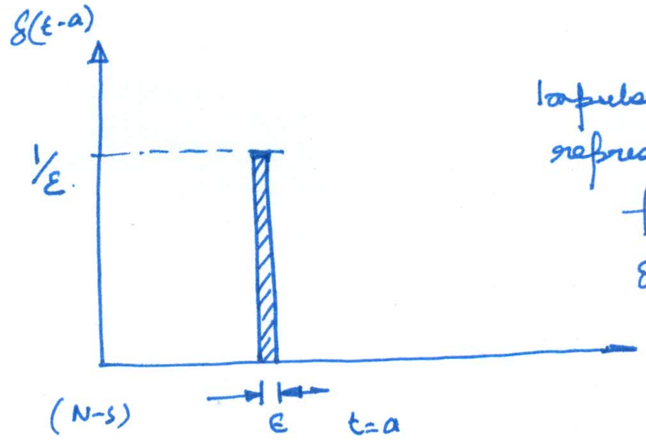
Response to arbitrary loading (non-harmonic, non-periodic)

Impulse response



Impulse is mathematically represented by a Dirac-delta function

$\delta(t-a) = 0 \;\forall\; t \neq a$

$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 0$$

Impulse load = $\hat{F}\,\delta(t-a)\;s^{-1}$

unit of impulse

Impulsive response: The response of a system with zero initial condition due to an unit impulse acting at $t=0$ is called impulsive response. and referred as $h(t)$

Eqn. of motion $\quad m\ddot{u} + c\dot{u} + ku = \hat{F}\,\delta(t)$

The effect of impulse acting at $t=0$ is equivalent to imparting an initial velocity $\frac{\hat{F}}{m}$

$\therefore$ The impulse response is equivalent to the free vibration response with initial condn $\quad u(0)=0$ and $\dot{u}(0) = \frac{\hat{F}}{m}$

$\therefore$ The effect of applying an impulse load at $t=0$ is equivalent of giving an initial velocity $\dot{u}_0 = \frac{\hat{F}}{m}$.

Response of an under-damped SDOF system due to an ~~impu~~ impulse.

$$u(t) = e^{-\xi\omega_n t}\left[u_0 \cos\omega_d t + \frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_d}\sin\omega_d t\right]$$

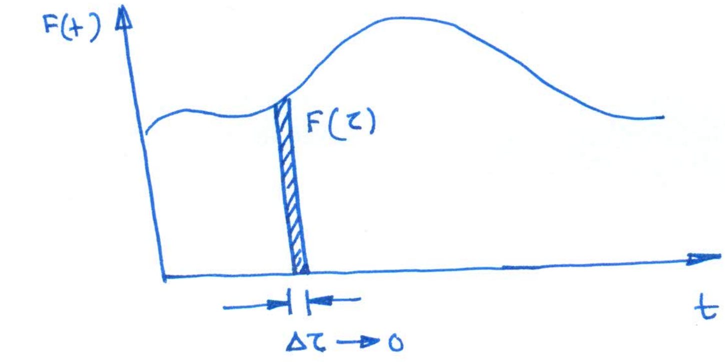
where, $u_0 = 0$ & $\dot{u}_0 = \hat{F}/m$

$$\Rightarrow \quad u(t) = e^{-\xi\omega_n t}\,\frac{\hat{F}}{m\omega_d}\sin\omega_d t$$

Response due to unit impulse load, $h(t) = \frac{1}{m\omega_d}\,e^{-\xi\omega_n t}\sin\omega_d t$.

$$h(t-\tau) = \frac{1}{m\omega_d}\,e^{-\xi\omega_n (t-\tau)}\sin\omega_d (t-\tau)$$

Arbitrary loading



The shaded element of the force can be written as,

$$F(\tau)\,\Delta\tau\,\delta(t-\tau)$$

The response due to the shaded element of the load.

$$\bar{u}(t) = \frac{F(\tau)\,\Delta\tau\, e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)}{m\omega_d}$$

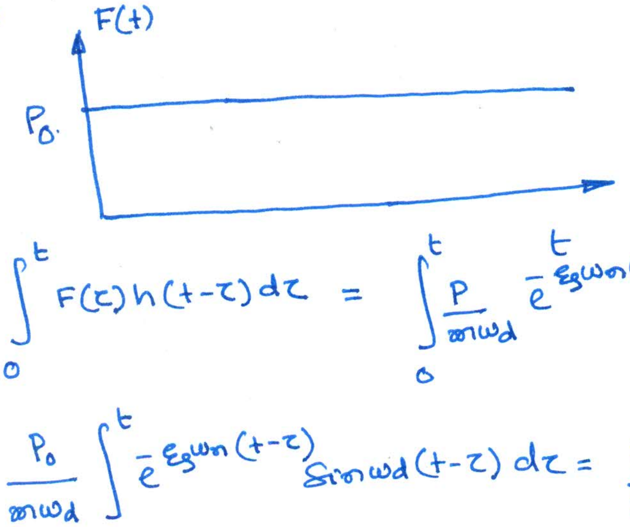
$$= F(\tau)\,\Delta\tau\, h(t-\tau)$$

∴ The response due to the entire load,

$$u(t) = \sum_{\Delta\tau\to 0} F(\tau)\,\Delta\tau\, h(t-\tau)$$

$$= \int_0^t F(\tau)\,h(t-\tau)\,d\tau \longrightarrow \text{Convolution integral / Duhamel integral}$$

Example Response of an under-damped SDOF due to step loading using convolution integral



$$u(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau = \int_0^t \frac{P}{m\omega_d} e^{-\xi\omega_n(t-\tau)} \sin\omega_d(t-\tau)\,d\tau$$

$$= \frac{P_0}{m\omega_d}\int_0^t e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau = \frac{P_0}{m\omega_d} I$$

$$I = \int_0^t e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$$

$$= e^{-\xi\omega_n(t-\tau)}\frac{1}{\omega_d}\cos\omega_d(t-\tau)\Big|_0^t - \int_0^t \xi\omega_n e^{-\xi\omega_n(t-\tau)}\frac{1}{\omega_d}\cos\omega_d(t-\tau)\,d\tau$$

$$= \left[\frac{1}{\omega_d} - \frac{1}{\omega_d}e^{-\xi\omega_n t}\cos\omega_d t\right] + \frac{\xi\omega_n}{\omega_d^2}e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)\Big|_0^t$$

$$- \int_0^t \frac{\xi^2\omega_n^2}{\omega_d^2} e^{-\xi\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau.$$

$$= \frac{1}{\omega_d}\left[1 - e^{-\xi\omega_n t}\cos\omega_d t\right] - \frac{\xi\omega_n}{\omega_d^2} e^{-\xi\omega_n t}\sin\omega_d t - \frac{\xi^2}{1-\xi^2} I$$

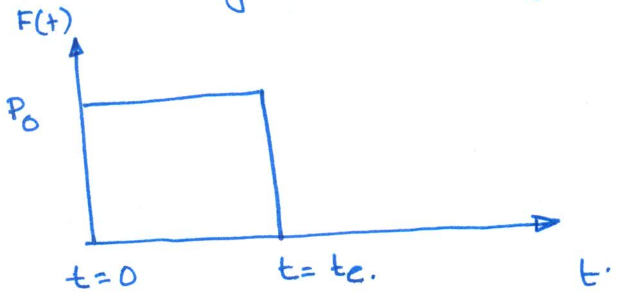
$$\Rightarrow \quad \left[1 + \frac{\xi^2}{1-\xi^2}\right] I = \frac{1}{\omega_d}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right]$$

$$\Rightarrow \quad I = \frac{1-\xi^2}{\omega_d^2}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right]$$

$$u(t) = \frac{P_0(1-\xi^2)}{m\omega_d^2}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right]$$

$$= \frac{P_0}{m\omega_n^2}\left[1 - e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right)\right]$$

Example Find the response of an undamped SDOF system due to the following load using convolution integral



For $t \le t_c$

$$u(t) = \int_0^t F(\tau)\,h(t-\tau)\,d\tau = \int_0^t \underbrace{P_0\frac{1}{m\omega_n}\sin\omega_n(t-\tau)}_{\text{Impulse response for undamped system}}\,d\tau$$

$$= \frac{P_0}{m\omega_n}\frac{1}{\omega_n}\cos\omega_n(t-\tau)\Big|_0^t$$

$$= \frac{P_0}{m\omega_n^2}\left[1-\cos\omega_n t\right]$$

For $t \ge t_c$

$$u(t) = \int_0^{t_c} \underbrace{F(\tau)}_{=P_0} h(t-\tau)\,d\tau + \int_{t_c}^{t} \underbrace{F(\tau)}_{=0} h(t-\tau)\,d\tau$$

$$= \frac{P_0}{m\omega_n}\int_0^{t_c}\sin\omega_n(t-\tau)\,d\tau = \frac{P_0}{m\omega_n^2}\cos\omega_n(t-\tau)\Big|_0^{t_c}$$

$$= \frac{P_0}{m\omega_n^2}\left[\cos\omega_n(t-t_c) - \cos\omega_n t\right]$$