

Hamilton principle: Euler-Bernoulli beam.

Displacement field

$$u(x, y, z, t) = -z \frac{\partial w(x, t)}{\partial x}$$

$$v(x, y, z, t) = 0$$

$$w(x, y, z, t) = w(x, t)$$

Strain-displacement relation

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}$$

$$\sigma_{xx} = E \epsilon_{xx}$$

Strain energy $V = \frac{1}{2} \int_0^L \int_A \sigma_{xx}^T \epsilon_{xx} dx dA = \frac{1}{2} \int_0^L \int_A E \epsilon_{xx}^2 dx dA$

$$= \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx$$

Kinetic energy $T = \frac{1}{2} \int_0^L \int_A \rho \dot{w}^2 dx dA = \frac{1}{2} \int_0^L \rho A(x) \dot{w}^2 dx$

$$\int_{t_1}^{t_2} \delta L dt = \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \delta \left[\rho A(x) \dot{w}^2 - EI(x) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx dt = 0$$

$$= \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \left\{ \rho A(x) \frac{\partial w}{\partial t} \frac{\partial}{\partial t} (\delta w) - EI(x) \frac{\partial^2 w}{\partial x^2} \frac{\partial^2}{\partial x^2} (\delta w) \right\} dx dt = 0$$

Integrating by parts
 (points $\delta w(t_1) = \delta w(t_2) = 0$)

$$\int_0^L \rho A(x) \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^L \rho A(x) \frac{\partial^2 w}{\partial t^2} \delta w dx dt - \int_{t_1}^{t_2} EI(x) \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_0^L dt$$

$$+ \int_{t_1}^{t_2} \int_0^L \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] \delta \left(\frac{\partial w}{\partial x} \right) dx dt = 0$$

$$\Rightarrow - \int_{t_1}^{t_2} \int_0^L \rho A(x) \frac{\partial^2 w}{\partial t^2} \delta w dx dt - \int_{t_1}^{t_2} \int_0^L EI(x) \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_0^L dt + \int_{t_1}^{t_2} \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] \delta w \Big|_0^L dt$$

$$- \int_{t_1}^{t_2} \int_0^L \frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] \delta w dx dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \int_0^L \left\{ \frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] - \rho A(x) \frac{\partial^2 w}{\partial t^2} \right\} \delta w dx dt - \int_{t_1}^{t_2} EI(x) \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \Big|_0^L dt$$

$$+ \int_{t_1}^{t_2} \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] \delta w \Big|_0^L dt = 0$$

Since δw is arbitrary and cannot be zero always

$$\frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] + \rho A(x) \frac{\partial^2 w}{\partial t^2} = 0 \quad \text{--- (1)}$$

and the boundary conditions are,

$$EI(x) \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} \text{ is specified at } x=0/x=L$$

$$\text{and } \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] = 0 \quad \text{or } w \text{ is specified at } x=0/x=L$$

Timoshenko beam / Higher order shear deformation beam.

Displacement field

$$u(x, y, z, t) = -z \overset{\substack{\text{rotational} \\ \text{dof}}}{\phi(x, t)} \left(\frac{\partial w}{\partial x} + \beta(x) \right)$$

$$v(x, y, z, t) = 0$$

$$w(x, y, z, t) = w(x, t)$$

Strain-displacement relation

$$\epsilon_{xx} = -z \frac{\partial \phi}{\partial x}$$

$$\gamma_{xz} = -\phi + \frac{\partial w}{\partial x}$$

Strain energy

$$V = \frac{1}{2} \int_0^L \int_A \left(\sigma_{xx}^T \epsilon_{xx} + \tau_{xz}^T \gamma_{xz} \right) dx dA$$

$$= \frac{1}{2} \int_0^L \left[EI \left(\frac{\partial \phi}{\partial x} \right)^2 + GA \left(-\phi + \frac{\partial w}{\partial x} \right)^2 \right] dx$$

Kinetic energy

$$T = \frac{1}{2} \int_0^L \int_A \left[\rho \left(\frac{\partial w}{\partial t} \right)^2 + \rho z^2 \left(\frac{\partial \phi}{\partial t} \right)^2 \right] dx dA$$

$$= \frac{1}{2} \int_0^L \left[\rho A(x) \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial \phi}{\partial t} \right)^2 \right] dx$$

$$\int_{t_1}^{t_2} L dt = \frac{1}{2} \int_{t_1}^{t_2} \int_0^L \delta \left\{ \rho A(x) \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial \phi}{\partial t} \right)^2 - EI \left(\frac{\partial \phi}{\partial x} \right)^2 - GA \left(-\phi + \frac{\partial w}{\partial x} \right)^2 \right\} dx dt = 0$$

Integrating by parts:

$$\int_0^L \rho A(x) \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^L \rho A(x) \frac{\partial w}{\partial t^2} \delta w dx dt + \int_0^L \rho A \frac{\partial \phi}{\partial x} \delta \phi \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^L \rho I \frac{\partial^2 \phi}{\partial x^2} \delta \phi dx dt$$

$$- \int_{t_1}^{t_2} EI(x) \frac{\partial \phi}{\partial x} \delta \phi \Big|_0^L dt + \int_{t_1}^{t_2} \int_0^L \frac{\partial}{\partial x} \left[EI(x) \frac{\partial \phi}{\partial x} \right] \delta \phi dx dt - \int_{t_1}^{t_2} GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) \delta w \Big|_0^L dt$$

$$+ \int_{t_1}^{t_2} \int_0^L \frac{\partial}{\partial x} \left[GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) \right] \delta w dx dt$$

$$- \int_{t_1}^{t_2} \int_0^L GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) (-\delta \phi) dx dt = 0$$

$$\therefore \int_{t_1}^{t_2} \int_0^L \left\{ -\rho A(x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left[GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) \right] \right\} \delta w dx dt$$

$$+ \int_{t_1}^{t_2} \int_0^L \left\{ -\rho I(x) \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial x} \left[EI(x) \frac{\partial \phi}{\partial x} \right] + GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) \right\} \delta \phi dx dt$$

$$- \int_{t_1}^{t_2} EI(x) \frac{\partial \phi}{\partial x} \delta \phi \Big|_0^L dt - \int_{t_1}^{t_2} GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) \delta w \Big|_0^L dt = 0$$

\(\therefore\) Governing Eqn. of motions are,

$$\rho A(x) \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x} \left[GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) \right] = 0$$

and

$$\rho I(x) \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial x} \left[EI(x) \frac{\partial \phi}{\partial x} \right] - GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) = 0$$

Boundary conditions

$$EI(x) \frac{\partial \phi}{\partial x} = 0 \quad \text{or } \phi \text{ is specified at } x=0/x=L$$

$$GA(x) \left(-\phi + \frac{\partial w}{\partial x} \right) = 0 \quad \text{or } w \text{ is specified at } x=0/x=L$$