fiber will account for 70% of the total load.

Example 3-2: Calculate the fraction of load carried by the fibers in the composites indicated in Example 3-1.

The desired fractions can be obtained easily using Eq. (3.21). The results are

	P_f/P_c	
System (E_f/E_m)	V_f = 10%	V_f = 50%
Glass–epoxy (20)	0.69	0.952
Carbon–epoxy (100)	0.917	0.99

3.2.3 Behavior beyond Initial Deformation

The rule of mixtures accurately predicts the stress–strain behavior of a unidirectional composite subjected to longitudinal loads, provided that Eq. (3.13) is used for the stress and Eq. (3.14) for the slope of the stress–strain curve. However, the simplification of Eq. (3.14) to Eq. (3.15) through the replacement of slopes by the elastic moduli is possible only when both the constituents deform elastically. This may constitute only a small portion of the composite stress–strain behavior and is applicable primarily for glass- or ceramic-fiber-reinforced thermosetting plastics. In general, the deformation of a composite may proceed in four stages [1], summarized as follows:

1. Both the fibers and the matrix deform in a linear elastic fashion.
2. The fibers continue to deform elastically, but the matrix now deforms nonlinearly or plastically.
3. The fibers and the matrix both deform nonlinearly or plastically.
4. The fibers fracture followed by the composite fracture.

Stage 2 may occupy the largest portion of the composite stress–strain curve, particularly for a metal matrix composite, and in this stage the matrix stress–strain curve is no longer linear, so the composite modulus must be predicted at each strain level by

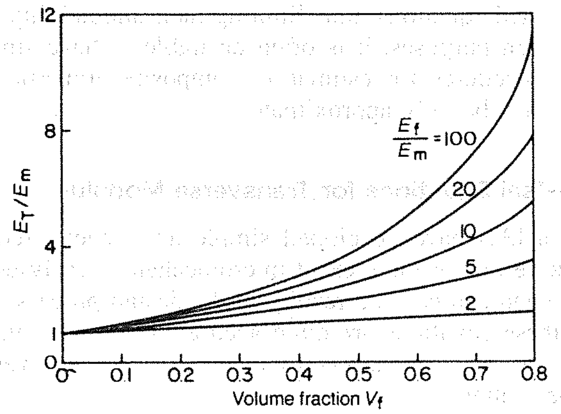

Figure 3-11. Transverse modulus predicted by Halpin–Tsai equation [Eq. (3.36)].

equation than they are to Eq. (3.34). This comparison further demonstrate applicability of the Halpin–Tsai equations for transverse modulus predictio

It is suggested that the Halpin–Tsai equations are quite adequate to satisf the practical requirements for the predictions of transverse composite modu lus, particularly because the variations in composite materials manufacturin processes always cause a variation in the composite moduli. Therefore, or cannot hope to precisely predict composite moduli.

Example 3-3: Calculate, using the Halpin–Tsai equation, the ratios c transverse modulus of the composite to the matrix modulus for the con posites given in Example 3-1. Compare these ratios with those obtained i Example 3-1.

$$\xi = 2 \quad \text{for all cases}$$

Glass–Epoxy System

$$\frac{E_f}{E_m} = 20 \qquad \eta = \frac{20 - 1}{20 + 2} = \frac{19}{22}$$

$$V_f = 10\% \qquad \frac{E_T}{E_m} = \frac{1 + 2 \times (19/22) \times 0.1}{1 - (19/22) \times 0.1} = 1.28$$

$$V_f = 50\% \qquad \frac{E_T}{E_m} = \frac{1 + 2 \times (19/22) \times 0.5}{1 - (19/22) \times 0.5} = 3.28$$

Carbon–Epoxy System

$$\frac{E_f}{E_m} = 100 \qquad \eta = \frac{100 - 1}{100 + 2} = \frac{99}{102}$$

$$V_f = 10\% \qquad \frac{E_T}{E_m} = \frac{1 + 2 \times (99/102) \times 0.1}{1 - (99/102) \times 0.1} = 1.32$$

$$V_f = 50\% \qquad \frac{E_T}{E_m} = \frac{1 + 2 \times (99/102) \times 0.5}{1 - (99/102) \times 0.5} = 3.83$$

For a better comparison, these results, along with the results of Example 3-1, can be tabulated as follows:

	$V_f = 10\%$		$V_f = 50\%$	
System (E_f/E_m)	E_L/E_m	E_T/E_m	E_L/E_m	E_T/E_m
Glass–epoxy (20)	2.9	1.28	10.5	3.28
Carbon–epoxy (100)	10.9	1.32	50.5	3.83

It is easily observed that under these conditions, the transverse modulus of a unidirectional composite is much smaller than its longitudinal modulus. Further, while an increase in fiber volume fraction results in an increase of transverse modulus similar to the longitudinal modulus, an increase in fiber modulus only marginally increases the transverse modulus, unlike the longitudinal modulus.

3.3.4 Transverse Strength

So far in this discussion it is seen that the composite longitudinal strength and stiffness and transverse stiffness are improvements over the corresponding matrix properties owing to the presence of fibers. The longitudinal strength and stiffness are improved as a result of the predominant role played by the fibers. The response of composites to longitudinal loading is determined by the fact that the load is shared between the fibers and the matrix. However, because of their higher strength and stiffness, fibers carry a major portion of the load and thus cause composite properties that are significantly improved over the matrix properties.

When a unidirectional composite is subjected to transverse loads, the fibers, as a result of the geometry, are unable to take as large a proportion of the load as they do in the case of longitudinal loading. The high-modulus fibers serve as effective constraints on the deformation of the matrix, which results in the transverse composite modulus being higher than the matrix modulus, duction, or transport of electrical and magnetic fields. Accordingly, it is important to know the relationship between composite structure and such physical constants as thermal conductivity, mass diffusivity, electrical conductivity, dielectric constants, and magnetic permeability. It has been suggested [51–53] that a transport coefficient of a unidirectional composite in the longitudinal direction k_L can be calculated by the rule of mixtures as

$$k_L = V_f k_f + V_m k_m \tag{3.72}$$

The transverse coefficient k_T can be computed by invoking an analogy from classical physics between the in-plane shear field equations and boundary conditions to the transverse transport phenomenon [52,53]. Thus the transverse transport coefficient k_T may be computed by the Halpin–Tsai equation:

$$\frac{k_T}{k_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \tag{3.73}$$

where

$$\eta = \frac{(k_f/k_m) - 1}{(k_f/k_m) + \xi} \tag{3.74}$$

$$\log \xi = \sqrt{3} \log \frac{a}{b} \tag{3.75}$$

where k_f and k_m are the appropriate transfer coefficients for fibers and matrix and a and b are the dimensions of the fiber along and perpendicular to the direction of measurement of the transfer coefficient. For circular cross-sectional fibers, the ratio a/b is 1 if transverse coefficients are to be estimated.

Example 3-4: Find the thermal conductivities of unidirectional glass-fiber- and carbon-fiber-reinforced epoxy composites in the longitudinal and transverse directions. Fiber volume fraction is 60% in both cases. Following are the thermal conductivities for the fibers and the matrix (note that the carbon fiber itself is anisotropic):

$$\text{Epoxy matrix } K_m = 0.25 \text{ W/m/°C}$$

$$\text{Glass fibers } K_f = 1.05 \text{ W/m/°C}$$

$$\text{Carbon fibers } (K_f)_L = 80 \text{ W/m/°C}$$

$$(K_f)_T = 12.5 \text{ W/m/°C}$$

Glass–Epoxy Composite

$$K_L = 0.6 \times 1.05 + 0.4 \times 0.25 = 0.73\ \text{W/m/°C}$$

For circular or square cross-sectional fibers

$$\xi = 1$$

$$\eta = \frac{(1.05/0.25) - 1}{(1.05/0.25) + 1} = 0.615$$

$$\frac{K_T}{K_m} = \frac{1 + 0.615 \times 0.6}{1 - 0.615 \times 0.6} = 2.17$$

$$K_T = 0.543\ \text{W/m/°C}$$

Carbon-Fiber–Epoxy Composite

$$K_L = 0.6 \times 80 + 0.4 \times 0.25 = 48.1\ \text{W/m/°C}$$

$$\eta = \frac{(12.5/0.25) - 1}{(12.5/0.25) + 1} = 0.961$$

$$\frac{K_T}{K_m} = \frac{1 + 0.961 \times 0.6}{1 - 0.961 \times 0.6} = 3.72$$

$$K_T = 0.93\ \text{W/m/°C}$$

Longitudinal and transverse thermal conductivities of high-modulus (HMS and high-strength (HTS) carbon-fiber–epoxy composites are shown as a function of fiber volume fraction in Figs. 3-34 and 3-35 [54]. The linearity of the longitudinal thermal conductivities is consistent with Eq. (3.72). The plots show that the thermal conductivity of the matrix may be neglected for prediction of the longitudinal thermal conductivity. The transverse thermal conductivity of these composites also increases with increasing fiber volume fraction. The experimental data could not be compared with the theoretical predictions because of the absence of pertinent data, that is, transverse thermal conductivities for fibers. Longitudinal and transverse electrical conductivities of these composites are shown in Figs. 3-36 and 3-37. The trends exhibited by the electrical conductivities are similar to those exhibited by the corresponding thermal conductivities.

The time required to reach the maximum moisture content is insensitive to the moisture content of the environment but depends on the temperature because D depends on temperature.

Example 3-5: A 12.5-mm-thick plate of graphite–epoxy composite with an initial moisture content of 0.5% is exposed on both sides to air at 25°C and 90% relative humidity. Estimate the time required to reach 1% moisture content. For the composite, at 25°C, assume $D_T = 2.6 \times 10^{-7}$ mm^2/s.

Solution: Substitution of Eq. (3.79) in Eq. (3.77) gives

$$t = \frac{S^2}{D}\left[-\frac{1}{\pi^2}\ln\frac{\pi^2}{8}\left(\frac{C_m - C}{C_m - C_0}\right)\right]$$

In the composite laminate, diffusion is taking place in the thickness direction. Therefore, the transverse diffusivity D_T is to be used for calculation, which is given. Also given are $S = 12.5$ mm, $C = 1.0\%$, and $C_0 = 0.5\%$; C_m may be calculated using Eq. (3.80):

$$C_m = 0.018\left(\frac{90}{100}\right) = 0.0162 \quad \text{or} \quad 1.62\%$$

Therefore,

$$t = \frac{(12.5)^2}{2.6 \times 10^{-7}}\left[-\frac{1}{\pi^2}\ln\left(\frac{\pi^2}{8}\cdot\frac{1.62 - 1.0}{1.62 - 0.5}\right)\right]$$
$$= 2.32 \times 10^7 \text{ s} \quad \text{or} \quad 269 \text{ days}$$

Example 3-6: The plate specified in Example 3-5, having attained the moisture content of 1%, is exposed on both sides to humid air at 15°C and 10% relative humidity. Estimate the moisture content after 10 days. Assume $D_T = 1.13 \times 10^{-7}$ mm^2/s at 15°C.

Solution: Given $S = 12.5$ mm, $C_0 = 1.0\%$. At 10% relative humidity,

$$C_m = 0.018\left(\frac{10}{100}\right) = 0.0018 \quad \text{or} \quad 0.18\%$$

From Eq. (3.77),

$$C = \left[1 - \frac{8}{\pi^2}\exp\left(-\frac{\pi^2 \times 10 \times 24 \times 3600 \times 1.13 \times 10^{-7}}{12.5 \times 12.5}\right)\right] \times (0.18 - 1.0) + 1.0 = 0.84\%$$

Thus in 10 days the moisture content would be reduced from 1% to 0.84%.

3.8 TYPICAL UNIDIRECTIONAL FIBER COMPOSITE PROPERTIES

The preceding discussions on property-prediction methods and failure modes should be of use in comparing various physical properties of unidirectional composites. It is also valuable to appreciate the difference between different types of fiber composites and their respective properties. From the known values of fiber and matrix properties (see Chap. 2), the interested reader can try to predict the properties shown in Table 3-1. A more detailed insight into mechanical behavior can be gained by reviewing the stress–strain curves presented in Appendix 4 for unidirectional composites.

A few points should be made concerning the data presented in Table 3-1. The measured compression strength in the fiber direction of a unidirectional composite generally is less than the tensile strength. Since compression strength of this type of composite is so difficult to measure, the reported value

Table 3-2 Summary of influence of constituents on properties of unidirectional polymer composites[a]

Composite Property	Fibers	Matrix	Interface
	Tensile Properties		
Longitudinal modulus	S	W	N
Longitudinal strength	S	W	N
Transverse modulus	W	S	N
Transverse strength	W	S	S
	Compression Properties		
Longitudinal modulus	S	W	N
Longitudinal strength	S	S	N
Transverse modulus	W	S	N
Transverse strength	W	S	N
	Shear Properties		
In-plane shear modulus	W	S	N
In-plane shear strength	W	S	S
Interlaminar shear strength	N	S	S

[a] S = strong influence; W = weak influence; N = negligible influence.

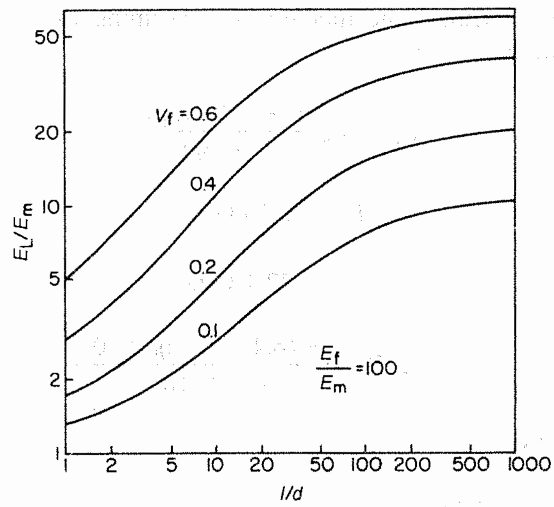

Figure 4-9. Dependence of longitudinal modulus of short-fiber composites on fiber aspect ratio (l/d) and fiber volume fraction (E_f/E_m = 100).

of the averaged properties also can be obtained through the analysis of a quasi-isotropic laminate made from unidirectional laminae. Construction of quasi-isotropic laminates and the analysis procedures will be discussed in Chap. 6.

The following empirical equations are often used to predict the elastic modulus and shear modulus of composites containing fibers that are randomly oriented in a plane:

$$E_{\text{random}} = \tfrac{3}{8}E_L + \tfrac{5}{8}E_T$$
$$G_{\text{random}} = \tfrac{1}{8}E_L + \tfrac{1}{4}E_T \qquad (4.15)$$

where E_L and E_T are, respectively, the longitudinal and transverse moduli of an aligned short-fiber composite having the same fiber aspect ratio and fiber volume fraction as the composite under consideration. Moduli E_L and E_T either can be determined experimentally or can be calculated using Eqs. (4.11) and (4.12).

Example 4-1: A glass-fiber-reinforced nylon with a fiber volume fraction of 20% is injection-molded to produce a random fiber orientation. The fiber length is 3.2 mm, and the fiber diameter is 10 μm. Calculate the elastic modulus, shear modulus, and Poisson's ratio of the random fiber composite.

Solution: First calculate the modulus E_L assuming that the fibers were oriented. Using Eq. (4.11),

$$E_L = E_m \frac{1 + (2l/d)\eta_L V_f}{1 - \eta_L V_f}$$

For nylon: $E_m = 2.76 \text{ GPa}$

Glass: $E_f = 72.4 \text{ GPa}$

$$E_L = 2.76 \frac{1 + (6.4/10^{-2})\eta_L \times 0.2}{1 - 0.2\eta_L}$$

Using Eq. (4.13),

$$\eta_L = \frac{(E_f/E_m) - 1}{(E_f/E_m) + (2l/d)}$$

$$= \frac{(72.4/2.76) - 1}{(72.4/2.76) + (6.4/10^{-2})} = \frac{25.23}{666.23} = 0.03787$$

Thus

$$E_L = 2.76 \frac{1 + 128 \times 0.03787}{1 - 0.03787 \times 0.2}$$

$$= 16.26 \text{ GPa}$$

Now calculate the transverse modulus E_T using Eq. (4.12):

$$E_T = E_m \frac{1 + 2\eta_T V_f}{1 - \eta_T V_f}$$

where

$$\eta_T = \frac{(E_f/E_m) - 1}{(E_f/E_m) + 2} = \frac{26.2 - 1}{26.2 + 2} = 0.89$$

$$E_T = 2.76 \frac{1 + 1.78 \times 0.2}{1 - 0.2 \times 0.89} = 4.53 \text{ GPa}$$

Now the elastic modulus and shear modulus of the random fiber composite can be calculated using Eq. (4.15):

$$E_R = \tfrac{3}{8} \times 16.26 + \tfrac{5}{8} \times 4.53 = 8.93 \text{ GPa}$$

$$G_R = \tfrac{1}{8} \times 16.26 + \tfrac{1}{4} \times 4.53 = 3.17 \text{ GPa}$$

Since a random fiber composite is considered isotropic in its plane, its in-plane Poisson's ratio can be calculated using the following isotropic relationship between E_R, G_R, and Poisson's ratio ν_R:

$$G_R = \frac{E_R}{2(1 + \nu_R)}$$

or

$$\nu_R = \frac{E_R}{2G_R} - 1$$

$$\nu_R = \frac{8.93}{2 \times 3.17} - 1 = 0.41$$

4.3.2 Prediction of Strength

The average longitudinal stress on an aligned short-fiber composite can be calculated by the rule of mixtures:

$$\sigma_c = \overline{\sigma}_f V_f + \sigma_m V_m \tag{4.16}$$

where $\overline{\sigma}_f$ is the average fiber stress given by Eq. (4.9). For a linear stress distribution at the fiber ends as shown in Fig. 4-3, values of $\overline{\sigma}_f$ are given by Eq. (4.10). Thus the average composite stress can be written as

$$\sigma_c = \tfrac{1}{2}(\sigma_f)_{max} V_f + \sigma_m V_m, \qquad l < l_t \tag{4.17}$$

and

$$\sigma_c = (\sigma_f)_{max}\left(1 - \frac{l_t}{2l}\right)V_f + \sigma_m V_m \qquad l > l_t \tag{4.18}$$

If the fiber length is much greater than the load-transfer length (e.g., $l = 100l_t$), the factor $1 - (l_t/l)$ approaches 1, and Eq. (4.18) can be written as

$$\sigma_c = (\sigma_f)_{max} V_f + \sigma_m V_m \qquad l \gg l_t \tag{4.19}$$

$$\epsilon_T = \frac{\sigma_T}{E_T} \tag{5.4}$$

$$\epsilon_L = -\nu_{TL}\epsilon_T = -\nu_{TL}\frac{\sigma_T}{E_T} \tag{5.5}$$

$$\gamma_{LT} = 0 \tag{5.6}$$

3. When τ_{LT} is the only nonzero stress ($\sigma_L = \sigma_T = 0$), the strains produced are

$$\epsilon_L = 0 \tag{5.7}$$

$$\epsilon_T = 0 \tag{5.8}$$

$$\gamma_{LT} = \frac{\tau_{LT}}{G_{LT}} \tag{5.9}$$

Superposition of these three states of stresses gives a most general state of stress on the lamina consisting of σ_L, σ_T, and τ_{LT}. In view of the assumption of linearly elastic material, the strains given by Eqs. (5.1)–(5.9) can be superposed to give the following relations:

$$\begin{aligned} \epsilon_L &= \frac{\sigma_L}{E_L} - \nu_{TL}\frac{\sigma_T}{E_T} \\ \epsilon_T &= \frac{\sigma_T}{E_T} - \nu_{LT}\frac{\sigma_L}{E_L} \\ \gamma_{LT} &= \frac{\tau_{LT}}{G_{LT}} \end{aligned} \tag{5.10}$$

Equations (5.10) are the stress–strain relations for a specially orthotropic lamina in terms of engineering constants. It may be noted that Eqs. (5.10) are similar to the stress–strain relations of an isotropic material under plane stress conditions. However, Eqs. (5.10) involve four independent elastic constants, whereas isotropic stress–strain relations under plane stress conditions require only two constants. Stress–strain relations of a generally orthotropic lamina (i.e., an orthotropic lamina referred to arbitrary axes) differ from Eqs. (5.10) and are discussed in the next section.

Example 5-1: A unidirectional composite is subjected to the following stresses:

$$\sigma_L = 3.0 \text{ MPa}, \quad \sigma_T = 0.5 \text{ MPa}, \quad \text{and} \quad \tau_{LT} = 3.5 \text{ MPa}$$

Find the normal and shear strains. Engineering constants are

$$E_L = 14.0 \text{ GPa}, \quad E_T = 3.5 \text{ GPa}, \quad G_{LT} = 4.2 \text{ GPa}$$

$$\nu_{LT} = 0.4 \quad \text{and} \quad \nu_{TL} = 0.1$$

Strains can be obtained by using Eq. (5.10):

$$\varepsilon_L = \frac{3.0}{14 \times 10^3} - 0.1 \times \frac{0.5}{3.5 \times 10^3} = 200 \times 10^{-6}$$

$$\varepsilon_T = \frac{0.5}{3.5 \times 10^3} - 0.4 \times \frac{3.0}{14 \times 10^3} = 57 \times 10^{-6}$$

$$\gamma_{LT} = \frac{3.5}{4.2 \times 10^3} = 833 \times 10^{-6}$$

5.2.2 Stress–Strain Relations for Generally Orthotropic Lamina

A generally orthotropic lamina is shown in Fig. 5-4, which has the principa material axes (L and T) oriented at an angle θ to the reference coordinate axes (x and y). In this case, engineering constants associated with the x anc y axes are required to relate stresses and strains. As explained earlier, thi: lamina responds to the loads like an anisotropic material. That is, norma stresses (σ_x and σ_y) produce shear strains (γ_{xy}) in addition to the normal strain: (ϵ_x and ϵ_y), and shear stress (τ_{xy}) produces normal strains (ϵ_x and ϵ_y) alon; with the shear strain (γ_{xy}). In other words, the shear and normal component: of stresses and strains are coupled. Therefore, in this case, in addition to th usual engineering constants associated with the x and y axes (E_x, E_y, G_{xy}, ν_{xy} and ν_{yx}), cross-coefficients m_x and m_y are also required to relate stresses an strains. These cross-coefficients relate the shear and normal components o stresses and strains. The significance of cross-coefficients will become clea from the stress–strain relations described below. It may be pointed out tha

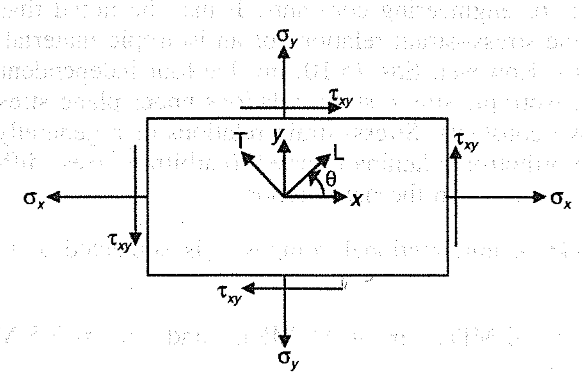

Figure 5-4. Generally orthotropic lamina with applied stresses.

$$G_{LT} < \frac{E_L}{2[(E_L/E_T) + \nu_{LT}]} \tag{5.39}$$

The inequality in Eq. (5.38) is not violated by any of the systems chosen earlier, but the inequality in Eq. (5.39) is violated by the boron–epoxy system and hence the behavior shown in Fig. 5-7. Similar conditions can be obtained for other material properties as well.

It is of practical interest to consider a lamina having identical properties in the two principal material directions (i.e., $E_L = E_T$ and $\nu_{LT} = \nu_{TL}$). Such a lamina is called a *balanced orthotropic lamina,* an example of which is a glass-fabric-reinforced material with equal volume fractions of fibers in two mutually perpendicular directions. Typical variations in the elastic constants of a balanced lamina are shown in Fig. 5-8. The elastic constants show symmetry in their variations about an orientation of 45° to the principal material axes. Fabric-reinforced laminae are of practical significance because with them almost any ratio of E_L/E_T can be established through the fabric weave (i.e., by changing the ratio of fiber volume fractions in the two mutually perpendicular directions).

Example 5-2: For the lamina shown in Fig. 5-9, find the strains in the xy directions. Lamina has the same engineering constants in the longitudinal and transverse directions as given in Example 5-1.

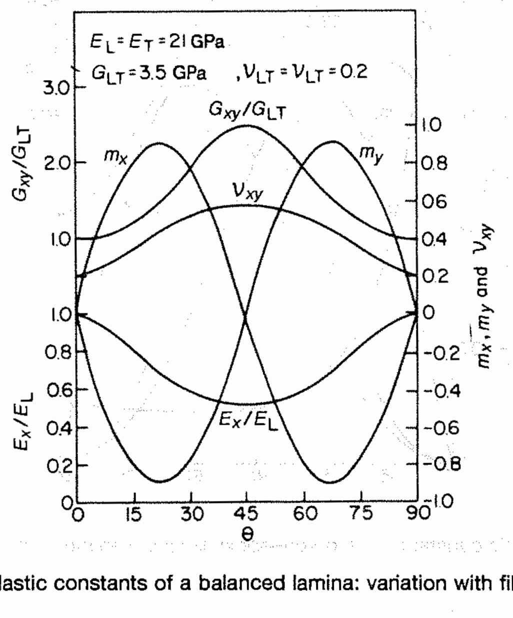

Figure 5-8. Elastic constants of a balanced lamina: variation with fiber orientation.

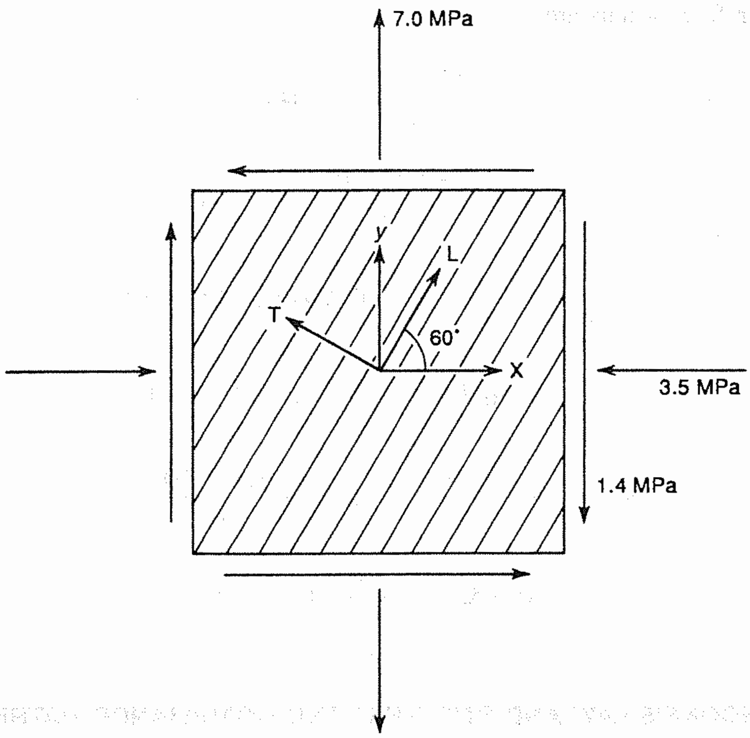

Figure 5-9. State of stress on a lamina for Example 5-2.

From Fig. 5-9, the stresses are

$$\sigma_x = -3.5 \text{ MPa}$$

$$\sigma_y = 7.0 \text{ MPa}$$

$$\tau_{xy} = 1.4 \text{ MPa}$$

Engineering constants in the x and y directions are required to calculate strains. These are obtained by substituting $\theta = 60°$ in Eqs. (5.25)–(5.28) and (5.36). The transformed constants are

$$E_x = 5.02 \text{ GPa}, \quad E_y = 10.87 \text{ GPa}, \quad G_{xy} = 2.70 \text{ GPa}$$

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} = -0.00446 \text{ GPa}^{-1}$$

$$m_x = 1.833, \quad m_y = 0.765$$

Therefore, strains are

$$\epsilon_x = \frac{-3.5 \times 10^{-3}}{5.02} - (-0.00446)(7.0 \times 10^{-3})$$

$$- 1.833\left(\frac{-1.4 \times 10^{-3}}{14}\right) = -483 \times 10^{-6}$$

$$\epsilon_y = \frac{7.0 \times 10^{-3}}{10.87} - (-0.00446)(-3.5 \times 10^{-3})$$

$$- 0.765\left(\frac{-1.4 \times 10^{-3}}{14}\right) = 705 \times 10^{-6}$$

$$\gamma_{xy} = \frac{-1.4 \times 10^{-3}}{2.70} - 1.833\left(\frac{-3.5 \times 10^{-3}}{14}\right)$$

$$- 0.765\left(\frac{7.0 \times 10^{-3}}{14}\right) = -443 \times 10^{-6}$$

5.3 HOOKE'S LAW AND STIFFNESS AND COMPLIANCE MATRICES

5.3.1 General Anisotropic Material

In general, the state of stress at a point in a body is described by the ni components of the stress tensor σ_{ij}, as shown in Fig. 5-10. Correspondingl there is a strain tensor, ϵ_{ij}, with nine components.

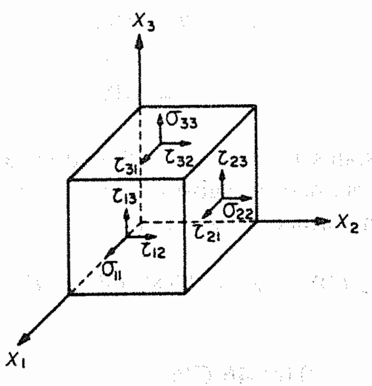

Figure 5-10. Components of stress tensor on a cube element.

$$
\begin{aligned}
Q_{11} &= \frac{E_L}{1 - \nu_{LT}\nu_{TL}} \\
Q_{22} &= \frac{E_T}{1 - \nu_{LT}\nu_{TL}} \\
Q_{12} &= \frac{\nu_{LT}E_T}{1 - \nu_{LT}\nu_{TL}} = \frac{\nu_{TL}E_L}{1 - \nu_{LT}\nu_{TL}} \\
Q_{66} &= G_{LT}
\end{aligned}
\tag{5.78}
$$

It may be pointed out that although five engineering constants have been mentioned, only four of them are independent. The following functional relationship, which is evident from Eq. (5.78), exists between four of the five constants:

$$\nu_{LT}E_T = \nu_{TL}E_L$$

or

$$\frac{\nu_{LT}}{E_L} = \frac{\nu_{TL}}{E_T} \tag{5.79}$$

Engineering constants for a number of commercially available composites are given in Appendix 4.

Example 5-3: Determine the stiffness and compliance matrices for a unidirectional AS4/3501-6 graphite–epoxy lamina that has the following engineering constants:

$$E_L = 148.0 \text{ GPa}, \quad E_T = 10.5 \text{ GPa}$$

$$G_{LT} = 5.61 \text{ GPa}, \quad \nu_{LT} = 0.3$$

From Eq. (5.79),

$$\nu_{TL} = \frac{0.3 \times 10.5}{148.0} = 0.021$$

Elements of the stiffness matrix are obtained by using Eq. (5.78):

$$Q_{11} = \frac{148.0}{1 - 0.3 \times 0.021} = 148.95 \text{ GPa}$$

$$Q_{22} = \frac{10.5}{1 - 0.3 \times 0.021} = 10.57 \text{ GPa}$$

$$Q_{12} = \frac{0.3 \times 10.5}{1 - 0.3 \times 0.021} = 3.17 \text{ GPa}$$

$$Q_{66} = 5.61 \text{ GPa}$$

Thus

$$[Q] = \begin{bmatrix} 148.95 & 3.17 & 0 \\ 3.17 & 10.57 & 0 \\ 0 & 0 & 5.61 \end{bmatrix} \text{GPa}$$

Elements of the compliance matrix are obtained by using Eq. (5.77):

$$S_{11} = \frac{1}{148.0} = 0.0068 \text{ GPa}^{-1}$$

$$S_{22} = \frac{1}{10.5} = 0.0952 \text{ GPa}^{-1}$$

$$S_{12} = -\frac{0.3}{148.0} = -0.0020 \text{ GPa}^{-1}$$

$$S_{66} = \frac{1}{5.61} = 0.1783 \text{ GPa}^{-1}$$

Thus

$$[S] = \begin{bmatrix} 0.0068 & -0.0020 & 0 \\ -0.0020 & 0.0952 & 0 \\ 0 & 0 & 0.1783 \end{bmatrix} \text{GPa}^{-1}$$

5.3.8 Restrictions on Elastic Constants

It was pointed out in an earlier section that for an orthotropic material, three-dimensional stress–strain relations require nine independent elastic constants and two-dimensional relations require four constants. For isotropic materials, the number of independent elastic constants is only two for both the two- and three-dimensional stress–strain relations. Consequently, for characterization purposes, more measurements have to be made for an orthotropic material

The $[\overline{Q}]$ matrix is now fully populated and similar in appearance to th
$[Q]$ matrix for a fully anisotropic lamina ($\overline{Q}_{16} \neq 0$, $\overline{Q}_{26} \neq 0$). It would see
that there are now six elastic constants that govern the behavior of the lamin
However, $\overline{Q}_{16}$ and $\overline{Q}_{26}$ are not independent but merely linear combinations
the four basic elastic constants. Sometimes Eq. (5.74) is referred to as th
constitutive equation for "specially" orthotropic lamina because $Q_{16} = Q$
$= 0$, and Eq. (5.94), as the constitutive equation for a generally" orthotrop
lamina, although both of them apply to the same lamina.

The inverse stress–strain relations referred to arbitrary axes now can b
written as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{5.96}$$

In a manner similar to the one adopted for obtaining the elements of th
$[\overline{Q}]$ matrix in terms of the elements of the $[Q]$ matrix, the elements of th
$[\overline{S}]$ matrix also can be obtained in terms of the elements of the complianc
matrix. The result is as follows:

$$\begin{aligned}
\overline{S}_{11} &= S_{11} \cos^4\theta + S_{22} \sin^4\theta + (2S_{12} + S_{66}) \sin^2\theta \cos^2\theta \\
\overline{S}_{22} &= S_{11} \sin^4\theta + S_{22} \cos^4\theta + (2S_{12} + S_{66}) \sin^2\theta \cos^2\theta \\
\overline{S}_{12} &= (S_{11} + S_{22} - S_{66}) \cos^2\theta \sin^2\theta + S_{12} (\cos^4\theta + \sin^4\theta) \\
\overline{S}_{66} &= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66}) \cos^2\theta \sin^2\theta + S_{66} (\cos^4\theta + \sin^4\theta) \\
\overline{S}_{16} &= (2S_{11} - 2S_{12} - S_{66}) \cos^3\theta \sin\theta - (2S_{22} - 2S_{12} - S_{66}) \cos\theta \sin^3\theta \\
\overline{S}_{26} &= (2S_{11} - 2S_{12} - S_{66}) \cos\theta \sin^3\theta - (2S_{22} - 2S_{12} - S_{66}) \cos^3\theta \sin\theta
\end{aligned} \tag{5.97}$$

Example 5-4: For the lamina considered in Example 5-2 (Fig. 5-9), firs
find the stresses and strains in the longitudinal and transverse direction
and then transform the strains to the x and y directions. Compare result
with those of Example 5-2.

Stresses in the longitudinal and transverse directions may be obtained fron
Eq. (5.86) by substituting $\theta = 60°$:

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} 0.25 & 0.75 & 0.866 \\ 0.75 & 0.25 & -0.866 \\ -0.433 & 0.433 & -0.5 \end{bmatrix} \begin{Bmatrix} -3.5 \\ 7.0 \\ -1.4 \end{Bmatrix} = \begin{Bmatrix} 3.16 \\ 0.34 \\ 5.24 \end{Bmatrix}$$

$$\sigma_L = 3.16 \text{ MPa}$$

$$\sigma_T = 0.34 \text{ MPa}$$

$$\tau_{LT} = 5.24 \text{ MPa}$$

Strains in the longitudinal and transverse directions may be obtained from Eq. (5.10) as follows:

$$\varepsilon_L = \frac{3.16 \times 10^6}{14 \times 10^9} - 0.1\left(\frac{0.34 \times 10^6}{3.5 \times 10^9}\right) = 216 \times 10^{-6}$$

$$\varepsilon_T = \frac{0.34 \times 10^6}{3.5 \times 10^9} - 0.4\left(\frac{3.16 \times 10^6}{14 \times 10^9}\right) = 6.9 \times 10^{-6}$$

$$\gamma_{LT} = \frac{5.24 \times 10^6}{4.2 \times 10^9} = 1248 \times 10^{-6}$$

The preceding strains now can be transformed to obtain strains in the x and y directions using the inverse of Eq. (5.88) as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 0.25 & 0.75 & -0.433 \\ 0.75 & 0.25 & 0.433 \\ 0.866 & -0.866 & -0.5 \end{bmatrix} \begin{Bmatrix} 216 \times 10^{-6} \\ 6.9 \times 10^{-6} \\ 1248 \times 10^{-6} \end{Bmatrix}$$

$$\varepsilon_x = -481 \times 10^{-6}$$

$$\varepsilon_y = 704 \times 10^{-6}$$

$$\gamma_{xy} = -442 \times 10^{-6}$$

These strains differ slightly from those calculated in Example 5-2 only because of rounding-off errors.

Example 5-5: If longitudinal and transverse axes of the lamina considered in Example 5-3 make a counterclockwise angle of 30° with the reference axes, determine $\overline{Q}$ and $\overline{S}$ matrices.

Substitution of $\theta = 30°$ in Eqs. (5.87) and (5.89) gives stress- and strain transformation matrices as

$$[T_1] = \begin{bmatrix} 0.750 & 0.250 & 0.866 \\ 0.250 & 0.750 & -0.866 \\ -0.433 & 0.433 & 0.500 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} 0.750 & 0.250 & 0.433 \\ 0.250 & 0.750 & -0.433 \\ -0.866 & 0.866 & 0.500 \end{bmatrix}$$

Inversion of $[T_1]$ and $[T_2]$ gives

$$[T_1]^{-1} = \begin{bmatrix} 0.750 & 0.250 & -0.866 \\ 0.250 & 0.750 & 0.866 \\ 0.433 & -0.433 & 0.500 \end{bmatrix}$$

$$[T_2]^{-1} = \begin{bmatrix} 0.750 & 0.250 & -0.433 \\ 0.250 & 0.750 & 0.433 \\ 0.866 & -0.866 & 0.500 \end{bmatrix}$$

From Eq. (5.93),

$$[\bar{Q}] = [T_1]^{-1}\,[Q]\,[T_2]$$

Substitution of $[T_1]^{-1}$, $[Q]$, and $[T_2]$ gives

$$[\bar{Q}] = \begin{bmatrix} 89.84 & 27.68 & 44.11 \\ 27.68 & 20.65 & 15.81 \\ 44.11 & 15.81 & 30.12 \end{bmatrix} \text{GPa}$$

Similarly,

$$[\bar{S}] = [T_2]^{-1}\,[S][T_1]$$

Substitution of $[T_2]^{-1}$, $[S]$, and $[T_1]$ gives

$$[\bar{S}] = \begin{bmatrix} 0.0424 & -0.0156 & -0.0539 \\ -0.0156 & 0.0867 & -0.0227 \\ -0.0539 & -0.0227 & 0.1241 \end{bmatrix} \text{GPa}^{-1}$$

5.3.10 Invariant Forms of Stiffness and Compliance Matrices

Design of laminates invariably requires a decision on constituent lamina orientations to meet stiffness and strength requirements in different directions

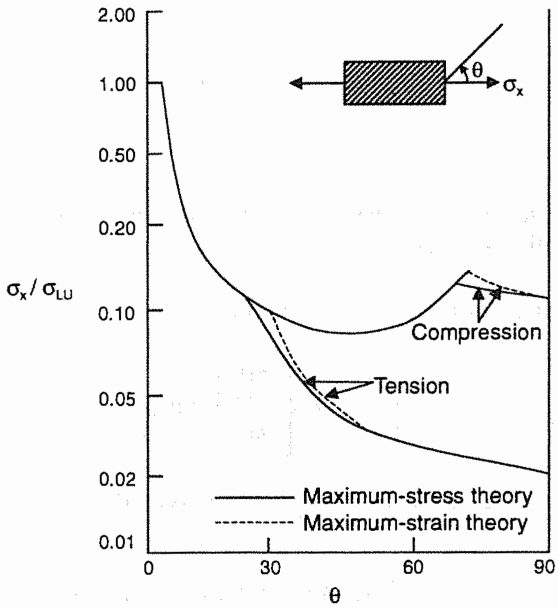

Figure 5-13. Off-axis strength predicted by maximum-stress and maximum-strain theories of failure.

Example 5-6: A unidirectional glass–epoxy lamina, shown in Fig. 5-14, has the following allowable stresses:

$$\sigma_{LU} = 1062 \text{ MPa}$$

$$\sigma'_{LU} = 610 \text{ MPa}$$

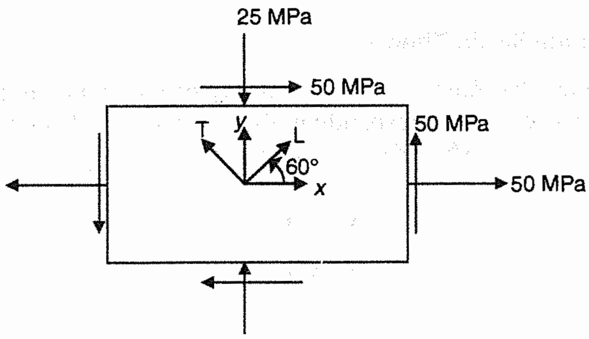

Figure 5-14. State of stress on a glass–epoxy lamina for Example 5-6.

$$\sigma_{TU} = 31 \text{ MPa}$$

$$\sigma'_{TU} = 118 \text{ MPa}$$

$$\tau_{LTU} = 72 \text{ MPa}$$

Determine if, according to the maximum-stress theory, the lamina will fail under the applied stresses.

The off-axis stresses applied to the lamina are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 50 \\ -25 \\ 50 \end{Bmatrix} \text{ MPa}$$

Since the fibers are oriented at 60° to the x axis, the stress-transformation matrix is

$$[T_1] = \begin{bmatrix} 0.250 & 0.750 & 0.866 \\ 0.750 & 0.250 & -0.866 \\ -0.433 & 0.433 & -0.500 \end{bmatrix}$$

Stress transformation gives

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} 0.250 & 0.750 & 0.866 \\ 0.750 & 0.250 & -0.866 \\ -0.433 & 0.433 & -0.500 \end{bmatrix} \begin{Bmatrix} 50 \\ -25 \\ 50 \end{Bmatrix} = \begin{Bmatrix} 37.05 \\ -12.05 \\ -57.48 \end{Bmatrix} \text{ MPa}$$

Comparison of allowable stresses σ_{LU}, σ'_{TU}, and τ_{LTU} with the calculated stresses σ_L, σ_T, and τ_{LT} shows that, according to the maximum-stress theory, the lamina will not fail under the applied stresses.

5.4.2 Maximum-Strain Theory

This theory states that failure will occur if any of the strains in the principal material axes exceed the corresponding allowable strain. Thus the following inequalities must be satisfied for "no failure":

$$\begin{aligned} \epsilon_L &< \epsilon_{LU} \\ \epsilon_T &< \epsilon_{TU} \\ \gamma_{LT} &< \gamma_{LTU} \end{aligned} \qquad (5.105)$$

If normal strains are compressive, ϵ_{LU} and ϵ_{TU} in Eq. (5.105) must be replaced by the allowable compressive strains:

Example 5-7: The lamina considered in Example 5-6 has the following elastic constants:

$$E_L = 38.6 \text{ GPa}$$

$$E_T = 8.27 \text{ GPa}$$

$$\nu_{LT} = 0.26$$

$$G_{LT} = 4.14 \text{ GPa}$$

Determine if, according to the maximum-strain theory, the lamina will fail. Assume that the lamina deforms linearly up to failure.

Lamina strains can be obtained using Eq. (5.10) and the stresses σ_L, σ_T, and τ_{LT} calculated in Example 5-6:

$$\varepsilon_L = \frac{37.05}{38.6 \times 10^3} - \frac{0.26}{38.6 \times 10^3} \times (-12.05) = 0.00104$$

$$\varepsilon_T = -\frac{12.05}{8.27 \times 10^3} - \left(\frac{0.26}{38.6 \times 10^3}\right) \times (37.05) = -0.00170$$

$$\gamma_{LT} = -\frac{57.48}{4.14 \times 10^3} = -0.01388$$

The lamina allowable strains can be obtained from allowable stresses and elastic constants as follows:

$$\varepsilon_{LU} = \frac{\sigma_{LU}}{E_L} = 0.0275$$

$$\varepsilon'_{LU} = \frac{\sigma'_{LU}}{E_L} = 0.0158$$

$$\varepsilon_{TU} = \frac{\sigma_{TU}}{E_T} = 0.0037$$

$$\varepsilon'_{TU} = \frac{\sigma'_{TU}}{E_T} = 0.0143$$

$$\gamma_{LTU} = \frac{\tau_{LTU}}{G_{LT}} = 0.0174$$

Comparison of allowable strains ε_{LU}, ε'_{TU}, and γ_{LTU} with the calculated strains ε_L, ε_T, and γ_{LT} shows that, according to the maximum-strain theory, the lamina will not fail.

5.4.3 Maximum-Work Theory

This theory states that in plane-stress states the failure initiates when the following inequality is violated:

$$\left(\frac{\sigma_L}{\sigma_{LU}}\right)^2 - \left(\frac{\sigma_L}{\sigma_{LU}}\right)\left(\frac{\sigma_T}{\sigma_{LU}}\right) + \left(\frac{\sigma_T}{\sigma_{TU}}\right)^2 + \left(\frac{\tau_{LT}}{\tau_{LTU}}\right)^2 < 1 \qquad (5.109)$$

When normal stresses are compressive, the corresponding compressive strengths are to be used in Eq. (5.109).

The theory was derived in this form by Tsai [6] from a yield criterion for anisotropic materials proposed by Hill [7]. Therefore, it is sometimes referred to as the *Tsai–Hill theory.* Application of this theory can be illustrated by the example used in the previous cases where an off-axis stress σ_x acts on an orthotropic lamina. The stresses in the principal material directions given by Eq. (5.104) can be substituted directly into Eq. (5.109) to obtain the failure criterion as

$$\frac{\cos^4\theta}{\sigma_{LU}^2} - \frac{\cos^2\theta\sin^2\theta}{\sigma_{LU}^2} + \frac{\sin^4\theta}{\sigma_{TU}^2} + \frac{\sin^2\theta\cos^2\theta}{\tau_{LTU}^2} < \frac{1}{\sigma_x^2} \qquad (5.110)$$

Thus the maximum-work theory provides a single function to predict strength. This criterion does take into consideration the interaction between strengths, which was not considered in the maximum-stress or maximum-strain theory. Predictions of the maximum-work theory for the glass–epoxy composite considered earlier are shown in Fig. 5-15. Also shown in the figure are the predictions of the maximum-stress theory. The maximum-work theory predicts slightly lower strength compared with that predicted by the maximum-stress theory. The largest differences occur at the points where the maximum-stress theory predicts a change in the failure mode, that is, from the shear mode to the longitudinal or transverse tensile failure mode. The maximum-work theory has found wider acceptability compared with the other two theories primarily because of the smooth variation of strength according to a single equation [Eq. (5.109)]. Experimental support for this theory has been reported by many investigators [8–12].

The theories discussed in the preceding paragraphs are applicable to cases where the state of stress is biaxial. When all three stresses σ_x, σ_y, and τ_{xy} are acting at a point, the theories may be applied in a manner similar to the one used earlier for a uniaxial off-axis stress. The first step in the application of the theories is to transform the stresses σ_x, σ_y, and τ_{xy} to the stresses σ_L, σ_T, three different values of $(\sigma_{LU}/\sigma_{TU})$ are shown in Fig. 5-16. When strengths in the longitudinal and transverse directions are equal $(\sigma_{LU}/\sigma_{TU} = 1)$, the failure ellipse represents a case of an isotropic material. In the other extreme case as the ratio $(\sigma_{LU}/\sigma_{TU})$ approaches an infinite value (e.g., when the transverse strength is negligibly small), the failure envelope becomes a circle. For a practical case of an orthotropic material in which the ratio of strengths has a value between one and infinity, the failure envelope will be an ellipse lying between the ellipse for an isotropic case and the circle. Influence of a nonzero shear stress τ_{LT} is shown in Fig. 5-17. For a fixed value of σ_{LU}/σ_{TU}), the effect of shear stress is to reduce the size of the failure envelope. As the value of τ_{LT}/τ_{LTU} increases, the major and minor axes of the failure ellipse become smaller with no change in orientations.

Example 5-8: Determine if, according to the maximum-work theory, the lamina in Example 5-6 will fail under the applied stresses.

To apply the maximum-work theory, the left-hand side of Eq. (5.109) can be evaluated as follows:

$$\left(\frac{37.05}{1062}\right)^2 - \left(\frac{37.05}{1062}\right)\left(\frac{12.05}{610}\right) + \left(\frac{12.05}{118}\right)^2 + \left(\frac{57.48}{72}\right)^2 = 0.65 < 1$$

Therefore, according to the maximum-work theory, the lamina will not fail under the applied stresses.

5.4.4 Importance of Sign of Shear Stress on Strength of Composites

A sign convention for stresses that is almost universally accepted is discussed in Appendix 2. It can be stated as "on a plane where the outward normal is in the positive direction of a coordinate axis, all the stress components acting in the positive directions of the axes are positive." According to this conven-

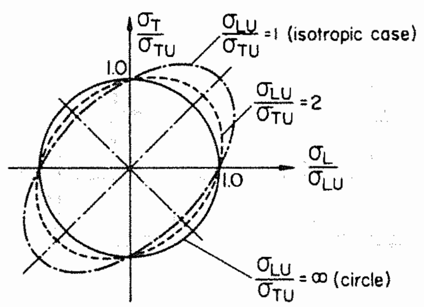

Figure 5-16. Failure envelopes on a normalized stress plane for zero shear stress.

transverse direction. Since the stresses in the fiber direction and perpendicul to it are of equal magnitude in both cases, it is reasonable to assume that tl shear strength is largely controlled by the transverse strength of the lamii (longitudinal strength generally is much greater). Thus, for this 45° lamin the apparent shear strength for a negative shear stress will be lower than th for a positive shear stress because transverse tensile strength of a lamii generally is smaller than the compressive strength.

Similar arguments can be extended to other fiber orientations. Thus oi axis shear strength of a lamina depends not only on the fiber orientation b also on the sign of applied shear stress. The following numerical examp illustrates the point further.

Example 5-9: A unidirectional lamina of glass–epoxy composite sho\ the following strength properties:

$$\sigma_{LU} = 500 \text{ MPa}$$

$$\sigma'_{LU} = 350 \text{ MPa}$$

$$\sigma_{TU} = 5 \text{ MPa}$$

$$\sigma'_{TU} = 75 \text{ MPa}$$

$$\tau_{LTU} = 35 \text{ MPa}$$

Estimate off-axis shear strength of the lamina for fiber orientations of 1! 45° and 60°. Failure may be predicted by using the Tsai–Hill criterion.

When τ_{xy} is the only nonzero stress, the stress in the longitudinal aı transverse directions may be obtained as

$$\sigma_L = \tau_{xy} \sin 2\theta$$

$$\sigma_T = -\tau_{xy} \sin 2\theta$$

$$\tau_{LT} = \tau_{xy} \cos 2\theta$$

When the sign of applied shear stress is positive, σ_L will be tensile and ı compressive. In that case, the Tsai–Hill failure criterion may be written

$$\frac{\tau_{xy}^2 \sin^2 2\theta}{500 \times 500} - \frac{\tau_{xy}^2 \sin^2 2\theta}{500 \times 350} + \frac{\tau_{xy}^2 \sin^2 2\theta}{75 \times 75} + \frac{\tau_{xy}^2 \cos^2 2\theta}{35 \times 35} = 1$$

When the sign of applied stress is negative, σ_L will be compressive aı σ_T tensile. In that case, the failure criterion may be written as

$$\frac{\tau_{xy}^2 \sin^2 2\theta}{350 \times 350} - \frac{\tau_{xy}^2 \sin^2 2\theta}{500 \times 350} + \frac{\tau_{xy}^2 \sin^2 2\theta}{5 \times 5} + \frac{\tau_{xy}^2 \cos^2 2\theta}{35 \times 35} = 1$$

By substituting different values of θ, the following shear strengths are obtained:

$\theta =$	15°	45°	60°
$+\tau_{xy} =$	39.03	75.36	54.54
$-\tau_{xy} =$	9.71	5.00	5.75

EXERCISE PROBLEMS

5.1. Derive expressions for E_y, ν_{yx}, and m_y by assuming that σ_y, is the only nonzero stress acting on the lamina shown in Fig. 5-4. Compare your results with Eqs. (5.26), (5.28), and (5.32).

5.2. Derive expressions for m_x and m_y by assuming that τ_{xy} is the only nonzero stress acting on the lamina shown in Fig. 5-4. Compare your results with Eqs. (5.30) and (5.32).

5.3. Plot the variation of E_x, G_{xy}, ν_{xy}, m_x, and m_y for a lamina with the following properties:

$$E_L = 35 \text{ GPa} \qquad E_T = 3.5 \text{ GPa}$$

$$G_{LT} = 4 \text{ GPa} \qquad \nu_{LT} = 0.45$$

5.4. Plot the variations of E_x, G_{xy}, ν_{xy}, m_x, and m_y for a balanced lamina with the following properties:

$$E_L = E_T = 15 \text{ GPa}$$

$$G_{LT} = 2.5 \text{ GPa}$$

$$\nu_{LT} = \nu_{TL} = 0.20$$

5.5. Calculate E_x, G_{xy}, ν_{xy}, m_x, and m_y at 30°, 45°, and 60° for an orthotropic lamina having the following properties:

$$E_L = 14 \text{ GPa} \qquad E_T = 3.5 \text{ GPa}$$

$$G_{LT} = 4.2 \text{ GPa} \qquad \nu_{LT} = 0.4$$

By introducing definitions of three new matrices, Eqs. (6.16) and (6.17) can be rewritten in a relatively simple form as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{6.18}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{6.19}$$

where

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k - h_{k-1}) \\ B_{ij} &= \tfrac{1}{2} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2) \\ D_{ij} &= \tfrac{1}{3} \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3) \end{aligned} \tag{6.20}$$

Combining Eqs. (6.18) and (6.19), the total plate constitutive equation can be written as follows:

$$\begin{Bmatrix} N \\ \hdashline M \end{Bmatrix} = \left[\begin{array}{c:c} A & B \\ \hdashline B & D \end{array}\right] \begin{Bmatrix} \epsilon^0 \\ \hdashline k \end{Bmatrix} \tag{6.21}$$

In Eqs. (6.18)–(6.21), the matrices A, B, and D are called the *extensional stiffness matrix, coupling stiffness matrix,* and *bending stiffness matrix,* respectively. The extensional stiffness matrix relates the resultant forces to the midplane strains, and the bending stiffness matrix relates the resultant moments to the plate curvatures.

The presence of the coupling matrix $[B]$ in the plate constitutive equation implies coupling between bending and extension of a laminated plate. That is, normal and shear forces acting at the midplane of the plate result in not only the in-plane deformations, leading to the midplane strains, but also twisting and bending, producing plate curvatures. Similarly, bending and twisting moments are accompanied by midplane strains. Thus stretching a laminate that has nonzero B_{ij} terms will produce bending and/or twisting of the laminate in addition to the extensional and shear deformation.

Example 6-1: Consider a two-ply laminate with the ply orientations of 0° and 45° with the laminate axes as shown in Fig. 6-6. The bottom lamina is a 0° layer with a thickness of 5 mm, whereas the 45° top lamina is 3

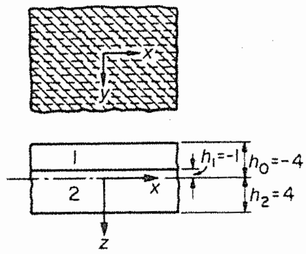

Figure 6-6. Two-ply laminate for Example 6-1.

mm thick. Evaluate A, B, and D matrices for the laminate if both the laminae have identical stiffness matrix Q as follows:

$$[Q] = \begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \text{GPa}$$

It may be pointed out that the units of the elements in the Q matrix are the same as the units of stress (e.g., gigapascals, as indicated in this example). The units of elements in the A matrix are those of stress times length; they are stress times length squared in the B matrix and stress times length cubed in the D matrix. In the examples considered here, laminae thicknesses are given in millimeters. Therefore, the units of the elements in the A, B, and D matrices will be GPa · mm, GPa · mm^2, and GPa · mm^3, respectively. These units are not being indicated in Examples 6-1–6-5 because these examples are used only to illustrate calculations that are not affected by the units. However, when laminate stresses and strains are to be calculated (as in Example 6-7), consistent units must be used.

Evaluation of matrices A, B, and D requires finding the $\overline{Q}$ matrices for the two layers. For the 0° lamina the Q and $\overline{Q}$ matrices are the same; that is,

$$[\overline{Q}]_{0^\circ} = [Q]_{0^\circ} = \begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

The $\overline{Q}_{ij}$ terms for the 45° lamina are found by using the transformation [Eq. (5.61)]:

$$\overline{Q}_{11} = 20(\cos 45)^4 + 2(\sin 45)^4 + 2(0.7 + 2 \times 0.7)(\sin 45)^2(\cos 45)^2$$

$$= 6.55$$

$$\overline{Q}_{22} = 20(\sin 45)^4 + 2(\cos 45)^4 + 2(0.7 + 2 \times 0.7)(\sin 45)^2(\cos 45)^2$$

$$= 6.55$$

$$\overline{Q}_{12} = \left(\frac{1}{\sqrt{2}}\right)^4 [(20 + 2 - 4 \times 0.7) + 2 \times 0.7] = 5.15$$

$$\overline{Q}_{66} = \left(\frac{1}{\sqrt{2}}\right)^4 [(20 + 2 - 2 \times 0.7 - 2 \times 0.7) + 2 \times 0.7] = 5.15$$

$$\overline{Q}_{16} = \left(\frac{1}{\sqrt{2}}\right)^4 [(20 - 0.7 - 2 \times 0.7) - (2 - 0.7 - 2 \times 0.7)] = 4.50$$

$$\overline{Q}_{26} = \left(\frac{1}{\sqrt{2}}\right)^4 [(20 - 0.7 - 2 \times 0.7) - (2 - 0.7 - 2 \times 0.7)] = 4.50$$

Therefore,

$$[\overline{Q}]_{45^\circ} = \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix}$$

Now the basic terms in Eq. (6.20) are known ($h_0 = -4.0$, $h_1 = -1.0$, and $h_2 = 4.0$, as shown in Fig. 6-6). Thus the terms of the laminate stiffness matrices A, B, and D can be calculated as follows:

$$A_{ij} = \sum_{k=1}^{n} (\overline{Q}_{ij})_k (h_k - h_{k-1}) = (\overline{Q}_{ij})_{45^\circ}[(-1) - (-4)] + (\overline{Q}_{ij})_{0^\circ}[4 - (-1)]$$

or

$$A_{ij} = 3(\overline{Q}_{ij})_{45^\circ} + 5(\overline{Q}_{ij})_{0^\circ}$$

Thus

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} = 3\begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix} + 5\begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 119.65 & 18.95 & 13.50 \\ 18.95 & 29.65 & 13.50 \\ 13.50 & 13.50 & 18.95 \end{bmatrix}$$

$$\begin{aligned} B_{ij} &= \tfrac{1}{2}\sum_{k=1}^{n} (\overline{Q}_{ij})_k(h_k^2 - h_{k-1}^2) = \tfrac{1}{2}(\overline{Q}_{ij})_{45^\circ}[(-1)^2 - (-4)^2] \\ &\quad + \tfrac{1}{2}(\overline{Q}_{ij})_{0^\circ}[(4)^2 - (-1)^2] \\ &= 7.5[-(\overline{Q}_{ij})_{45^\circ} + (\overline{Q}_{ij})_{0^\circ}] \end{aligned}$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} = 7.5\left[\begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} - \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix}\right]$$

$$[B] = \begin{bmatrix} 100.9 & -33.4 & -33.75 \\ -33.4 & -34.1 & -33.75 \\ -33.75 & -33.75 & -33.40 \end{bmatrix}$$

$$\begin{aligned} D_{ij} &= \tfrac{1}{3}\sum_{k=1}^{n} (\overline{Q}_{ij})_k(h_k^3 - h_{k-1}^3) = \tfrac{1}{3}(\overline{Q}_{ij})_{45^\circ}[(-1)^3 - (-4)^3] \\ &\quad + \tfrac{1}{3}(\overline{Q}_{ij})_{0^\circ}[(4)^3 - (-1)^3] \\ &= 21(\overline{Q}_{ij})_{45^\circ} + 21.67(\overline{Q}_{ij})_{0^\circ} \end{aligned}$$

Thus

$$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = 21\begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix} + 21.67\begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 571 & 123 & 94.5 \\ 123 & 181 & 94.5 \\ 94.5 & 94.5 & 123 \end{bmatrix}$$

Combining the preceding results, the total set of constitutive equatio for this particular two-ply laminate can be written as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \hdashline M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc:ccc} 119.6 & 18.9 & 13.5 & 100.9 & -33.4 & -33.8 \\ 18.9 & 29.6 & 13.5 & -33.4 & -34.1 & -33.8 \\ 13.5 & 13.5 & 18.9 & -33.8 & -33.8 & -33.4 \\ \hdashline 100.9 & -33.4 & -33.8 & 571 & 123 & 94.5 \\ -33.4 & -34.1 & -33.8 & 123 & 181 & 94.5 \\ -33.8 & -33.8 & -33.4 & 94.5 & 94.5 & 123 \end{array}\right] \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \hdashline k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

None of the elements of the A, B, and D matrices is zero for the laminate considered in this example. Such laminates are the most general laminates.

6.5 LAMINATE DESCRIPTION SYSTEM

Laminate properties and characteristics are influenced directly by the laminate makeup. It is therefore necessary to adopt a laminate description system that will provide a positive identification of the laminate makeup. A positive identification of a laminate requires the following:

1. Orientation of each lamina relative to a reference axis (the x axis in this text)
2. Number of laminae at each orientation
3. The exact geometric sequence of laminae

A laminate orientation code that provides a positive and concise identification of the laminates is described in Appendix 3. Basic features of the code are discussed here.

In the standard laminate code, it is assumed that all laminae are identical in thickness and properties. Following are the elements of the code:

1. Each lamina is denoted by a number representing the angle in degrees between its fiber direction and the x axis.
2. Individual adjacent laminae are separated in the code by a slash if their angles are different.
3. The laminae are listed in sequence from one laminate face to the other, starting with the first lamina laid up, with brackets indicating the beginning and end of the code.
4. Adjacent laminae of the same orientation are denoted by a numerical subscript.
5. The laminate possessing symmetry of laminae orientations about the geometric midplane requires specifying only one-half the laminate stacking sequence. A subscript s to the code signifies that only one-half

where E is the elastic modulus of the material, ν is the Poisson ratio, and is the plate thickness. It may be noted from Eq. (6.24) that for the isotrop plates,

$$A_{11} = A_{22}$$

$$A_{11} - A_{12} = 2A_{66} \qquad (6.2\text{:}$$

$$A_{16} = A_{26} = 0$$

Therefore, the $[A]$ matrix for a quasi-isotropic material also should satisfy E (6.25). It can be shown that a laminate constructed by meeting the followin conditions will be quasi-isotropic:

1. The total number of layers must be three or more.
2. The individual layers must have identical stiffness matrices $[Q]$ an thicknesses.
3. The layers must be oriented at equal angles. For example, if the tot number of layers is n, the angle between two adjacent layers should b π/n. If a laminate is constructed from identical sets of three or mo layers each, the condition on orientation must be satisfied by the laye in each set.

Since a laminate constructed according to the preceding design is isotrop with regard to extensional stiffness matrix $[A]$ and not, in general, with rega to coupling and bending stiffness matrices ($[B]$ and $[D]$), this design is calle *quasi-isotropic.*

The concept of a quasi-isotropic laminate is very helpful in predicting th properties of randomly oriented short-fiber composites, as discussed in Cha 4. A randomly oriented short-fiber composite may be modeled as a lamina having an infinite number of plies with continuously varying orientations. I practice, the properties are calculated from a quasi-isotropic laminate co sisting of three or four plies only. It is left to the reader to show that [0 ±60] and [0/±45/90] laminates are quasi-isotropic (the modulus an strength of randomly oriented short-fiber composites can be analyzed usin these types of laminates).

The reader is advised to attempt relevant exercise problems at the end o this chapter to further appreciate properties of quasiisotropic laminates.

Example 6-2: Consider a three-ply laminate as shown in Fig. 6-8. The to and bottom layers are each 3 mm thick and oriented at 45° to the lamina reference axis, whereas the 6-mm-thick middle layer is oriented at 0°. Ob tain the *A, B,* and *D* matrices if each lamina has the same properties a the lamina considered in Example 6-1.

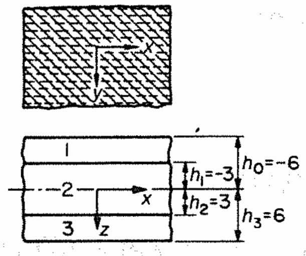

Figure 6-8. Three-ply laminate for Example 6-2.

Stiffness matrices for each lamina are

$$[\overline{Q}]_2 = \begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[\overline{Q}]_1 = [\overline{Q}]_3 = \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix}$$

For the laminate under consideration, $h_0 = -6$, $h_1 = -3$, $h_2 = 3$, and $h_3 = 6$, as shown in Fig. 6-8. Therefore,

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{3} (\overline{Q}_{ij})_k(h_k - h_{k-1}) = (\overline{Q}_{ij})_1(3) + (\overline{Q}_{ij})_2(6) + (\overline{Q}_{ij})_3(3) \\ &= (\overline{Q}_{ij})_1(6)^* + (\overline{Q}_{ij})_2(6) \\ &= [(\overline{Q}_{ij})_1 + (\overline{Q}_{ij})_2](6) \end{aligned}$$

Substitution of the values of $(\overline{Q}_{ij})_1$ and $(\overline{Q}_{ij})_2$ will give

$$[A] = \begin{bmatrix} 159.3 & 35.1 & 27.0 \\ 35.1 & 51.3 & 27.0 \\ 27.0 & 27.0 & 35.1 \end{bmatrix}$$

$$\begin{aligned} 2B_{ij} &= \sum_{k=1}^{3} (\overline{Q}_{ij})_k(h_k^2 - h_{k-1}^2) \\ &= (\overline{Q}_{ij})_1[(-3)^2 - (-6)^2] + (\overline{Q}_{ij})_2[(3)^2 - (-3)^2] + (\overline{Q}_{ij})_3[(6)^2 - (3)^2] \\ &= [(\overline{Q}_{ij})_3 - (\overline{Q}_{ij})_1](27) \end{aligned}$$

*Since $(\overline{Q}_{ij})_1 = (\overline{Q}_{ij})_3$.

But $(\overline{Q}_{ij})_1 = (\overline{Q}_{ij})_3$; therefore,

$$B_{ij} = 0$$

$$\begin{aligned} D_{ij} &= \tfrac{1}{3} \sum_{k=1}^{3} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3) \\ &= \tfrac{1}{3}(\overline{Q}_{ij})_1[(-3)^3 - (-6)^3] + \tfrac{1}{3}(\overline{Q}_{ij})_2[(3)^3 - (-3)^3] \\ &\quad + \tfrac{1}{3}(\overline{Q}_{ij})_3[(6)^3 - (3)^3] \\ &= 126(\overline{Q}_{ij})_1 + 18(\overline{Q}_{ij})_2 \end{aligned}$$

Substitution of values of $(\overline{Q}_{ij})_1$ and $(\overline{Q}_{ij})_2$ will give

$$[D] = \begin{bmatrix} 1185.3 & 661.5 & 567.0 \\ 661.5 & 861.3 & 567.0 \\ 567.0 & 567.0 & 661.5 \end{bmatrix}$$

This three-ply laminate does not exhibit bending–stretching coupling be cause of the midplane symmetry of the laminate but does exhibit both in plane and bending anisotropy because the A_{16}, A_{26}, D_{16}, and D_{26} terms ar all nonzero.

Example 6-3: Consider a four-ply laminate $[\pm 45]_S$. Each layer is assume to have a thickness of 3 mm and the same orthotropic properties as i Example 6-1.

In this laminate, the stiffness matrices are as follows:

$$[\overline{Q}]_1 = [\overline{Q}]_4 = \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix}$$

$$[\overline{Q}]_2 = [\overline{Q}]_3 = \begin{bmatrix} 6.55 & 5.15 & -4.50 \\ 5.15 & 6.55 & -4.50 \\ -4.50 & -4.50 & 5.15 \end{bmatrix}$$

$$\begin{aligned} A_{ij} &= 3[(\overline{Q}_{ij})_1 + (\overline{Q}_{ij})_2 + (\overline{Q}_{ij})_3 + (\overline{Q}_{ij})_4] \\ &= 6[(\overline{Q}_{ij})_1 + (\overline{Q}_{ij})_2] \end{aligned}$$

Thus

$$[A] = \begin{bmatrix} 78.6 & 61.8 & 0 \\ 61.8 & 78.6 & 0 \\ 0 & 0 & 61.8 \end{bmatrix}$$

Since the laminate possess midplane symmetry, $B_{ij} = 0$.

$$D_{ij} = \tfrac{1}{3}\{(\overline{Q}_{ij})_1[(-3)^3 - (-6)^3] + (\overline{Q}_{ij})_2[(0)^3 - (-3)^3] + (\overline{Q}_{ij})_3[(3)^3 - (0)^3] + (\overline{Q}_{ij})_4[(6)^3 - (3)^3]\}$$

$$= 126(\overline{Q}_{ij})_1 + 18(\overline{Q}_{ij})_2$$

Substitution of $(\overline{Q}_{ij})_1$ and $(\overline{Q}_{ij})_2$ gives

$$[D] = \begin{bmatrix} 943.2 & 741.6 & 486.0 \\ 741.6 & 943.2 & 486.0 \\ 486.0 & 486.0 & 741.6 \end{bmatrix}$$

Owing to symmetry, this laminate does not show bending–stretching coupling ($B_{ij} = 0$). Further, since $A_{16} = A_{26} = 0$, it is also free from normal stress–shear strain coupling.

Example 6-4: The effect of alternating-angle lamination can be illustrated by considering the following eight-ply laminates, with each lamina having a thickness of 3 mm and the properties the same as considered in Example 6-1: **(a)** all laminae at +45, **(b)** $[(45)_2/(-45)_2]_S$, **(c)** $[(\pm 45)_2]_S$, and **(d)** $[\pm\mp 45]_S$.

(a) This laminate is equivalent to a single lamina of thickness 24 mm. The stiffness matrices can be easily found to be

$$A = \begin{bmatrix} 157.2 & 123.6 & 108.0 \\ 123.6 & 157.2 & 108.0 \\ 108.0 & 108.0 & 123.6 \end{bmatrix}$$

$$B_{ij} = 0$$

$$D_{ij} = \tfrac{1}{3}(\overline{Q}_{ij})[(12)^3 - (-12)^3] = 1152(\overline{Q}_{ij})$$

$$[D] = 10^3\begin{bmatrix} 7.55 & 5.93 & 5.18 \\ 5.93 & 7.55 & 5.18 \\ 5.18 & 5.18 & 5.93 \end{bmatrix}$$

$$\frac{D_{16}}{D_{11}} = 0.686$$

(b) The terms A_{11}, A_{22}, A_{12}, and A_{66} are the same as calculated in case (a) and A_{16} and A_{26} are zero. Thus

$$[A] = \begin{bmatrix} 157.2 & 123.6 & 0 \\ 123.6 & 157.2 & 0 \\ 0 & 0 & 123.6 \end{bmatrix}$$

$$B_{ij} = 0$$

Calculation of the D_{ij} terms is simplified by noting that because of the symmetry of orientations, the contribution of layers above the midplane is equal to the contribution of the layers below the midplane. Further, since $\overline{Q}_{11}$, $\overline{Q}_{22}$, $\overline{Q}_{12}$, and $\overline{Q}_{66}$ are the same for all layers, D_{11}, D_{22}, D_{12}, and D_{66} remain the same as in case (a). Now

$$D_{16} = D_{26} = 2 \times \tfrac{1}{3}\{(4.50)[(12)^3 - (6)^3] - (4.50)[(6)^3 - (0)^3]\}$$
$$= 3.89 \times 10^3$$

$$[D] = 10^3 \begin{bmatrix} 7.55 & 5.93 & 3.89 \\ 5.93 & 7.55 & 3.89 \\ 3.89 & 3.89 & 5.93 \end{bmatrix}$$

$$\frac{D_{16}}{D_{11}} = 0.515$$

(c) In this case, the $[A]$ and $[B]$ matrices and the D_{11}, D_{22}, D_{12}, and D_{66} terms of the $[D]$ matrix remain the same as in case (b). Now, D_{16} and D_{26} are calculated as follows:

$$D_{16} = D_{26} = 2 \times \tfrac{1}{3}[(4.5)(12^3 - 9^3) - (4.5)(9^3 - 6^3)$$
$$+ (4.5)(6^3 - 3^3) - (4.5)(3^3 - 0^3)]$$
$$= 1.94 \times 10^3$$

Thus

$$[D] = 10^3 \times \begin{bmatrix} 7.55 & 5.93 & 1.94 \\ 5.93 & 7.55 & 1.94 \\ 1.94 & 1.94 & 5.93 \end{bmatrix}$$

$$\frac{D_{16}}{D_{11}} = 0.257$$

(d) In this case, D_{16} and D_{26} are also the only terms affected by changing the stacking sequence and can be calculated as follows:

$$D_{16} = D_{26} = 2 \times \tfrac{1}{3}[(4.5)(12^3 - 9^3) - (4.5)(9^3 - 3^3) + (4.5)(3^3 - 0^3)]$$
$$= 0.97 \times 10^3$$

Thus

$$[D] = 10^3\begin{bmatrix} 7.55 & 5.93 & 0.97 \\ 5.93 & 7.55 & 0.97 \\ 0.97 & 0.97 & 5.93 \end{bmatrix}$$

$$\frac{D_{16}}{D_{11}} = 0.129$$

This simple example has demonstrated the effect of stacking sequence on the $[A]$, $[B]$, and $[D]$ matrices. Stacking sequence has no effect on the $[A]$ matrix. If all laminae are oriented at equal positive and negative angles, the D_{11}, D_{22}, D_{12}, and D_{66} terms of the $[D]$ matrix also remain unaffected by stacking sequence. Stacking sequence has no effect on the $[B]$ matrix as long as the symmetry about the midplane is maintained. However, if the number of laminae forming the laminate is large, it is possible, by selection of the proper stacking sequence, to minimize the D_{16} and D_{26} terms of the $[D]$ matrix without disturbing the laminate symmetry.

Example 6-5: Using an analysis for a quasi-isotropic laminate [0/ ±45/ 90] made up of the composite considered in Example 4-1, predict elastic modulus, shear modulus, and Poisson's ratio for a randomly oriented fiber composite. Compare results with those obtained earlier.

The following moduli values were obtained in Example 4-1:

$$E_L = 16.26 \text{ GPa}$$

$$E_T = 4.53 \text{ GPa}$$

Shear modulus for the aligned-fibers composite can be calculated using Halpin–Tsai equation [Eq. (3.53)]. For this purpose, we may assume the Poisson ratios of glass fibers and nylon matrix as 0.2 and 0.35, respectively, so that their shear moduli may be obtained as follows:

$$G_f = \frac{72.4}{2(1 + 0.2)} = 30.17 \text{ GPa}$$

$$G_m = \frac{2.76}{2(1 + 0.35)} = 1.02 \text{ GPa}$$

From Eq. (3.54),

$$\eta = \frac{(30.17/1.02) - 1}{(30.17/1.02) + 1} = 0.935$$

From Eq. (3.53),

$$\frac{G_{LT}}{G_m} = \frac{1 + \xi\eta V_f}{1 - \eta V_f}$$

$$G_{LT} = 1.02\left(\frac{1 + 0.935 \times 0.2}{1 - 0.935 \times 0.2}\right) = 1.49 \text{ GPa}$$

Poisson's ratio ν_{LT} may be obtained from the rule of mixtures [Eq. (3.59)]:

$$\nu_{LT} = 0.8 \times 0.35 + 0.2 \times 0.2 = 0.32$$

Minor Poisson's ratio ν_{TL} is obtained through Eq. (3.60):

$$\nu_{TL} = 0.32(4.53/16.26) = 0.089$$

Now the stiffness matrix of the laminae may be obtained from Eq. (5.78):

$$Q_{11} = \frac{16.26}{1 - 0.32 \times 0.089} = 16.74 \text{ GPa}$$

$$Q_{22} = \frac{4.53}{1 - 0.32 \times 0.089} = 4.66 \text{ GPa}$$

$$Q_{12} = \frac{0.32 \times 4.53}{1 - 0.32 \times 0.089} = 1.49 \text{ GPa}$$

$$Q_{66} = 1.49 \text{ GPa}$$

$$[\overline{Q}_{ij}]_{0^\circ} = \begin{bmatrix} 16.74 & 1.49 & 0 \\ 1.49 & 4.66 & 0 \\ 0 & 0 & 1.49 \end{bmatrix}$$

$$[\overline{Q}_{ij}]_{90^\circ} = \begin{bmatrix} 4.66 & 1.49 & 0 \\ 1.49 & 16.74 & 0 \\ 0 & 0 & 1.49 \end{bmatrix}$$

Transformation equations [Eq. (5.61)] give

$$[\overline{Q}_{ij}]_{45^\circ} = \begin{bmatrix} 7.585 & 4.605 & 3.02 \\ 4.605 & 7.585 & 3.02 \\ 3.02 & 3.02 & 4.605 \end{bmatrix}$$

$$[\overline{Q}_{ij}]_{-45^\circ} = \begin{bmatrix} 7.585 & 4.605 & -3.02 \\ 4.605 & 7.585 & -3.02 \\ -3.02 & -3.02 & 4.605 \end{bmatrix}$$

Assuming unit thickness of the laminate, the $[A]$ matrix is obtained as follows:

$$A_{ij} = \tfrac{1}{4}[(\overline{Q}_{ij})_{0^\circ} + (\overline{Q}_{ij})_{90^\circ} + (\overline{Q}_{ij})_{45^\circ} + (\overline{Q}_{ij})_{-45^\circ}]$$

$$[A] = \begin{bmatrix} 0.1425 & 3.0475 & 0 \\ 3.04751 & 9.1425 & 0 \\ 0 & 0 & 3.0475 \end{bmatrix}$$

Using the results of Exercise 6-7, the following elastic constants can be obtained easily:

$$E_R = \frac{9.1425^2 - 3.0475^2}{9.1425} = 8.13 \text{ GPa}$$

$$G_R = 3.0475 \text{ GPa}$$

$$\nu_R = \frac{3.0475}{9.1425} = 0.33$$

The values of E_R and G_R obtained here compare well with those obtained in Example 4-1, but the value of ν_R is lower than that obtained earlier.

$$\begin{Bmatrix} \epsilon^0 \\ k \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A' & B' \\ B' & D' \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{6.33}$$

where

$$[A'] = [A^*] - [B^*][D^{*-1}][C^*] = [A^*] + [B^*][D^{*-1}][B^*]^T$$

$$[B'] = [B^*][D^{*-1}]$$

$$[C'] = -[D^{*-1}][C^*] = [B']^T = [B']$$

$$[D'] = [D^{*-1}]$$

Thus a fully inverted form of the laminate constitutive equations is obtained by inverting an additional 3 × 3 matrix and carrying out two more matrix multiplications.

The laminate constitutive equations in one of the three forms discussed earlier can be used to calculate the laminae stresses and strains. The choice of a particular form depends on the loading condition of the laminate. It is important to note that each form can be obtained through the basic elastic properties of the laminae (i.e., $\bar{Q}_{ij}$ matrices) and their stacking sequence.

For symmetric laminates, the constitutive equations [Eqs. (6.22) and (6.23)] can be written as

$$\{N\} = [A]\{\varepsilon^0\} \tag{6.34}$$

$$\{M\} = [D]\{k\} \tag{6.35}$$

Each of the Eqs. (6.34) and (6.35) is a set of three algebraic equations with three unknowns. Solution of these equations is relatively simple. Equation (6.34) is solved by premultiplying both sides by $[A^{-1}]$, and Eq. (6.35), by premultiplying by $[D^{-1}]$. The solutions are

$$\{\varepsilon^0\} = [A^{-1}]\{N\} \tag{6.36}$$

$$\{k\} = [D^{-1}]\{M\} \tag{6.37}$$

Thus the inverted form of the constitutive equations for symmetric laminates can be obtained by inversion of only two 3 × 3 matrices.

Example 6-6: Obtain the partially inverted and fully inverted forms of the laminate constitutive equations for the laminate considered in Example 6-1. [A], [B], and [D] matrices for the laminate are

$$[A] = \begin{bmatrix} 119.6 & 18.9 & 13.5 \\ 18.9 & 29.6 & 13.5 \\ 13.5 & 13.5 & 18.9 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 100.9 & -33.4 & -33.8 \\ -33.4 & -34.1 & -33.8 \\ -33.8 & -33.8 & -33.4 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 571.0 & 123.0 & 94.5 \\ 123.0 & 181.0 & 94.5 \\ 94.5 & 94.5 & 123.0 \end{bmatrix}$$

First, $[A^{-1}]$ can be found to be (see Appendix 1 for the procedure):

$$[A^{-1}] = [A^*] = 10^{-2}\begin{bmatrix} 0.95 & -0.44 & -0.36 \\ -0.44 & 5.21 & -3.41 \\ -0.36 & -3.41 & 7.99 \end{bmatrix}$$

Now the other matrices in the semi-inverted form of the constitutive equations can be obtained easily by matrix multiplications and subtraction as follows:

$$[B^*] = -10^{-2}\begin{bmatrix} 0.95 & -0.44 & -0.36 \\ -0.44 & 5.21 & -3.41 \\ -0.36 & -3.41 & 7.99 \end{bmatrix}\begin{bmatrix} 100.9 & -33.4 & -33.8 \\ -33.4 & -34.1 & -33.8 \\ -33.8 & -33.8 & -33.4 \end{bmatrix}$$

$$[B^*] = \begin{bmatrix} -1.224 & 0.044 & 0.050 \\ 1.032 & 0.479 & 0.475 \\ 1.926 & 1.415 & 1.392 \end{bmatrix}$$

$$[C^*] = -[B^*]^T = \begin{bmatrix} 1.224 & -1.032 & -1.926 \\ -0.044 & -0.479 & -1.415 \\ -0.050 & -0.475 & -1.392 \end{bmatrix}$$

$$\begin{aligned} [D^*] &= [D] - [B][A^{-1}][B] \\ &= [D] + [B][B^*] \\ &= \begin{bmatrix} 571.0 & 123.0 & 94.5 \\ 123.0 & 181.0 & 94.5 \\ 94.5 & 94.5 & 123.0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 100.9 & -33.4 & -33.8 \\ -33.4 & -34.1 & -33.8 \\ -33.8 & -33.8 & -33.4 \end{bmatrix}\begin{bmatrix} -1.224 & 0.044 & 0.500 \\ 1.032 & 0.479 & 0.475 \\ 1.926 & 1.415 & 1.392 \end{bmatrix} \end{aligned}$$

$$[D^*] = \begin{bmatrix} 347.95 & 63.61 & 36.68 \\ 63.61 & 115.38 & 29.57 \\ 36.68 & 29.57 & 58.75 \end{bmatrix}$$

Thus the partially inverted form of the constitutive equations becomes

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \hline M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0.0095 & -0.0044 & -0.0036 & -1.224 & 0.044 & 0.050 \\ -0.0044 & 0.0521 & -0.0341 & 1.032 & 0.479 & 0.475 \\ -0.0036 & -0.0341 & 0.0799 & 1.926 & 1.415 & 1.392 \\ \hline 1.224 & -1.032 & -1.926 & 347.95 & 63.61 & 36.68 \\ -0.044 & -0.479 & -1.415 & 63.61 & 115.38 & 29.57 \\ -0.050 & -0.475 & -1.392 & 36.68 & 29.57 & 58.75 \end{array}\right] \cdot \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

Now, to find the fully inverted form of the constitutive equations, begin with finding the inverse of matrix $[D^*]$.

$$[D^{*-1}] = [D'] = \begin{bmatrix} 0.0033 & -0.0015 & -0.0013 \\ -0.0015 & 0.0106 & -0.0044 \\ -0.0013 & -0.0044 & 0.0201 \end{bmatrix}$$

The other matrices can be obtained by matrix multiplications and subtractions:

$$[B'] = [B^*][D^{*-1}]$$

$$= \begin{bmatrix} -1.224 & 0.044 & 0.050 \\ 1.032 & 0.479 & 0.475 \\ 1.926 & 1.415 & 1.392 \end{bmatrix} \begin{bmatrix} 0.0033 & -0.0015 & -0.0013 \\ -0.0015 & 0.0106 & -0.0044 \\ -0.0013 & -0.0044 & 0.0201 \end{bmatrix}$$

$$[B'] = \begin{bmatrix} -0.0041 & 0.0021 & 0.0024 \\ 0.0021 & 0.0015 & 0.0060 \\ 0.0024 & 0.0060 & 0.0192 \end{bmatrix}$$

$$[C'] = [B']^{\mathrm{T}} = \begin{bmatrix} -0.0041 & 0.0021 & 0.0024 \\ 0.0021 & 0.0015 & 0.0060 \\ 0.0024 & 0.0060 & 0.0192 \end{bmatrix}$$

$$[A'] = [A^*] - [B^*][D^{*-1}][C^*]$$

$$= [A^*] - [B'][C^*]$$

$$= \begin{bmatrix} 0.0095 & -0.0044 & -0.0036 \\ -0.0044 & 0.0521 & -0.0341 \\ -0.0036 & -0.0341 & 0.0799 \end{bmatrix}$$

$$+ \begin{bmatrix} -0.0041 & 0.0021 & 0.0024 \\ 0.0021 & 0.0015 & 0.0060 \\ 0.0024 & 0.0060 & 0.0192 \end{bmatrix} \begin{bmatrix} -1.224 & 1.032 & 1.926 \\ 0.044 & 0.479 & 1.415 \\ 0.050 & 0.475 & 1.392 \end{bmatrix}$$

$$[A'] = \begin{bmatrix} 0.0148 & -0.0065 & -0.0053 \\ -0.0065 & 0.0578 & -0.0196 \\ -0.0053 & -0.0196 & 0.1197 \end{bmatrix}$$

Thus the fully inverted form of the constitutive equation is

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \hline k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} 0.0148 & -0.0065 & -0.0053 & -0.0041 & 0.0021 & 0.0024 \\ -0.0065 & 0.0578 & -0.0196 & 0.0021 & 0.0015 & 0.0060 \\ -0.0053 & -0.0196 & 0.1197 & 0.0024 & 0.0060 & 0.0192 \\ \hline -0.0041 & 0.0021 & 0.0024 & 0.0033 & -0.0015 & -0.0013 \\ 0.0021 & 0.0015 & 0.0060 & -0.0015 & 0.0106 & -0.0044 \\ 0.0024 & 0.0060 & 0.0192 & -0.0013 & -0.0044 & 0.0201 \end{array}\right] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

It should be noted that the preceding compliance matrix in the fully inverted form of the constitutive equation is a symmetric matrix, as should be expected because of the symmetry of the original stiffness matrix. Small errors caused by the rounding off of the numbers in the intermediate steps should be ignored. It may be pointed out that this compliance matrix could be obtained by directly inverting the original 6 × 6 stiffness matrix.

Example 6-7: Let the three-ply laminate considered in Example 6-2 be subjected to the forces $N_x = 1000$ N/mm, $N_y = 200$ N/mm, and $N_{xy} = 0$, as shown in Fig. 6-9. Calculate the stresses and strains in the individual plies.

The extensional stiffness matrix for the laminate was found to be

$$[A] = \begin{bmatrix} 159.3 & 35.1 & 27.0 \\ 35.1 & 51.3 & 27.0 \\ 27.0 & 27.0 & 35.1 \end{bmatrix}$$

The coupling matrix $[B]$ for this laminate is zero, as shown in Example 6-2. Therefore, the given loading would produce only in-plane normal and shear strains, and no plate curvatures would be produced. This also implies, from Eq. (6.6), that the midplane strains are also the strains for individual plies because there is no strain gradient through the thickness. However, the stresses in each ply will be different and have to be evaluated by taking into consideration the corresponding stiffness matrix.

To obtain the midplane strains, first, $[A^{-1}]$ can be found to be

$$[A^{-1}] = \begin{bmatrix} 0.00759 & -0.00356 & -0.00309 \\ -0.00356 & 0.03441 & -0.02373 \\ -0.00309 & -0.02373 & 0.04911 \end{bmatrix}$$

Since the coupling matrix $[B]$ is equal to zero, Eq. (6.36) can be used to calculate the in-plane strains as

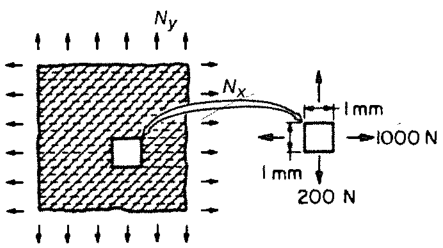

Figure 6-9. Definition of applied forces on laminate for Example 6-7.

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = 10^{-3} \begin{bmatrix} 0.00759 & -0.00356 & -0.00309 \\ -0.00356 & 0.03441 & -0.02373 \\ -0.00309 & -0.02373 & 0.04911 \end{bmatrix} \begin{Bmatrix} 1000 \\ 200 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.00685 \\ 0.00332 \\ -0.00784 \end{Bmatrix}$$

It may be noted that the factor 10^{-3} has been placed before the $[A^{-1}]$ matrix to make its units consistent with those of N_x, N_y, and N_{xy}. The reader is advised to verify this. The preceding midplane strains are also the lamina strains in the xy reference coordinates. The reference coordinates for each lamina are explained in Fig. 6-10. The lamina stresses in the xy coordinates can be obtained from the stress–strain relation [Eq. (594)]. Using the $[\bar{Q}]$ matrices obtained in Example 6-2, stresses are found as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{0^\circ \text{ ply}} = \begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2.0 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{Bmatrix} 0.00685 \\ 0.00332 \\ -0.00784 \end{Bmatrix} = \begin{Bmatrix} 139.3 \\ 11.4 \\ -5.5 \end{Bmatrix} 10^{-3} \text{ GPa}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{45^\circ \text{ ply}} = \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix} \begin{Bmatrix} 0.00685 \\ 0.00332 \\ -0.00784 \end{Bmatrix} = \begin{Bmatrix} 26.7 \\ 21.7 \\ 5.4 \end{Bmatrix} 10^{-3} \text{ GPa}$$

The laminae stresses and strains in the xy reference coordinates are represented graphically in Fig. 6-11.

For purposes of the laminate strength analysis, it is desirable that the laminae stresses and strains be obtained along their natural (longitudinal and transverse) axes. These now can be obtained easily by using the transformation equations [Eqs. (5.86) and (5.88)]. For the 0° ply, the lamina natural axes coincide with the laminate xy coordinates. Therefore, the stresses and strains obtained in the preceding paragraphs for the 0° ply are also the stresses and strains in the longitudinal and transverse directions; in other words,

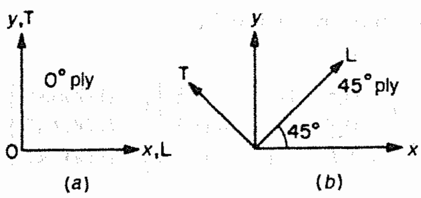

Figure 6-10. Reference coordinate axes for (*a*) 0° lamina and (*b*) 45° lamina.

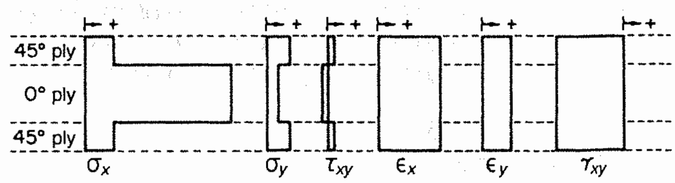

Figure 6-11. Lamina stresses and strains along the reference axes (Example 6-7).

$$\begin{Bmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{Bmatrix}_{0^\circ \text{ ply}} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}_{0^\circ \text{ ply}} = \begin{Bmatrix} 0.00685 \\ 0.00332 \\ -0.00784 \end{Bmatrix}$$

and

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{0^\circ \text{ ply}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{0^\circ \text{ ply}} = \begin{Bmatrix} 139.3 \\ 11.4 \\ -5.5 \end{Bmatrix} \text{ MPa}$$

The lamina stress and strains for the 45° plies in the longitudinal an(transverse axes may be obtained using one of two procedures. In the firs procedure, both the stresses and strains are transformed using Eqs. (5.86 and (5.88), whereas in the second procedure, only strains are transformed and then the stresses are calculated from the stress–strain relations [Eq (5.74)]. Resulting stresses in the two cases will be the same. In problem where the stresses in the xy coordinates are not needed, the second pro cedure should be preferred because it would save an intermediate step ii the calculations.

The following is the transformation matrix for the 45° orientation:

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1.0 & 1.0 & 0 \end{bmatrix}$$

The strains are obtained by using Eq. (5.88):

$$\begin{Bmatrix} \varepsilon_L \\ \varepsilon_T \\ \gamma_{LT} \end{Bmatrix}_{45^\circ \text{ ply}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1.0 & 1.0 & 0 \end{bmatrix} \begin{Bmatrix} 0.00685 \\ 0.00332 \\ -0.00784 \end{Bmatrix} = \begin{Bmatrix} 0.00116 \\ 0.00900 \\ -0.00352 \end{Bmatrix}$$

Similarly, the stresses are obtained by using Eq. (5.86):

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1.0 \\ 0.5 & 0.5 & -1.0 \\ -0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} 26.7 \\ 21.7 \\ 5.4 \end{Bmatrix} = \begin{Bmatrix} 29.6 \\ 18.8 \\ -2.5 \end{Bmatrix} \text{MPa}$$

The laminae stresses and strains in the longitudinal and transverse reference coordinates are represented graphically in Fig. 6-12. These stress–strain variations can be compared with the allowable stresses and strains in each lamina, and thus the load at which the failure initiates in one of the laminae may be calculated. The procedure for calculating the load at failure initiation and the laminae stresses and strains after failure initiation is discussed in the next section.

6.8 ANALYSIS OF LAMINATES AFTER INITIAL FAILURE

Procedures for calculating stresses and strains in individual laminae owing to external loads on the laminate were discussed in the preceding section. The stresses and strains in each lamina may be compared with the corresponding allowable values to predict failure. Commonly employed failure theories were discussed in Chap. 5. Thus, for a given load, it may be determined easily whether any of the plies in the laminate will fail. Conversely, the load at which the first ply failure (FPF) will occur may be calculated. Since the strength of a ply is a function of its orientation, it is expected that all plies will not fail at the same load. Plies will fail successively in the increasing order of strength in the direction of loading. Moreover, the transverse strength of unidirectional laminae is known to be much smaller than the longitudinal strength, so the plies with fibers perpendicular to the load will fail first. Thus the FPF may occur at relatively small loads at which the laminate is in no real danger of fracture. Sometimes the effect of the FPF may not be evident from the macroscopic response of the laminate, but as the number of ply failures increases, the loss of laminate stiffness becomes evident, and the overall response of the laminate deviates from its original straight-line behavior. However, the laminate is still able to carry additional loads, although

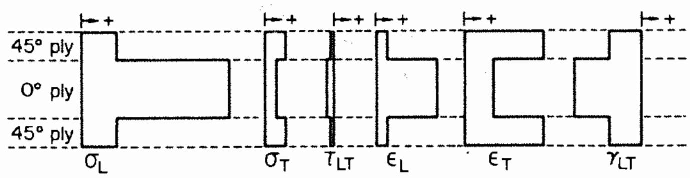

Figure 6-12. Lamina stresses and strains along the longitudinal and transverse axes (Example 6-7).

ysis gives a conservative estimate of the load-carrying capacity of the laminate because lamina failure in one direction does not, in general, result in complete stress relaxation in all directions. For example, when failure occurs in the transverse direction, the lamina stresses in the longitudinal direction may not be affected significantly. Some studies [3,4] have been carried out to establish experimentally the quantitative influence on lamina properties resulting from failure in one direction. These studies are inconclusive, and the rationale in reducing elastic constants to specific values has not been established. Thus, in the present circumstances, the practice of setting all lamina properties equal to zero may be continued.

Example 6-8: A 5-mm-thick symmetric cross-ply laminate is constructed from 15 identical laminae having the following stiffness matrix and strengths:

$$[Q] = \begin{bmatrix} 56 & 4.6 & 0 \\ 4.6 & 18.7 & 0 \\ 0 & 0 & 8.9 \end{bmatrix} \text{GPa}$$

$$\sigma_{LU} = 1050 \text{ MPa}$$

$$\sigma_{TU} = 28 \text{ MPa}$$

$$\tau_{LTU} = 42 \text{ MPa}$$

A uniaxial load is applied, and the laminate construction is such that nine laminae are in the load direction. Calculate the load at which the 90° plies fail and the load-carrying capacity of the laminate.

Solution: $[\overline{Q}]$ matrices for the 0° and 90° laminae can be written as

$$[\overline{Q}]_{0^\circ} = \begin{bmatrix} 56.0 & 4.6 & 0 \\ 4.6 & 18.7 & 0 \\ 0 & 0 & 8.9 \end{bmatrix} \text{GPa}$$

$$[\overline{Q}]_{90^\circ} = \begin{bmatrix} 18.7 & 4.6 & 0 \\ 4.6 & 56.0 & 0 \\ 0 & 0 & 8.9 \end{bmatrix} \text{GPa}$$

The $[A]$ matrix for the laminate can be obtained if the thicknesses of all 0° and 90° plies are known. Since all plies have the same thickness, thicknesses of all 0° and 90° plies are proportional to their number. Therefore,

$$\text{Thickness of } 0^\circ \text{ plies (9 in number)} = \tfrac{5}{15} \times 9 = 3 \text{ mm}$$

$$\text{Thickness of } 90^\circ \text{ plies (6 in number)} = \tfrac{5}{15} \times 6 = 2 \text{ mm}$$

The terms of the [A] matrix are given by

$$A_{ij} = 3(\overline{Q}_{ij})_{0^\circ} + 2(\overline{Q}_{ij})_{90^\circ}$$

Therefore the [A] matrix is obtained as

$$[A] = \begin{bmatrix} 205.4 & 23 & 0 \\ 23 & 168.1 & 0 \\ 0 & 0 & 44.5 \end{bmatrix} \text{GPa} \cdot \text{mm}$$

For analysis of this cross-ply laminate, we shall use the maximum-strain theory to predict failure of the laminae. Maximum allowable strains can be obtained from the given strength values, and the moduli values can be calculated from the given stiffness matrix. The moduli values are

$$E_{\text{L}} = 54.87 \text{ GPa}$$

$$E_{\text{T}} = 18.32 \text{ GPa}$$

Fracture strains in the longitudinal and transverse directions become

$$\varepsilon_{\text{LU}} = \frac{1050 \times 10^{-3}}{54.87} = 0.01914$$

$$\varepsilon_{\text{TU}} = \frac{28 \times 10^{-3}}{18.32} = 0.00153$$

Therefore, the 90° plies will fail when $\varepsilon_x = 0.00153$. The load N_x at failure of the 90° plies can be obtained as follows:

$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 205.4 & 23 & 0 \\ 23 & 168.1 & 0 \\ 0 & 0 & 44.5 \end{bmatrix} \begin{Bmatrix} 0.00153 \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Solution of this matrix equation gives

$$N_x = 0.3094 \text{ GPa} \cdot \text{mm} = 309.4 \text{ MPa} \cdot \text{mm}$$

The 90° plies will fail when $N_x = 309.4$ MPa·mm. After failure of the 90° plies, the $[A]$ matrix is modified according to Eq. (6.38). The modified $[A]$ matrix is obtained by substituting $(Q_{ij})_{90°} = 0$. Therefore,

$$[A] = \begin{bmatrix} 168 & 13.8 & 0 \\ 13.8 & 56.1 & 0 \\ 0 & 0 & 26.7 \end{bmatrix} \text{GPa}\cdot\text{mm}$$

The laminate will fail at a strain of $\varepsilon_x = 0.01914$, that is, an additional strain $\Delta\varepsilon_x = 0.01914 - 0.00153 = 0.01761$. Now, additional load at fracture can be obtained from the following:

$$\begin{Bmatrix} \Delta N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 168 & 13.8 & 0 \\ 13.8 & 56.1 & 0 \\ 0 & 0 & 26.7 \end{bmatrix} \begin{Bmatrix} 0.01761 \\ \Delta\varepsilon_y \\ \Delta\gamma_{xy} \end{Bmatrix}$$

Solution of this equation gives

$$\Delta N_x = 2.8987 \text{ GPa mm} = 2898.7 \text{ MPa}\cdot\text{mm}$$

Therefore, the total load at fracture or the load-carrying capacity of the laminate is

$$N_x = 2898.7 + 309.4 = 3208.1 \text{ MPa}\cdot\text{mm}$$

Example 6-9: Following are the elastic constants and strengths of laminae in a quasi-isotropic laminate $[0/\pm45/90]_S$:

Elastic constants

$$E_L = 40 \text{ GPa} \qquad E_T = 10 \text{ GPa}$$

$$G_{LT} = 4 \text{ GPa} \qquad \nu_{LT} = 0.285$$

Strengths

$$\sigma_{LU} = 1050 \text{ MPa} \qquad \sigma'_{LU} = 650 \text{ MPa}$$

$$\sigma_{TU} = 20 \text{ MPa} \qquad \sigma'_{TU} = 140 \text{ MPa}$$

$$\tau_{LTU} = 65 \text{ MPa}$$

From the laminate, a rectangular specimen with the dimensions 250 mm × 20 mm × 2 mm is tested in uniaxial tension. Predict the load elongation curve for the specimen if the grips are initially 200 mm apart. Assume th the laminae fail according to the maximum-stress theory and that all t elastic constants of a lamina become zero when it fails. Calculate the fra ture load of the specimen.

Solution: The $[Q]$ matrix for the laminae can be obtained using Eq. (5.7

$$[Q] = \begin{bmatrix} 40.83 & 2.91 & 0 \\ 2.91 & 10.21 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{GPa}$$

The $[\overline{Q}]$ matrices for different ply orientations can be obtained using t transformation equations [Eq. (5.95)]:

$$[\overline{Q}]_{0^\circ} = \begin{bmatrix} 40.83 & 2.91 & 0 \\ 2.91 & 10.21 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{GPa}$$

$$[\overline{Q}]_{90^\circ} = \begin{bmatrix} 10.21 & 2.91 & 0 \\ 2.91 & 40.83 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{GPa}$$

$$[\overline{Q}]_{45^\circ} = \begin{bmatrix} 18.215 & 10.215 & 7.655 \\ 10.215 & 18.215 & 7.655 \\ 7.655 & 7.655 & 11.305 \end{bmatrix} \text{GPa}$$

$$[\overline{Q}]_{-45^\circ} = \begin{bmatrix} 18.215 & 10.215 & -7.655 \\ 10.215 & 18.215 & -7.655 \\ -7.655 & -7.655 & 11.305 \end{bmatrix} \text{GPa}$$

Analysis of Initial Behavior The $[A]$ matrix is obtained by using E (6.20) and noting that the thickness of each ply is $\frac{2}{8} = 0.25$ mm:

$$A_{ij} = 2 \times \tfrac{2}{8}[(\overline{Q}_{ij})_{0^\circ} + (\overline{Q}_{ij})_{90^\circ} + (\overline{Q}_{ij})_{45^\circ} + (\overline{Q}_{ij})_{-45^\circ}]$$

Therefore,

$$[A] = \begin{bmatrix} 43.735 & 13.125 & 0 \\ 13.125 & 43.735 & 0 \\ 0 & 0 & 15.305 \end{bmatrix} \text{GPa} \cdot \text{mm}$$

Owing to the symmetry of the laminate, the $[B]$ matrix vanishes, init stress–strain relation for uniaxial tension ($N_y = N_{xy} = 0$) may be writt as

$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 43.735 & 13.125 & 0 \\ 13.125 & 43.735 & 0 \\ 0 & 0 & 15.305 \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}$$

Solution of this matrix equation yields

$$\varepsilon_y^0 = -0.3\varepsilon_x^0 \qquad \gamma_{xy}^0 = 0$$

$$N_x = \varepsilon_x^0/0.0254 \text{ GPa} \cdot \text{mm}$$

For $\varepsilon_y^0 = -0.3\varepsilon_x^0$, and $\gamma_{xy}^0 = 0$, stresses in the laminae in the longitudinal and transverse directions may be obtained in terms of ε_x^0 using Eqs. (5.94) and (5.74):

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} 39.96 & \varepsilon_x^0 \\ -0.153 & \varepsilon_x^0 \\ 0 & \end{Bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{90^\circ} = \begin{Bmatrix} -9.339 & \varepsilon_x^0 \\ 9.337 & \varepsilon_x^0 \\ 0 & \end{Bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{45^\circ} = \begin{Bmatrix} 15.31 & \varepsilon_x^0 \\ 4.59 & \varepsilon_x^0 \\ -5.20 & \varepsilon_x^0 \end{Bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{-45^\circ} = \begin{Bmatrix} 15.31 & \varepsilon_x^0 \\ 4.59 & \varepsilon_x^0 \\ -5.20 & \varepsilon_x^0 \end{Bmatrix} \text{GPa}$$

(Notice that the uniaxial stress on the quasi-isotropic laminate produces complex stresses and strains in the constituent laminae.)

It can be shown easily that for the preceding state of stress, 90° plies will fail first when $\sigma_T = \sigma_{TU}$. That is,

$$9.337\varepsilon_x^0 = 20 \times 10^{-3}$$

$$\varepsilon_x^0 = 0.002142$$

Following are the load and elongation at FPF:

$$\text{Elongation } \delta = 200 \times 0.002142 = 0.4284 \text{ mm}$$

$$N_x = \frac{0.002142}{0.0254} \text{ GPa} \cdot \text{mm} = 0.08433 \text{ kN/mm}$$

For the 20-mm-wide specimen, the load is

$$P = 20 \times 0.08433 = 1.687 \text{ kN}$$

Analysis during Second Segment After FPF, the $[A]$ matrix is modified by neglecting the contributions of 90° plies. The modified $[A]$ matrix is calculated as follows:

$$\bar{A}_{ij} = 2 \times \tfrac{2}{8}[(\bar{Q}_{ij})_{0^\circ} + (\bar{Q}_{ij})_{45^\circ} + (\bar{Q}_{ij})_{-45^\circ}]$$

Thus

$$[\bar{A}] = \begin{bmatrix} 38.63 & 11.67 & 0 \\ 11.67 & 23.32 & 0 \\ 0 & 0 & 13.305 \end{bmatrix} \text{GPa} \cdot \text{mm}$$

The load–strain relationship for the second segment can be written as

$$\begin{Bmatrix} \Delta N_x \\ 0 \\ 0 \end{Bmatrix}_2 = \begin{bmatrix} 38.63 & 11.67 & 0 \\ 11.67 & 23.32 & 0 \\ 0 & 0 & 13.305 \end{bmatrix}_2 \begin{Bmatrix} \Delta\varepsilon_x^0 \\ \Delta\varepsilon_y^0 \\ \Delta\gamma_{xy}^0 \end{Bmatrix}_2$$

Solution of the matrix equation gives

$$\Delta\varepsilon_y^0 = -0.5\,\Delta\varepsilon_x^0 \qquad \Delta\gamma_{xy}^0 = 0$$

$$\Delta N_x = 32.795\,\Delta\varepsilon_x^0 \text{ GPa} \cdot \text{mm}$$

Incremental stresses in the laminae now may be obtained in terms of incremental strain $\Delta\varepsilon_x^0$ using Eqs. (5.94) and (5.74):

$$\begin{Bmatrix} \Delta\sigma_L \\ \Delta\sigma_T \\ \Delta\tau_{LT} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} 39.375\,\Delta\varepsilon_x^0 \\ -2.195\,\Delta\varepsilon_x^0 \\ 0 \end{Bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \Delta\sigma_L \\ \Delta\sigma_T \\ \Delta\tau_{LT} \end{Bmatrix}_{\pm 45^\circ} = \begin{Bmatrix} 10.94\,\Delta\varepsilon_x^0 \\ 3.28\,\Delta\varepsilon_x^0 \\ \mp 6.00\,\Delta\varepsilon_x^0 \end{Bmatrix} \text{GPa}$$

Instantaneous stresses may be obtained by adding these incremental stresses to the stresses at FPF. Thus

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{0^\circ} = \begin{Bmatrix} 0.0856 + 39.3751\ \Delta\varepsilon_x^0 \\ -0.00033 - 2.195\ \Delta\varepsilon_x^0 \\ 0 \end{Bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix}_{\pm 45^\circ} = \begin{Bmatrix} 0.0328 + 10.94\ \Delta\varepsilon_x^0 \\ 0.0098 + 3.28\ \Delta\varepsilon_x^0 \\ \mp 0.0111 \mp 6.00\ \Delta\varepsilon_x^0 \end{Bmatrix} \text{GPa}$$

It can be shown easily that $\pm 45^\circ$ plies will fail next when $(\sigma_T)_{45^\circ} = \sigma_{TU}$. That is

$$0.0098 + 3.28\ \Delta\varepsilon_x^0 = 20 \times 10^{-3} \quad \text{or} \quad \Delta\varepsilon_x^0 = 0.00311$$

The incremental elongation and load at the second ply failure are

$$\Delta\delta = 200 \times 0.00311 = 0.622 \text{ mm}$$

$$\Delta P = 20 \times 0.00311 \times 32.795 \text{ GPa} \cdot \text{mm}^2 = 2.04 \text{ kN}$$

The total load and elongation at the second ply failure are

$$P = 1.687 + 2.04 = 3.721 \text{ kN}$$

$$\delta = 0.4284 + 0.622 = 1.054 \text{ mm}$$

Analysis during Third Segment During the third segment, only 0° plies are acting, and therefore, the analysis can be carried out by assuming it to be a unidirectional composite. Failure will occur when

$$(\sigma_L)_{0^\circ} = \sigma_{LU} = 1050 \text{ MPa}$$

Therefore, strain at the laminate failure will be

$$\varepsilon_x = \frac{1050 \times 10^{-3}}{40} = 0.02625$$

Incremental strain at laminate failure will be

$$\Delta\varepsilon_x = 0.02625 - 0.002142 - 0.00311 = 0.0210$$

Incremental elongation and load are

$$\Delta\delta = 200 \times 0.021 = 4.2 \text{ mm}$$

$$\Delta P = 20 \times 2 \times \tfrac{2}{8} \times 0.021 \times 40 \text{ GPa}\cdot\text{mm}^2$$

$$\Delta P = 8.4 \text{ kN}$$

Therefore,

$$\text{Load at fracture} = 8.4 + 3.721 = 12.121 \text{ kN}$$

$$\text{Elongation at fracture} = 4.2 + 1.054 = 5.254 \text{ mm}$$

The complete load–elongation curve for the laminate is shown in Fig. 6-18. Each ply failure causes a change in the slope of—that is, produces a knee in—the stress–strain curve. Load levels at the ply failure have been marked. It may be noted that when 90° plies fail, the change in slope of load–elongation curve is hardly noticeable.

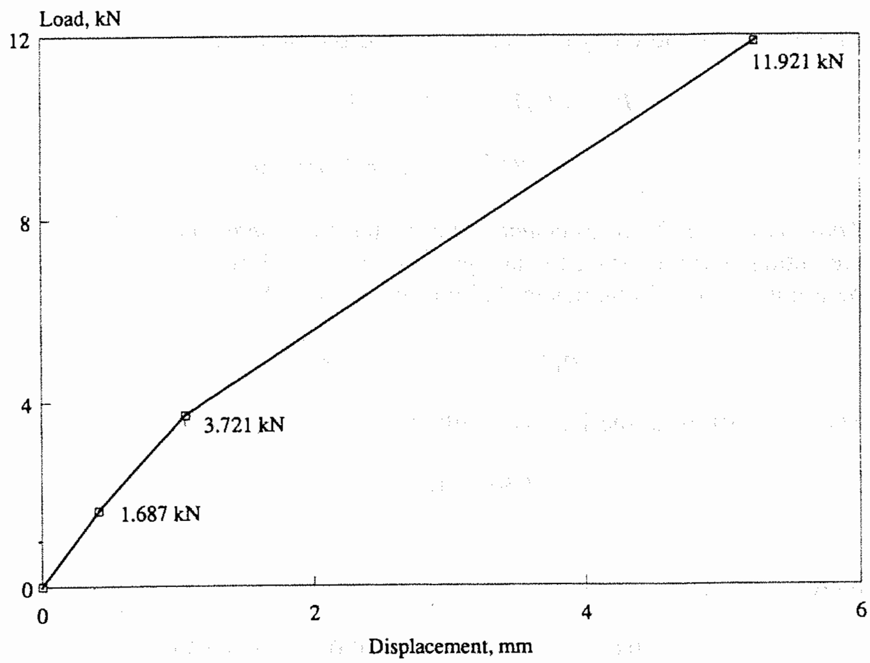

Figure 6-18. Predicted load–elongation curve for the quasi-isotropic laminate in Example 6-9.

warpage due to hygrothermal change during fabrication is avoided by the use of symmetric laminates. A more specific discussion on the residual stresses is given in refs. 5–7.

Example 6-10: Calculate the residual stresses in the laminate considered in Example 6-1 that is fabricated at 125°C and cooled to room temperature of 25°C, given

$$\alpha_L = 7.0 \times 10^{-6}/°C \qquad \text{and} \qquad \alpha_T = 23 \times 10^{-6}/°C$$

First, transform the coefficients of thermal expansion in the xy coordinate axes:

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0°} = \begin{Bmatrix} \alpha_L \\ \alpha_T \\ 0 \end{Bmatrix} = 10^{-6}\begin{Bmatrix} 7 \\ 23 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{45°} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 7 \times 10^{-6} \\ 23 \times 10^{-6} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{45°} = \begin{Bmatrix} 15 \\ 15 \\ -16 \end{Bmatrix} 10^{-6}$$

Now thermal forces and moments may be calculated by means of Eqs. (6.65) and (6.66), where the $[\overline{Q}]$ matrices for the two plies were obtained in Example 6-1. The calculations may be carried out in the following sequence:

$$\Delta T = 25 - 125 = -100°C$$

$$\Delta T \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{0°} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{0°} = 10^{-3}\begin{Bmatrix} -15.61 \\ -5.09 \\ 0 \end{Bmatrix}$$

$$\Delta T \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_{45°} \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix}_{45°} = 10^{-3}\begin{Bmatrix} -10.35 \\ -10.35 \\ -5.26 \end{Bmatrix}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = [(4) - (-1)]10^{-3}\begin{Bmatrix} -15.61 \\ -5.09 \\ 0 \end{Bmatrix} + [(-1) - (-4)]10^{-3}\begin{Bmatrix} -10.35 \\ -10.35 \\ -5.26 \end{Bmatrix}$$

$$\begin{Bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{Bmatrix} = 10^{-3}\begin{Bmatrix} -109.10 \\ -56.50 \\ -15.78 \end{Bmatrix} \text{GPa} \cdot \text{mm}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = \tfrac{1}{2}[(4)^2 - (-1)^2]10^{-3}\begin{Bmatrix} -15.61 \\ -5.09 \\ 0 \end{Bmatrix}$$

$$+ \tfrac{1}{2}[(-1)^2 - (-4)^2]10^{-3}\begin{Bmatrix} -10.35 \\ -10.35 \\ -5.26 \end{Bmatrix}$$

$$\begin{Bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{Bmatrix} = 10^{-3}\begin{Bmatrix} -39.45 \\ 39.45 \\ 39.45 \end{Bmatrix} \text{GPa} \cdot \text{mm}$$

The midplane strains and plate curvatures may be obtained from Eqs. (6.63) and (6.64). It may be noted, however, that Eqs. (6.63) and (6.64) may be written in an inverted form similar to Eq. (6.33) as follows:

$$\begin{Bmatrix} \epsilon^0 \\ \hline k \end{Bmatrix} = \left[\begin{array}{c|c} A' & B' \\ \hline B' & D' \end{array}\right]\begin{Bmatrix} N^T \\ \hline M^T \end{Bmatrix}$$

Further, the matrices $[A']$, $[B']$, and $[D']$ have the same meaning as in Eq. (6.33), and for the laminate under consideration, these were evaluated in Example 6-6. Therefore, the midplane strains and plate curvatures can be found to be

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = 10^{-4}\begin{Bmatrix} -8.14 \\ -20.20 \\ 6.99 \end{Bmatrix}$$

and

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = 10^{-4}\begin{Bmatrix} 0.58 \\ -1.00 \\ -2.35 \end{Bmatrix}$$

The nonzero values of plate curvatures $\{k\}$ in the preceding calculations show that warping of the laminate will occur when the laminate is cooled from the curing temperature (125°C) to room temperature (25°C). Mechanical strains that cause the residual stresses are calculated by Eq. (6.61):

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{0°} = 10^{-4}\begin{Bmatrix} -8.14 + 0.58z + 7.0 \\ -20.20 - 1.00z + 23.0 \\ 6.99 - 2.35z + 0 \end{Bmatrix} = 10^{-4}\begin{Bmatrix} -1.14 + 0.58z \\ 2.80 - 1.00z \\ 6.99 - 2.35z \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix}_{45°} = 10^{-4}\begin{Bmatrix} -8.14 + 0.58z + 15.0 \\ -20.20 - 1.00z - 15.0 \\ 6.99 - 2.35z - 16.0 \end{Bmatrix} = 10^{-4}\begin{Bmatrix} 6.86 + 0.58z \\ -5.20 - 1.00z \\ -9.01 - 2.35z \end{Bmatrix}$$

The residual stress distribution may be obtained by substituting the preceding strains into Eq. (6.62). Because the strain variation and hence the stress variation are linear across the thickness of a ply, it is sufficient to calculate the stresses only at the ply surfaces to complete the residual stress distribution. The required stresses are calculated as follows:

0° ply, z = 4

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = 10^{-4}\begin{Bmatrix} 1.18 \\ -1.20 \\ -2.41 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = 10^{-4}\begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{Bmatrix} 1.18 \\ -1.20 \\ -2.41 \end{Bmatrix} \text{ GPa}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = \begin{Bmatrix} \sigma_L^T \\ \sigma_T^T \\ \tau_{LT}^T \end{Bmatrix} = \begin{Bmatrix} 2.28 \\ -0.16 \\ -0.17 \end{Bmatrix} \text{ MPa}$$

0° ply, z = −1

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = 10^{-4}\begin{Bmatrix} -1.72 \\ 3.80 \\ 9.34 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = 10^{-4}\begin{bmatrix} 20 & 0.7 & 0 \\ 0.7 & 2 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{Bmatrix} -1.72 \\ 3.80 \\ 9.34 \end{Bmatrix} \text{ GPa}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = \begin{Bmatrix} \sigma_L^T \\ \sigma_T^T \\ \tau_{LT}^T \end{Bmatrix} = \begin{Bmatrix} -3.17 \\ 0.64 \\ 0.65 \end{Bmatrix} \text{ MPa}$$

45° ply, z = −1

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 6.28 \\ -4.20 \\ -6.66 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = 10^{-4} \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix} \begin{Bmatrix} 6.28 \\ -4.20 \\ -6.66 \end{Bmatrix} = 10^{-3} \begin{Bmatrix} -1.05 \\ -2.51 \\ -2.49 \end{Bmatrix} \text{ GPa}$$

$$\begin{Bmatrix} \sigma_L^T \\ \sigma_T^T \\ \tau_{LT}^T \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ -0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} -1.05 \\ -2.51 \\ -2.49 \end{Bmatrix} = \begin{Bmatrix} -4.27 \\ 0.71 \\ -0.73 \end{Bmatrix} \text{ MPa}$$

45° ply, z = −4

$$\begin{Bmatrix} \epsilon_x^M \\ \epsilon_y^M \\ \gamma_{xy}^M \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 4.54 \\ -1.20 \\ 0.39 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x^T \\ \sigma_y^T \\ \tau_{xy}^T \end{Bmatrix} = 10^{-4} \begin{bmatrix} 6.55 & 5.15 & 4.50 \\ 5.15 & 6.55 & 4.50 \\ 4.50 & 4.50 & 5.15 \end{bmatrix} \begin{Bmatrix} 4.54 \\ -1.20 \\ 0.39 \end{Bmatrix} = 10^{-3} \begin{Bmatrix} 2.53 \\ 1.73 \\ 1.70 \end{Bmatrix} \text{ GPa}$$

$$\begin{Bmatrix} \sigma_L^T \\ \sigma_T^T \\ \tau_{LT}^T \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ -0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} 2.53 \\ 1.73 \\ 1.70 \end{Bmatrix} = \begin{Bmatrix} 3.83 \\ 0.43 \\ -0.40 \end{Bmatrix} \text{ MPa}$$

The variations of the residual stresses across the laminate thickness ar shown in Fig. 6-20 for the *xy* reference axes, as well as the longitudinal an transverse axes. It may be noted from the variations of σ_x, σ_y, and τ_{xy} tha the resultant forces N_x, N_y, and N_{xy} and resultant moments, M_x, M_y, and M_x are zero; that is, the net area in each plot and the moment of the area abou any point are zero. This shows the self-equilibrating nature of the residua stresses.

6.10 LAMINATE ANALYSIS THROUGH COMPUTERS

Laminate analysis procedures were discussed in this chapter. It is assume that the laminae elastic properties are known either through the predictio