Orthogonality of mode shapes for an Euler-Bernoulli beam

For the rth normal mode

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right] = \omega_r^2\, m(x)\,\phi_r(x) \qquad (1)$$

For the sth normal mode

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_s}{dx^2}\right] = \omega_s^2\, m(x)\,\phi_s(x) \qquad (2)$$

Multiplying both sides of Eqn. (1) with $\phi_s(s)$ and integrating, we get,

$$\int_0^L \phi_s \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right]dx = \omega_r^2 \int_0^L \phi_s\, m(x)\,\phi_r\, dx$$

$$\Rightarrow \overbrace{\cancel{\phi_s \frac{d}{dx}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right]\Bigg|_0^L}}^{=0 \text{ (for free, ss, fixed BCs)}} - \int_0^L \frac{d\phi_s}{dx}\frac{d}{dx}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right]dx = \omega_r^2\int_0^L \phi_s\, m(x)\,\phi_r\, dx$$

$$\Rightarrow \overbrace{\cancel{-\frac{d\phi_s}{dx}EI(x)\frac{d^2\phi_r}{dx^2}\Bigg|_0^L}}^{=0 \text{ (for free, fixed, ss BCs)}} + \int_0^L \frac{d^2\phi_s}{dx^2}EI(x)\frac{d^2\phi_r}{dx^2}dx = \omega_r^2\int_0^L \phi_s\, m(x)\,\phi_r\, dx$$

$$\Rightarrow \int_0^L \frac{d^2\phi_s}{dx^2}EI(x)\frac{d^2\phi_r}{dx^2}dx = \omega_r^2\int_0^L \phi_s(x)\, m(x)\,\phi_r(x)\, dx \qquad (3)$$

Similarly, we can get, (by multiplying both sides of Eqn. 2 with $\phi_r(x)$)

$$\int_0^L \frac{d^2\phi_r}{dx^2}EI(x)\frac{d^2\phi_s}{dx^2}dx = \omega_s^2\int_0^L \phi_r(x)\, m(x)\,\phi_s(x)\, dx \qquad (4)$$

Subtracting Eqn. (4) from Eqn. (3), we get,

$$(\omega_r^2 - \omega_s^2)\int_0^L \phi_r(x)\, m(x)\,\phi_s(x)\, dx = 0$$

for $r \neq s$, $\omega_r^2 \neq \omega_s^2 \Rightarrow \int_0^L \phi_r(x)\, m(x)\,\phi_s(x)\, dx = 0$, $\int_0^L \phi_r(x)\, m(x)\,\phi_r(x)\, dx = 1$ (for mass normalized mode shapes)

and

$$\int_0^L \frac{d^2\phi_r}{dx^2}EI(x)\frac{d^2\phi_s}{dx^2}dx = 0 \quad \text{and} \quad \int_0^L \frac{d^2\phi_r}{dx^2}EI(x)\frac{d^2\phi_r}{dx^2}dx = \omega_r^2$$

Fixed-free Euler-Bernoulli beam :

$$\phi(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

@ $x=0$ $\quad y(0,t) = \phi(0)\,T(t) = 0 \Rightarrow \phi(0) = 0$

@ $x=0$ $\quad y'(0,t) = \phi'(0)\,T(t) = 0 \Rightarrow \phi'(0) = 0$

$\phi(0) = 0 \Rightarrow C_2 + C_4 = 0$ and $\phi'(0) = C_1 + C_3 = 0$

$$\therefore \quad \phi(x) = C_1(\sin\beta x - \sinh\beta x) + C_2(\cos\beta x - \cosh\beta x)$$

@ $x=L$, $\quad M = EI\dfrac{\partial^2 y}{\partial x^2} = EI\,\phi''(L)\,T(t) = 0 \Rightarrow \phi''(L) = 0$

@ $x=L$ $\quad V = EI\dfrac{\partial^3 y}{\partial x^3} = EI\,\phi'''(L)\,T(t) = 0 \Rightarrow \phi'''(L) = 0$

$$-\beta^2 C_1(\sin\beta L + \sinh\beta L) - \beta^2 C_2(\cos\beta L + \cosh\beta L) = 0$$

$$-\beta^3 C_1(\cos\beta L + \cosh\beta L) + \beta^3 C_2(\sin\beta L - \sinh\beta L) = 0$$

$$\Rightarrow \begin{bmatrix} \sin\beta L + \sinh\beta L & \cos\beta L + \cosh\beta L \\ \cos\beta L + \cosh\beta L & -(\sin\beta L - \sinh\beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

$$\Rightarrow \boxed{\cos\beta L \cosh\beta L = -1} \Rightarrow \beta L = 1.8751$$

4.6941

7.8548

10.996

Modal analysis : Solution of continuous system

Eqn. of motion of rod under axial vibration

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] + f(x,t) = m(x)\frac{\partial^2 u}{\partial t^2} \quad \text{——} (1)$$

Modal expansion theorem $\quad u(x,t) = \sum_{r=1}^{\infty} \phi_r(x)\,\eta_r(t) \quad$ ——(2)

Substituting Eqn.(2) into Eqn.(1), we get,

$$\frac{d}{dx}\left[EA(x)\sum_r \phi_r'(x)\,\eta_r(t)\right] + f(x,t) = m(x)\sum_r \phi_r(x)\,\ddot{\eta}_r(t) \quad \text{——}(3)$$

Multiplying both sides of Eqn. (3) with $\phi_s(x)$ and Integrating, we get,

$$\int_0^L \phi_s(x)\frac{d}{dx}\left[EA(x)\sum_r \phi_r'(x)\,\eta_r(t)\right]dx + \int_0^L \phi_s(x) f(x,t)\,dx$$

$$= \underbrace{\int_0^L \phi_s(x)\,m(x)\sum \phi_r(x)\,\ddot{\eta}_r(t)\,dx}_{\substack{=\ddot{\eta}_r(t) \text{ for } r=s \\ =0 \text{ for } r\neq s}}$$

$$\Rightarrow \underbrace{\phi_s(x)\,EA(x)\sum \phi_r'(x)\,\eta_r(t)\Big|_0^L}_{\substack{=0 \text{ for all BCs} \\ (\text{fixed/free/spring})}} - \int_0^L \frac{d\phi_s}{dx}EA(x)\frac{d\phi_r}{dx}\eta_r(t)\,dx = \ddot{\eta}_r(t)$$

$$+ \int_0^L \phi_s(x) f(x,t)\,dx$$

$$\Rightarrow -\int_0^L \overbrace{\frac{d\phi_s}{dx}EA(x)\frac{d\phi_r}{dx}}^{\substack{=\omega_r^2\eta_r(t)\ (\text{for } s=r) \\ =0\ (\text{for } s\neq r)}}\eta_r(t)\,dx + \int_0^L \phi_s(x) f(x,t)\,dx = \ddot{\eta}_r(t)$$

$$\Rightarrow -\eta_r(t)\,\omega_r^2 + \int_0^L \phi_r(x) f(x,t)\,dx = \ddot{\eta}_r(t)$$

$$\Rightarrow \ddot{\eta}_r(t) + \omega_r^2\eta_r = \int_0^L \phi_r(x) f(x,t)\,dx = Q_r(t)$$

Example Consider a fixed-fixed rod under axial vibration. Derive the expression for the response of the rod due to an initial condition given as $u(x,0) = Ax\left(1-\frac{x}{L}\right)$

Step 1 Natural frequencies & mode shapes.

$$\phi(x) = C_1 \sin\beta x + C_2 \cos\beta x$$

BCs

$@\, x=0, \quad \phi(0)=0 \Rightarrow C_2 = 0$

$@\, x=L, \quad \phi(L)=0 \Rightarrow C_1 \sin\beta L = 0 \Rightarrow \beta = \frac{\bar{\lambda} r}{L} \Rightarrow \omega_r = \frac{\bar{\lambda} r}{L}\sqrt{\frac{EA}{m}}$

and $\phi_r(x) = C_r \sin\frac{\bar{\lambda} r x}{L}$, Mass normalized mode shape. $\phi_r(x) = \sqrt{\frac{2}{mL}} \sin\frac{\bar{\lambda} r x}{L}$

Step 2 Eqn. of motion after modal analysis

$$\ddot{\eta}_r + \omega_r^2 \eta_r = 0$$

$$\Rightarrow \eta_r(t) = \eta_r(0)\cos\omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t$$

Step 3 $\eta_r(0) = ?$ and $\dot{\eta}_r(0) = \phi$

Initial conditions $\quad u(x,0) = \sum_{r=1}^{\infty} \phi_r(x)\eta_r(0) = \cancel{Ax}\; Ax\left(1-\frac{x}{L}\right).$

$$\sum_{r=1}^{\infty} \int_0^L \phi_s(x)\, m(x)\, \phi_r(x)\, \eta_r(0)\, dx = \int_0^L m(x)\, \phi_s(x)\, Ax\left(1-\frac{x}{L}\right) dx$$

$$\eta_r(0) = \int_0^L m\sqrt{\frac{2}{mL}} \sin\frac{\bar{\lambda} x}{L}\, Ax\left(1-\frac{x}{L}\right) dx = \frac{-2L^2}{(\lambda r)^3}\left[(-1)^r - 1\right] m\sqrt{\frac{2}{mL}}\, A$$

$$\doteq 0 \quad \text{for } r = 2, 4, \ldots$$

$$= \sqrt{\frac{2}{mL}} \cdot \frac{4mL^3 A}{\bar{\lambda} r^3} \quad r = 1, 3, \ldots$$

and $\dot{\eta}_r(0) = 0$ as $\dot{u}(x,0) = 0$

$$\therefore \quad u(x,t) = \sum_r \phi_r(x)\eta_r(t) = \frac{2}{mL} \cdot \frac{4mL^3 A}{\lambda r^3} \cos\frac{\bar{\lambda} r}{L}\sqrt{\frac{EA}{m}}\, t\, \sin\frac{\bar{\lambda} x r}{L}$$

$$= \sum_{r=1,3,\ldots} \frac{8L^2 A}{\lambda^3 r^3} \cos\omega_r t \sin\frac{\bar{\lambda} r x}{L}.$$