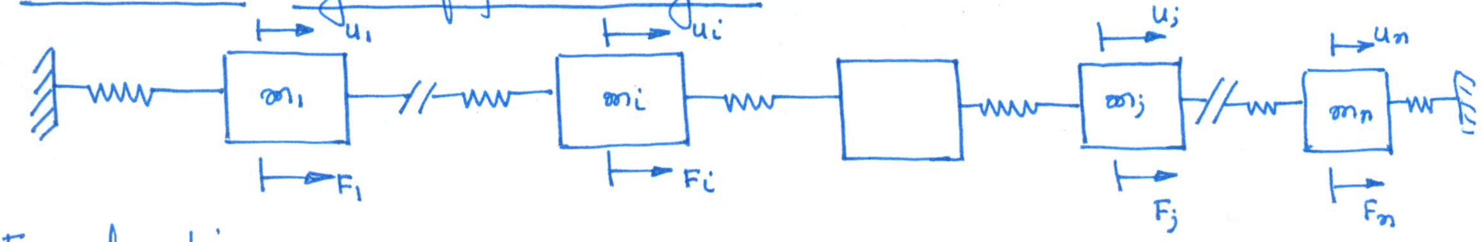
Multi-degree of freedom system : Modal analysis/natural coordinates

Properties of stiffness and mass matrices

Linear multi-degree of freedom system



Eqn. of motion

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_n \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{ij} & \cdots & k_{1n} \\ \vdots & & & & & \\ k_{i1} & k_{i2} & \cdots & k_{ij} & \cdots & k_{in} \\ \vdots & & & & & \\ k_{n1} & k_{n2} & \cdots & k_{nj} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{Bmatrix} = 0$$

k_{ij} → stiffness influence coefficient

→ force required at mass i to produce an unit displacement at mass j with the displacements of all other masses being zero.

$k_{ij} u_j$ → force reqd. at mass i to produce at displacement u_j at mass j with the displacements at all other masses equal to zero

Next, considering displacements at all other masses to be non-zero the the total force at mass m_i is,

$$F_i = \sum_{j=1}^{n} k_{ij} u_j$$ → written in matrix form for $i = 1$ to n as

$$\{F\} = [K]\{u\}$$

↓ stiffness matrix

a_{ij} → flexibility influence co-efficient.

→ displacement at i due to unit force at j

$u_i = a_{ij} F_j$ → displacement at i due to force F_j at j only.

Using superposition principle

$$u_i = \sum_{j=1}^{n} a_{ij} F_j \longrightarrow$$ can be written in a matrix form for $i = 1$ to n.

$$\{u\} = [a]\{F\}$$

$\downarrow$

flexibility matrix

Maxwell's reciprocity theorem:

Displacement at i due to an unit force at j = Displacement at j due to an unit force at i

a_{ij} a_{ji}

$$\Rightarrow [a]^T = [a] \Rightarrow [K]^T = [K] \Rightarrow$$ [M] and [K] matrices are symmetric for linear MDOF

Positive definiteness of [M] & [K] matrices

$$V_i = \frac{1}{2} F_i u_i$$

$$\sum_{i=1}^{n} V_i = \frac{1}{2}\sum_{i=1}^{n} F_i u_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} u_j u_i$$

$$= \frac{1}{2}\{u\}^T[K]\{u\} > 0$$] quadratic form of a matrix

$$T_i = \frac{1}{2}\sum_{j=1}^{n} m_{ij}\dot{u}_j^2$$

$$T = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} m_{ij}\dot{u}_j^2 = \frac{1}{2}\{\dot{u}\}^T[M]\{\dot{u}\} > 0$$

As the quadratic form of the matrices are positive, the matrices are positive definite.

The matrices are semi-positive definite if there are rigid body modes.