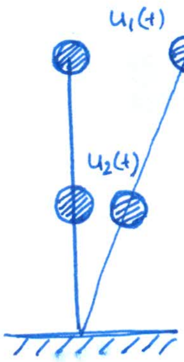
Normal mode of vibration (special case / solution)

The shape of the structure undergoing vibration remains unchanged with time ~~as~~ (it undergoes synchronous motion)



$$\frac{u_1(t)}{u_2(t)} = \text{constant} \neq g(t)$$

$$u_1(t) = \underbrace{r_1}_{\text{constant}}\, \eta(t) \quad \text{and} \quad u_2(t) = \underbrace{r_2}_{\text{constant}}\, \eta(t)$$

Let us consider $\eta(t) = \cos(\omega t - \phi)$

The eqn. of motion for free vibration

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

Under normal mode of vibration

$$-\omega^2 [M] \underbrace{\{r\}}_{\begin{Bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{Bmatrix}} \cos(\omega t - \phi) + [K]\{r\}\cos(\omega t - \phi) = 0$$

$$\Rightarrow \left([K] - \omega^2[M]\right)\{r\} = 0 \quad \Rightarrow \det\left([K] - \omega^2[M]\right) = 0 \quad \text{— (1)}$$

for non-trivial solution of $\{r\}$

Solution of Eqn. (1) gives the natural frequencies and corresponding $\{r\}$ is the mode shape

Orthogonality of mode shapes $\quad \{r\}_i^T [M] \{r\}_j = 0 \;\; \forall \;\; i \neq j$

$\{r\}_i \rightarrow$ mode shape at i^{th} normal mode

Mass normalization of $\{r\}_i$ $\quad \{r\}_i^T [M] \{r\}_i = 1$

For mass normalized mode shapes

$[\phi]^T [M] [\phi] = I \quad$ where, $[\phi]$ is the modal matrix containing mass normalized mode shapes.

Orthogonality of mode shape

For the p^{th} normal mode vibration

$$u(t) = C_p \{r\}_p \cos(\omega_p t - \theta_p) \quad \text{——— (1)}$$

is a solution of the eqn. of motion

where, ω_p and $\{r\}_p$ are the natural frequency & mode shape of the p^{th} normal mode of vibration

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \quad \text{——— (2)}$$

Substituting Eqn. (1) in Eqn. (2), we get,

$$\Rightarrow \quad -C_p\omega_p^2 [M]\{r\}_p \cos(\omega_p t - \theta_p) + C_p[K]\{r\}_p \cos(\omega_p t - \theta_p) = 0$$

$$\Rightarrow \quad \omega_p^2 [M]\{r\}_p = [K]\{r\}_p \quad \text{——— (3)}$$

Pre-multiplying both sides of Eqn. (3) with $\{r\}_q^T$, where, $\{r\}_q$ is the q^{th} mode shape, we get,

$$\omega_p^2 \{r\}_q^T [M]\{r\}_p = \{r\}_q^T [K]\{r\}_p \quad \text{——— (4)}$$

Following similar steps, we can write,

$$\omega_q^2 \{r\}_p^T [M]\{r\}_q = \{r\}_p^T [K]\{r\}_q \quad \text{——— (5)}$$

Taking the transpose on both sides of Eqn. (5), we get,

$$\omega_q^2 \{r\}_q^T [M]\{r\}_p = \{r\}_q^T [K]\{r\}_p \quad \text{——— (6)}$$

(Since [K] and [M] are symmetric matrices)

Subtracting Eqn. (6) from Eqn. (4), we get,

$$(\omega_p^2 - \omega_q^2)\{r\}_q^T [M]\{r\}_p = 0$$

$$\Rightarrow \quad \boxed{\{r\}_q^T [M]\{r\}_p = 0} \quad \text{as } \omega_p^2 \neq \omega_q^2 \text{ for } p \neq q.$$

and $\cancel{\{r\}_q^T [M]\{r\}_p \neq 0}$ $\quad \{r\}_q^T [M]\{r\}_q = 1$ for mass normalized mode shapes

Similarly, $\boxed{\{r\}_q^T [K]\{r\}_p = 0}$ for $p \neq q$.

and $\{r\}_q^T [K]\{r\}_q = \omega_q^2$ for mass normalized mode shapes