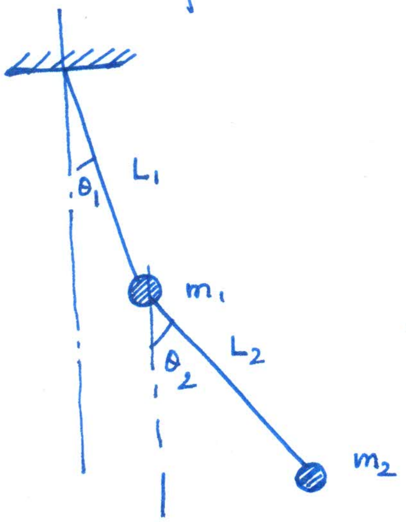
Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad Q_j \rightarrow \text{non-conservative forces.}$$

Example: Double pendulum



Kinetic energy T:

$$T = \frac{1}{2} m_1 (L_1\dot{\theta}_1)^2 + \frac{1}{2} m_2 \left[L_2\dot{\theta}_2 + L_1\dot{\theta}_1 \cos(\theta_2-\theta_1)\right]^2 + \frac{1}{2} m_2 \left[L_1\dot{\theta}_1 \sin(\theta_2-\theta_1)\right]^2.$$

$$= \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2-\theta_1) + \frac{1}{2} m_2 L_1^2 \dot{\theta}_1^2$$

$$= \frac{1}{2}(m_1+m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_1 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2-\theta_1)$$

Potential energy $\quad V = m_1 g L_1 (1-\cos\theta_1) + m_2 g L_2 (1-\cos\theta_2) + m_2 g L_1 (1-\cos\theta_1)$

$$\frac{\partial T}{\partial \dot{\theta}_1} = (m_1+m_2) L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_2-\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) - m_2 L_1 L_2 \dot{\theta}_2 \sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)$$

$$\frac{\partial L}{\partial \theta_1} = + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1) - m_1 g L_1 \sin\theta_1 - m_2 g L_1 \sin\theta_2.$$

$$\therefore \; (m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) - m_2 L_1 L_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1) + \cancel{m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1)}$$
$$\cancel{- m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1)} + m_1 g L_1 \sin\theta_1 + m_2 g L_1 \sin\theta_2 = 0$$

$$\Rightarrow \; (m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) - m_2 L_1 L_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1) + m_1 g L_1 \sin\theta_1 + m_2 g L_1 \sin\theta_1 = 0 \quad \text{——} (1)$$

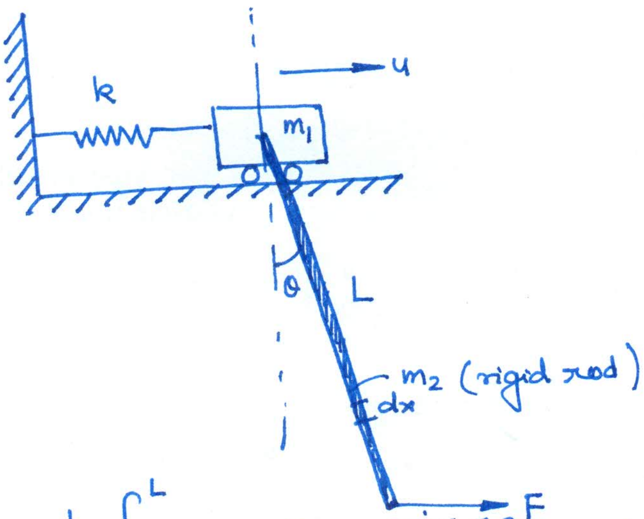
Similarly,

$$\cancel{m_2 \left[L_2 L_1 \ddot{\theta}_1 \cos(\theta_2-\theta_1) + L_2 L_1 \dot{\theta}_1^2 \sin(\theta_2-\theta_1) + L_2 \ddot{\theta}_2\right] + m_2}$$

$$m_2 \left[L_1 L_2 \ddot{\theta}_1 \cos(\theta_2-\theta_1) + L_1 L_2 \dot{\theta}_1^2 \sin(\theta_2-\theta_1) + L_2^2 \ddot{\theta}_2\right] + m_2 L_2 g \sin\theta_2 = 0$$

$$\text{——} (2)$$

Example



$$T = \frac{1}{2} m_1 \dot{u}^2 + \frac{1}{2}\int_0^L \frac{m_2}{L}\,dx\left[\dot{u} + \dot{\overline{x\sin\theta}}\right]^2 + \frac{1}{2}\int_0^L \frac{m_2}{L}\,dx\left[\dot{\overline{x\cos\theta}}\right]^2$$

$$= \frac{1}{2} m_1 \dot{u}^2 + \frac{1}{2}\frac{m_2}{L}\int_0^L \left[\dot{u}^2 + x^2\dot{\theta}^2 + 2\dot{u}x\cos\theta\,\dot{\theta}\right]dx$$

$$= \frac{1}{2} m_1 \dot{u}^2 + \frac{1}{2} m_2 \dot{u}^2 + \frac{1}{2}\frac{m_2 L^2}{3}\dot{\theta}^2 + \frac{m_2 L}{2}\dot{\theta}\dot{u}\cos\theta$$

$$= \frac{1}{2}(m_1 + m_2)\dot{u}^2 + \frac{1}{2}\frac{m_2 L^2}{3}\dot{\theta}^2 + \frac{m_2 L}{2}\dot{u}\dot{\theta}\cos\theta.$$

$$V = \frac{1}{2} k u^2 + m_2 g \frac{L}{2}(1 - \cos\theta)$$

$$\frac{\partial T}{\partial \dot{u}} = (m_1 + m_2)\dot{u} + \frac{m_2 L}{2}\dot{\theta}\cos\theta \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = (m_1 + m_2)\ddot{u} + \frac{m_2 L}{2}\ddot{\theta}\cos\theta - \frac{m_2 L}{2}\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial u} = -ku$$

$$\therefore\quad (m_1 + m_2)\ddot{u} + \frac{m_2 L}{2}\ddot{\theta}\cos\theta - \frac{m_2 L}{2}\dot{\theta}^2\sin\theta + ku = F \qquad (1)$$

Similarly,

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{m_2 L^2}{3}\dot{\theta} + \frac{m_2 L}{2}\dot{u}\cos\theta \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{m_2 L^2}{3}\ddot{\theta} + \frac{m_2 L}{2}\ddot{u}\cos\theta - \frac{m_2 L}{2}\dot{u}\sin\theta\,\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -m_2 g \frac{L}{2}\sin\theta$$

$$\therefore\quad \frac{m_2 L^2}{3}\ddot{\theta} + \frac{m_2 L}{2}\ddot{u}\cos\theta - \frac{m_2 L}{2}\dot{u}\dot{\theta}\sin\theta + m_2 g\frac{L}{2}\sin\theta = FL\cos\theta. \qquad (2)$$