Hamilton's principle

Principle of virtual work

Let us consider a system of N particles and denote by R_i $(i = 1, 2, \ldots, N)$ the resultant force acting on particle i. For a system in equilibrium, $R_i = 0$ and the same can be said about the dot product $R_i \cdot \delta r_i$, where $R_i \cdot \delta r_i$ is the virtual work performed by the force R_i through the virtual displacement δr_i

Virtual displacement δr_i —
1. It is consistent with the system constraints
2. arbitrary
3. infinitesimal changes to the actual displacement of the body at equilibrium.
4. virtual displacement takes place contemporaneously (it is part of a overall infinitesimal displacement)

$$\text{Virtual work} = \bar{\delta W} = \sum_{i=1}^{N} R_i \cdot \delta r_i = 0.$$

$$\underbrace{R_i}_{} = \underbrace{F_i}_{\text{applied forces}} + \underbrace{F_i'}_{\text{constraint forces}} \qquad i = 1 \text{ to } N$$

$$\therefore \quad \bar{\delta W} = \sum F_i \cdot \delta r_i + \underbrace{\sum F_i' \cdot \delta r_i}_{=0} = 0$$

0 as δr_i satisfies the constraints

$$\therefore \quad \boxed{\delta \bar{W} = \sum F_i \cdot \delta r_i = 0}$$

$\Rightarrow$ Principle of virtual work

The work done by the applied forces through virtual displacements compatible with system constraint is zero.

D'Alembert's Principle

The principle of virtual work is concerned with the static equilibrium of a system. It can be extended to dynamical systems by means of a principle attributed to D'Alembert.

For a dynamical system,

$$R_i = F_i + F_i' - m\ddot{r}_i = 0$$

$$\therefore \sum_{i=1}^{N} (F_i + F_i' - m\ddot{r}_i)\,\delta r_i = \sum_{i=1}^{N} (F_i - m\ddot{r}_i)\,\delta r_i = 0 \quad \text{—— (1)}$$

$$\frac{d}{dt}(\dot{r}_i \cdot \delta r_i) = \ddot{r}_i \cdot \delta r_i + \dot{r}_i \cdot \delta \dot{r}_i$$

$$= \ddot{r}_i \cdot \delta r_i + \delta\left(\tfrac{1}{2}\dot{r}_i \cdot \dot{r}_i\right)$$

$$m\frac{d}{dt}(\dot{r}_i \cdot \delta r_i) = m_i\ddot{r}_i \cdot \delta r_i + m_i\,\delta\left(\tfrac{1}{2}\dot{r}_i \cdot \dot{r}_i\right)$$

$$\Rightarrow \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{r}_i \cdot \delta r_i) = \sum_{i=1}^{N} m_i\ddot{r}_i \cdot \delta r_i + \sum_{i=1}^{N} m_i\,\delta\left(\tfrac{1}{2}\dot{r}_i \cdot \dot{r}_i\right)$$

$$\Rightarrow \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{r}_i \cdot \delta r_i) = \sum m_i\ddot{r}_i \cdot \delta r_i + \delta T \quad \text{—— (2)}$$

Substituting Eqn. (2) in Eqn. (1), we get,

$$\sum_{i=1}^{N} (F_i \cdot \delta r_i + \delta T) = \sum_{i=1}^{N} m_i \frac{d}{dt}(\ddot{r}_i \cdot \delta r_i)$$

$$\Rightarrow (\overline{\delta W} + \delta T) = \sum_{i=1}^{N} m_i \frac{d}{dt}(\ddot{r}_i \cdot \delta r_i)$$

$$\Rightarrow \int_{t_1}^{t_2} (\overline{\delta W} + \delta T)\,dt = \sum_{i=1}^{N} \int_{t_1}^{t_2} m_i \frac{d}{dt}(\ddot{r}_i \cdot \delta r_i)\,dt$$

$$= \overset{=0}{\cancel{\sum_{i=1}^{N} m_i \dot{r}_i \delta r_i \Big|_{t_1}^{t_2}}} \quad \text{as } \delta r_i(t_1) = \delta r_i(t_2) = 0$$

$$\Rightarrow \int_{t_1}^{t_2} (\overline{\delta W} + \delta T)\,dt = 0$$

Virtual work $\overline{\delta W} = \underbrace{\overline{\delta W}_{NC}}_{\text{work due to non-conservative force}} + \overline{\delta W}_C \rightarrow$ work due to conservative forces.

$\overline{\delta W}_C = -\delta V \rightarrow$ potential energy

$$\therefore \int_{t_1}^{t_2} (\overline{\delta W} + \delta T)\,dt = 0 \Rightarrow \int_{t_1}^{t_2} (\overline{\delta W}_{NC} - \delta V + \delta T)\,dt = 0$$

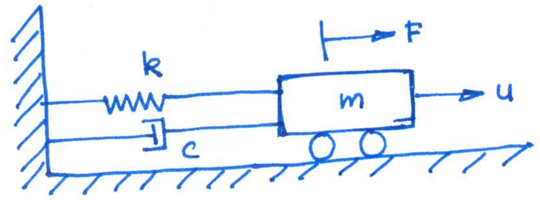
Defining Lagrangian $L = T - V$

$$\boxed{\int_{t_1}^{t_2} (\overline{\delta W}_{NC} + \delta L)\,dt = 0}$$ Extended Hamilton's principle

In absence of non-conservative forces $\overline{\delta W}_{NC} = 0$

and $$\boxed{\int_{t_1}^{t_2} \delta L\,dt = 0}$$ Hamilton's principle.

Example



$T = \frac{1}{2} m \dot{u}^2$

$V = \frac{1}{2} k u^2$

$\overline{\delta W}_{nc} = (F - c\dot{u})\delta u$

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left\{ \delta \left[\frac{1}{2} m\dot{u}^2 - \frac{1}{2} k u^2\right] + (F - c\dot{u})\delta u \right\} dt = 0$$

$$= \int_{t_1}^{t_2} \{ m\dot{u}\delta\dot{u} - k u \delta u + (F - c\dot{u})\delta u \} dt = 0$$

integrating by parts

$$= \cancel{m\dot{u}\delta u \Big|_{t_1}^{t_2}} + \int_{t_1}^{t_2} (-m\ddot{u} - ku + F - c\dot{u})\delta u\,dt = 0$$

$= 0 \quad (\because \delta u(t_1) = \delta u(t_2) = 0)$

$$\Rightarrow \int_{t_1}^{t_2} (-m\ddot{u} - ku + F - c\dot{u})\delta u\,dt = 0$$

Since, δu is arbitrary, $\delta u \neq 0$,

$$\therefore \boxed{m\ddot{u} + c\dot{u} + ku = F}$$

Example Axial vibration of rod.

Continuous System Kinetic energy $T = \frac{1}{2}\int_0^L m(x)\,\dot{u}^2 dx$

Strain energy $V = \frac{1}{2}\int_0^L EA(x)\left(\frac{\partial u}{\partial x}\right)^2 dx$

$$\delta\int_{t_1}^{t_2} L\,dt = \delta\int_{t_1}^{t_2}\int_0^L \frac{1}{2}\left[m(x)\dot{u}^2 - EA(x)\left(\frac{\partial u}{\partial x}\right)^2\right]dx\,dt = 0$$

$$\int_0^L\int_{t_1}^{t_2} m(x)\,\dot{u}\frac{d}{dt}(\delta u)\,dx\,dt - \int_0^L\int_{t_1}^{t_2} EA(x)\frac{\partial u}{\partial x}\frac{\partial}{\partial x}(\delta u)\,dx\,dt = 0$$

$$\Rightarrow \int_0^L m(x)\,\dot{u}\,\delta u\,dx\Big|_{t_1}^{t_2} - \int_0^L\int_{t_1}^{t_2} m(x)\,\ddot{u}\,\delta u\,dx\,dt - \int_0^L\int_{t_1}^{t_2} EA(x)\frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x}\,dt\,dx = 0$$

$$\Rightarrow -\int_0^L\int_{t_1}^{t_2} m(x)\,\ddot{u}\,\delta u\,dx\,dt - \underbrace{\int_{t_1}^{t_2} EA(x)\frac{\partial u}{\partial x}\delta u\Big|_0^L dt}_{=0\ (BCs)} + \int_0^L\int_{t_1}^{t_2}\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]\delta u\,dt\,dx = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\int_0^L\left\{-m(x)\,\ddot{u} + \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right]\right\}\delta u\,dx\,dt = 0$$

$\delta u \neq 0$ (arbitrary) $\therefore$ $\boxed{\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] + m(x)\,\ddot{u} = 0}$