Orthogonality of mode shapes for an Euler- Bernoulli beam

For the r^{th} normal mode
$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right] = \omega_r^2\, m(x)\,\phi_r(x) \qquad (1)$$

For the s^{th} normal mode
$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_s}{dx^2}\right] = \omega_s^2\, m(x)\,\phi_s(x) \qquad (2)$$

Multiplying both sides of Eqn. (1) with $\phi_s(s)$ and integrating, we get,

$$\int_0^L \phi_s \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right]dx = \omega_r^2 \int_0^L \phi_s\, m(x)\,\phi_r\, dx$$

$$\Rightarrow \overbrace{\cancel{\phi_s \frac{d}{dx}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right]\Bigg|_0^L}}^{=0 \text{ (for free, ss, fixed BCs)}} - \int_0^L \frac{d\phi_s}{dx}\frac{d}{dx}\left[EI(x)\frac{d^2\phi_r}{dx^2}\right]dx = \omega_r^2 \int_0^L \phi_s\, m(x)\,\phi_r\, dx$$

$$\Rightarrow \overbrace{\cancel{-\frac{d\phi_s}{dx} EI(x)\frac{d^2\phi_r}{dx^2}\Bigg|_0^L}}^{=0 \text{ (for free, fixed, ss BCs)}} + \int_0^L \frac{d^2\phi_s}{dx^2} EI(x)\frac{d^2\phi_r}{dx^2}dx = \omega_r^2 \int_0^L \phi_s\, m(x)\,\phi_r\, dx$$

$$\Rightarrow \int_0^L \frac{d^2\phi_s}{dx^2} EI(x)\frac{d^2\phi_r}{dx^2}dx = \omega_r^2 \int_0^L \phi_s(x)\, m(x)\,\phi_r(x)\, dx \qquad (3)$$

Similarly, we can get, (by multiplying both sides of Eqn. 2 with $\phi_r(x)$)

$$\int_0^L \frac{d^2\phi_r}{dx^2} EI(x)\frac{d^2\phi_s}{dx^2}dx = \omega_s^2 \int_0^L \phi_r(x)\, m(x)\,\phi_s(x)\, dx \qquad (4)$$

Subtracting Eqn. (4) from Eqn. (3), we get,

$$(\omega_r^2 - \omega_s^2)\int_0^L \phi_r(x)\, m(x)\,\phi_s(x)\, dx = 0$$

for $r \neq s$, $\omega_r^2 \neq \omega_s^2 \Rightarrow \displaystyle\int_0^L \phi_r(x)\, m(x)\,\phi_s(x)\, dx = 0$, $\displaystyle\int_0^L \phi_r(x)\, m(x)\,\phi_r(x)\, dx = 1$ (for mass normalized mode shapes)

and

$$\int_0^L \frac{d^2\phi_r}{dx^2} EI(x)\frac{d^2\phi_s}{dx^2}dx = 0 \quad \text{and} \quad \int_0^L \frac{d^2\phi_r}{dx^2} EI(x)\frac{d^2\phi_r}{dx^2}dx = \omega_r^2$$

Fixed-free Euler-Bernoulli beam:

$$\phi(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

@ $x=0$ $\quad y(0,t) = \phi(0)\,T(t) = 0 \Rightarrow \phi(0) = 0$

@ $x=0$ $\quad y'(0,t) = \phi'(0)\,T(t) = 0 \Rightarrow \phi'(0) = 0$

$\phi(0) = 0 \Rightarrow C_2 + C_4 = 0$ and $\phi'(0) = C_1 + C_3 = 0$

$$\therefore \quad \phi(x) = C_1(\sin\beta x - \sinh\beta x) + C_2(\cos\beta x - \cosh\beta x)$$

@ $x = L$, $\quad M = EI\dfrac{\partial^2 y}{\partial x^2} = EI\,\phi''(L)\,T(t) = 0 \Rightarrow \phi''(L) = 0$

@ $x = L$ $\quad V = EI\dfrac{\partial^3 y}{\partial x^3} = EI\,\phi'''(L)\,T(t) = 0 \Rightarrow \phi'''(L) = 0$

$$-\beta^2 C_1(\sin\beta L + \sinh\beta L) - \beta^2 C_2(\cos\beta L + \cosh\beta L) = 0$$

$$-\beta^3 C_1(\cos\beta L + \cosh\beta L) + \beta^3 C_2(\sin\beta L - \sinh\beta L) = 0$$

$$\Rightarrow \begin{bmatrix} \sin\beta L + \sinh\beta L & \cos\beta L + \cosh\beta L \\ \cos\beta L + \cosh\beta L & -(\sin\beta L - \sinh\beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0$$

$$\Rightarrow \boxed{\cos\beta L \cosh\beta L = -1} \Rightarrow \beta L = 1.8751$$

$$4.6941$$

$$7.8548$$

$$10.996$$