Hamilton principle : Euler-Benoulli beam.

Displacement field

$$u(x,y,z,t) = -z\frac{\partial w(x,t)}{\partial x}.$$
$$v(x,y,z,t) = 0$$
$$w(x,y,z,t) = w(x,t)$$

Strain-displacement relation

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}.$$
$$\sigma_{xx} = E\epsilon_{xx}$$

Strain energy $V = \frac{1}{2}\int_0^L\int_A \sigma_{xx}^T \epsilon_{xx}\,dx\,dA = \frac{1}{2}\int_0^L\int_A E\epsilon_{xx}^2\,dx\,dA$

$$= \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

Kinetic energy $T = \frac{1}{2}\int_0^L\int_A \rho\dot{w}^2\,dx\,dA = \frac{1}{2}\int_0^L \rho A(x)\dot{w}^2\,dx$

$$\int_{t_1}^{t_2}\delta L\,dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L \delta\left[\rho A(x)\dot{w}^2 - EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx\,dt = 0$$

$$= \cancel{\frac{1}{2}}\int_{t_1}^{t_2}\int_0^L\left\{\rho A(x)\frac{\partial w}{\partial t}\frac{\partial}{\partial t}(\delta w) - EI(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2}{\partial x^2}(\delta w)\right\}dx\,dt = 0$$

Integrating by parts $(\delta w(t_1) = \delta w(t_2) = 0)$

$$\underbrace{\cancel{\int_0^L \rho A(x)\frac{\partial w}{\partial t}\delta w\Big|_{t_1}^{t_2}dx}}_{=0} - \int_{t_1}^{t_2}\int_0^L \rho A(x)\frac{\partial^2 w}{\partial t^2}\delta w\,dx\,dt - \int_{t_1}^{t_2}EI(x)\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt$$
$$+\int_{t_1}^{t_2}\int_0^L\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]\delta\left(\frac{\partial w}{\partial x}\right)dx\,dt = 0$$

$$\Rightarrow -\int_{t_1}^{t_2}\int_0^L \rho A(x)\frac{\partial^2 w}{\partial t^2}\delta w\,dx\,dt - \int_{t_1}^{t_2}EI(x)\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt + \int_{t_1}^{t_2}\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]\delta w\Big|_0^L dt$$
$$-\int_{t_1}^{t_2}\int_0^L\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]\delta w\,dx\,dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2}\int_0^L \left\{ \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right] - \rho A(x)\frac{\partial^2 w}{\partial t^2} \right\} \delta w \, dx\, dt - \int_{t_1}^{t_2} EI(x)\frac{\partial^2 w}{\partial x^2}\,\delta\left(\frac{\partial w}{\partial x}\right)\Big|_0^L dt$$

$$+ \int_{t_1}^{t_2} \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]\delta w \Big|_0^L dt = 0$$

Since δw is arbitrary and cannot be zero always

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{——— (1)}$$

and the boundary conditions are,

$EI(x)\dfrac{\partial^2 w}{\partial x^2} = 0$ or $\dfrac{\partial w}{\partial x}$ is specified at $x=0/x=L$

and $\dfrac{\partial}{\partial x}\left[EI(x)\dfrac{\partial^2 w}{\partial x^2}\right] = 0$ or w is specified at $x=0/x=L$.

Timoshenko beam / Higher order shear deformation beam.

Displacement field

$$u(x,y,z,t) = -z\phi(x,t) \quad \text{rotational dof } \left(\frac{\partial w}{\partial x} + \beta(x)\right)$$

$$v(x,y,z,t) = 0$$

$$w(x,y,z,t) = w(x,t)$$

Strain-displacement relation

$$\epsilon_{xx} = -z\frac{\partial \phi}{\partial x}$$

$$\gamma_{xz} = -\phi + \frac{\partial w}{\partial x}.$$

Strain energy
$$V = \frac{1}{2}\int_0^L\int_A \left(\sigma_{xx}^T \epsilon_{xx} + \tau_{xz}^T \gamma_{xz}\right) dx\,dA$$

$$= \frac{1}{2}\int_0^L \left[EI\left(\frac{\partial \phi}{\partial x}\right)^2 + GA\left(-\phi + \frac{\partial w}{\partial x}\right)^2\right] dx$$

Kinetic energy
$$T = \frac{1}{2}\int_0^L\int_A \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + \rho z^2\left(\frac{\partial \phi}{\partial t}\right)^2\right] dx\,dA$$

$$= \frac{1}{2}\int_0^L \left[\rho A(x)\left(\frac{\partial w}{\partial t}\right)^2 + \rho I\left(\frac{\partial \phi}{\partial t}\right)^2\right] dx$$

$$\therefore\ \delta\int_{t_1}^{t_2} L\,dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L \delta\left\{\rho A(x)\left(\frac{\partial w}{\partial t}\right)^2 + \rho I\left(\frac{\partial \phi}{\partial x}\right)^2 - EI\left(\frac{\partial \phi}{\partial x}\right)^2 - GA\left(-\phi + \frac{\partial w}{\partial x}\right)^2\right\} dx\,dt = 0$$

Integrating by parts:

$$\cancel{\int_0^L \rho A(x)\frac{\partial w}{\partial t}\delta w\Big|_{t_1}^{t_2} dx}^{=0} - \int_{t_1}^{t_2}\int_0^L \rho A(x)\frac{\partial^2 w}{\partial t^2}\delta w\,dx\,dt + \cancel{\int_0^L \rho A\frac{\partial \phi}{\partial x}\delta\phi\Big|_{t_1}^{t_2} dx}^{=0} - \int_{t_1}^{t_2}\int_0^L \rho I\frac{\partial^2 \phi}{\partial x^2}\delta\phi\,dx\,dt$$

$$- \int_{t_1}^{t_2} EI(x)\frac{\partial \phi}{\partial x}\delta\phi\Big|_0^L dt + \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left[EI(x)\frac{\partial \phi}{\partial x}\right]\delta\phi\,dx\,dt - \int_{t_1}^{t_2} GA(x)\left(-\phi + \frac{\partial w}{\partial x}\right)\delta w\Big|_0^L dt$$

$$+ \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left[GA(x)\left(-\phi + \frac{\partial w}{\partial x}\right)\right]\delta w\,dx\,dt$$

$$- \int_{t_1}^{t_2}\int_0^L GA(x)\left(-\phi + \frac{\partial w}{\partial x}\right)(-\delta\phi)\,dx\,dt = 0$$