Modal analysis

Example 1

$$[M] = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \qquad [K] = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

Initial conditions: $\begin{Bmatrix} u_1(0) \\ u_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} \dot{u}_1(0) \\ \dot{u}_2(0) \end{Bmatrix} = \begin{Bmatrix} v_0 \\ 0 \end{Bmatrix}$

Find the response of the system due to the given initial conditions: using modal analysis.

Natural frequencies: $\omega_1 = 0.7962\sqrt{\frac{k}{m}}$, $\omega_2 = 1.5382\sqrt{\frac{k}{m}}$

$$\{r\}_1 = \begin{Bmatrix} 1.000 \\ 1.366 \end{Bmatrix} \quad \text{and} \quad \{r\}_2 = \begin{Bmatrix} 1.000 \\ -0.366 \end{Bmatrix}$$

Mass normalized mode shapes

$$\{r\}_1^T[M]\{r\}_1 = \{1.0 \quad 1.366\}\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix} = 0.4732m$$

$$\{r\}_1^{mn} = \frac{1}{2.175\sqrt{m}}\begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix} \qquad \text{Similarly,} \quad \{r\}_2^{m} = \frac{1}{0.856\sqrt{m}}\begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}$$

After modal analysis:

$$\ddot{\eta}_p + \omega_p^2\eta_p = 0 \qquad p = 1, 2. \Rightarrow \quad \begin{aligned} \ddot{\eta}_1 + \omega_1^2\eta_1 = 0 \\ \ddot{\eta}_2 + \omega_2^2\eta_2 = 0 \end{aligned}$$

$$\eta_1(t) = \eta_1(0)\cos\omega_1 t + \frac{\dot{\eta}_1(0)}{\omega_1}\sin\omega_1 t$$

$$\eta_2(t) = \eta_2(0)\cos\omega_2 t + \frac{\dot{\eta}_2(0)}{\omega_2}\sin\omega_2 t$$

Using modal expansion:

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = [\Phi]\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \Rightarrow [\Phi]^T[M]\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = [\Phi]^T[M][\Phi]\{\eta\}$$

$$\Rightarrow \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = [\Phi]^T[M]\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = [\phi]^T[M] \begin{Bmatrix} u_1(0) \\ u_2(0) \end{Bmatrix} = [\phi]^T[M] \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

and

$$\begin{Bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \end{Bmatrix} = [\phi]^T[M] \begin{Bmatrix} \dot{u}_1(0) \\ \dot{u}_2(0) \end{Bmatrix} = [\phi]^T[M] \begin{Bmatrix} v_0 \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1.0}{2.175\sqrt{m}} & \frac{1.366}{2.175\sqrt{m}} \\ \frac{1}{0.875\sqrt{m}} & \frac{-0.366}{0.875\sqrt{m}} \end{bmatrix} \begin{Bmatrix} m v_0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} m v_0 / 2.175\sqrt{m} \\ m v_0 / 0.875\sqrt{m} \end{Bmatrix} = \begin{Bmatrix} \frac{v_0\sqrt{m}}{2.175} \\ \frac{v_0\sqrt{m}}{0.875} \end{Bmatrix}$$

$$\Rightarrow \eta_1(t) = \frac{\dot{\eta}_1(0)}{\omega_1}\sin\omega_1 t = \frac{v_0\sqrt{m}}{2.175\,\omega_1}\sin\omega_1 t$$

$$\eta_2(t) = \frac{\dot{\eta}_2(0)}{\omega_2}\sin\omega_2 t = \frac{v_0\sqrt{m}}{0.875\,\omega_2}\sin\omega_2 t$$

$$\Rightarrow \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \frac{v_0\sqrt{m}}{2.175\,\omega_1}\cdot\frac{1}{2.175\sqrt{m}} \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix}\sin\omega_1 t + \frac{v_0\sqrt{m}}{0.875\,\omega_2}\cdot\frac{1}{0.875\sqrt{m}} \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}\sin\omega_2 t$$

$$= \frac{v_0}{2.175^2\,\omega_1} \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix}\sin\omega_1 t + \frac{v_0}{0.875^2\,\omega_2} \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}\sin\omega_2 t$$