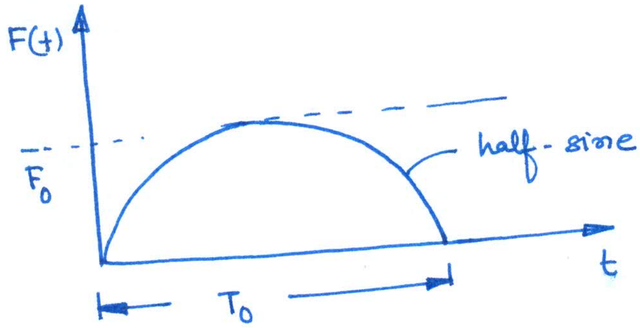
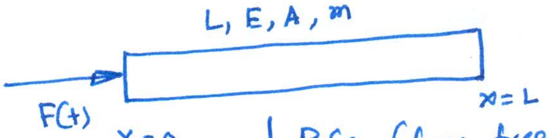
Modal analysis of continuous system

Example A missile in flight is excited longitudinally by a thrust $F(t)$ of it's rocket engine at the end $x=0$. Determine the expression for the displacement $u(x,t)$. The missile can be modeled as a rod of uniform c/s.



Eqn. of motion



$$EA\frac{\partial^2 u}{\partial x^2} + F(t)\,\delta(x-0) = m\frac{\partial^2 u}{\partial t^2}$$

thrust force acting at $x=0$

BCs (free-free condn)

@ $x=0$ $\quad \frac{\partial u}{\partial x}(x,t)=0 \Rightarrow \phi'(0)=0$

and

@ $x=L$ $\quad \frac{\partial u(x,t)}{\partial x} = 0 \Rightarrow \phi'(L)=0$

After modal analysis, we get,

$$\ddot{\eta}_r + \omega_r^2\eta_r = \int_0^L \phi_r(x)\,F(t)\,\delta(x-0)\,dx$$

$$= F(t)\,\phi_r(0)$$

Natural frequencies and mode shapes

$$\phi(x) = C_1\sin\beta x + C_2\cos\beta x$$

$$\phi'(x) = \beta C_1\cos\beta x - \beta C_2\sin\beta x$$

@ $x=0$, $\phi'(0)=0 \Rightarrow C_1=0$

@ $x=0$ $\quad \phi'(L)=0 \Rightarrow -\beta C_2\sin\beta L = 0$

$\Rightarrow \beta = 0$ (rigid body mode) and $\beta L = \pi r$

$\Rightarrow \beta = \frac{\pi r}{L}$

Natural frequencies

$\omega = 0$ (rigid body mode)

$$\omega_r = \frac{\pi r}{L}\sqrt{\frac{EA}{m}}$$

Mode shape

$$\phi_r(x) = \sqrt{\frac{2}{mL}}\cos\frac{\pi r x}{L}$$

$\therefore$ Eqn. after modal analysis,

$$\ddot{\eta}_r + \omega_r^2 \eta_r = \sqrt{\frac{2}{mL}} F(t) = \sqrt{\frac{2}{mL}} F_0 \sin \frac{2\pi t}{2T_0} = \sqrt{\frac{2}{mL}} F_0 \sin \frac{\pi t}{T_0} \quad \text{for flexible modes} \qquad (1)$$

From Eqn. (1), we get,

$$\eta_r(t) = \frac{1}{\omega_r} \sqrt{\frac{2}{mL}} F_0 \int_0^t \sin \frac{\pi \tau}{T_0} \sin \omega_r (t-\tau)\, d\tau \qquad t < T_0$$

$$= \frac{1}{\omega_r} \sqrt{\frac{2}{mL}} F_0 \int_0^{T_0} \sin \frac{\pi \tau}{T_0} \sin \omega_r (t-\tau)\, d\tau \qquad t \geqslant T_0$$

$$u(x,t) = \sum_{r=1}^{N} \phi_r(x) \eta_r(t)$$