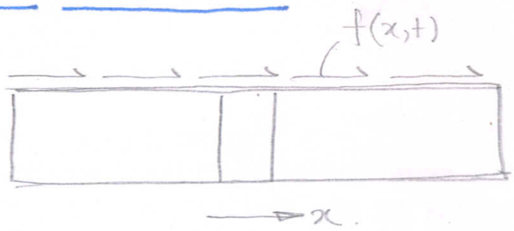
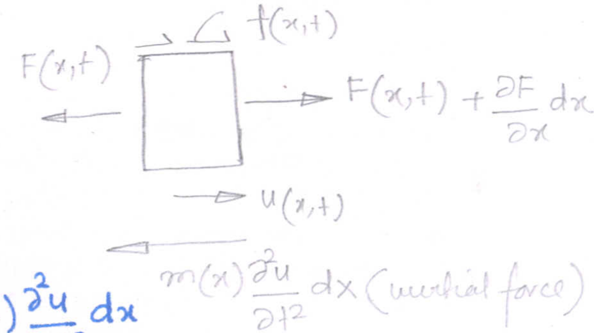
Lecture 13 — Continuous system

Axial vibration in a rod

Eqn. of motion



Free body diagram



Force balance

$$\left[F(x,t)+\frac{\partial F}{\partial x}dx\right]-F(x,t)-m(x)\frac{\partial^2 u}{\partial t^2}dx + f(x,t)dx = 0$$

$$\Rightarrow \quad \frac{\partial F}{\partial x}+f(x,t) = m(x)\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

and

$$F(x,t) = EA(x)\frac{\partial u}{\partial x} \qquad (2)$$

Substituting eqn. (2) into eqn (1), we get,

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] + \cancel{f(x,t)} = m(x)\frac{\partial^2 u}{\partial t^2} \qquad (3) \Rightarrow$$

$f(x,t) = 0$ for free vibration

Under normal mode of vibration

$$u(x,t) = \varphi(x)\,T(t)$$ [the shape of the deformation will not change with time].

$$\text{If} \quad \frac{u(x_1,t)}{u(x_2,t)} = \frac{\varphi(x_1)\,T(t)}{\varphi(x_2)\,T(t)} \neq f(t).$$ where $\varphi(x)$ is the mode shape.

or, under normal mode of vibration, the displacement $u(x,t)$ is separable in time and space.

$$u(x,t) = \underbrace{\varphi(x)}_{\text{mode shape}}\,T(t) \qquad (4)$$

Substituting eqn. (4) into (3), we get,

$$\frac{d}{dx}\left[EA(x)\frac{d\varphi}{dx}\right]T(t) = m(x)\,\varphi(x)\,\ddot{T}(t)$$

$$\Rightarrow \quad \underbrace{\frac{1}{m(x)\varphi(x)}\frac{d}{dx}\left[EA(x)\frac{d\varphi}{dx}\right]}_{\text{(function of } x \text{ only)}} = \underbrace{\frac{\ddot{T}(t)}{T(t)}}_{\text{(function of } t \text{ only)}} = -\omega^2 \ \text{(constant)}$$

$$\therefore \frac{1}{m(x)\varphi(x)} \frac{d}{dx}\left[EA(x)\frac{d\varphi}{dx}\right] = -\omega^2$$

$$\Rightarrow \frac{d}{dx}\left[EA(x)\frac{d\varphi}{dx}\right] + \omega^2 m(x)\varphi(x) = 0$$

Considering uniform c/s, we get.

$$EA\frac{d^2\varphi}{dx^2} + \omega^2 m\varphi = 0 \Rightarrow \frac{d^2\varphi}{dx^2} + \underbrace{\left(\frac{\omega^2 m}{EA}\right)}_{\beta^2}\varphi = 0$$

$$\Rightarrow \varphi(x) = C_1\sin\beta x + C_2\cos\beta x.$$

<u>Fixed - free beam</u>

<u>Boundary conditions</u>

$$@\ x=0,\quad u(0,t) = \varphi(0)\underbrace{T(t)}_{\neq 0} = 0$$

$$\Rightarrow \varphi(0) = 0$$

$$@\ x=L \quad EA\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \Rightarrow EA\varphi'(L)\underbrace{T(t)}_{\neq 0} = 0$$

$$\Rightarrow \cancel{\varphi'(0)} \Rightarrow \varphi'(L) = 0$$

$$\varphi(0) = 0 \Rightarrow C_2 = 0$$

and $\varphi'(L) = 0 \Rightarrow \beta C_1\cos\beta L = 0 \Rightarrow \beta L = \frac{(2r-1)\pi}{2}$

$$\Rightarrow \beta = \frac{(2r-1)\pi}{2L} \Rightarrow \boxed{\omega_r = \frac{(2r-1)\pi}{2L}\sqrt{\frac{EA}{m}}}$$

natural frequency.

$$\boxed{\varphi_r(x) = C_r\sin\frac{(2r-1)\pi x}{2L}}$$

<u>mode shape</u>

<u>Mass normalization of mode shapes</u>

$$\int_0^L \varphi_r(x)\,m(x)\,\varphi_r(x)\,dx = 1 \Rightarrow \int_0^L C_r^2\sin^2\beta_r x\; m\,dx = 0$$

$$\Rightarrow \frac{C_r^2 mL}{2} = 1 \Rightarrow C_r = \sqrt{\frac{2}{mL}}.$$

$$\therefore \boxed{\varphi_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{(2r-1)\pi x}{2L}}$$

mass normalized mode shape.

Fixed - fixed rod :

$$\varphi(x) = C_1 \sin\beta x + C_2 \cos\beta x.$$

Boundary Condns

@ $x=0$, $\varphi(0)=0 \Rightarrow C_2 = 0$

@ $x=L$, $\varphi(L)=0 \Rightarrow C_1 \sin\beta L = 0 \Rightarrow \beta L = \bar{\lambda} r$

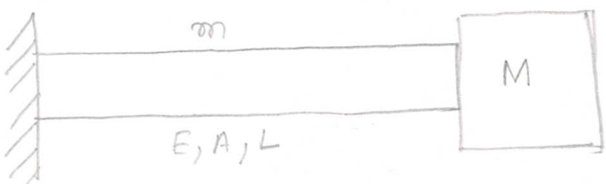
$$\Rightarrow \beta = \frac{\bar{\lambda} r}{L}.$$

$$\boxed{\varphi_r(x) = C_r \sin\frac{\bar{\lambda} r x}{L}}$$ mode shape.

$$\Rightarrow \boxed{\omega_r = \frac{\bar{\lambda} r}{L}\sqrt{\frac{EA}{m}}}$$ natural frequency.

Mass normalized mode shapes $\quad \varphi_r = \boxed{\sqrt{\frac{2}{mL}} \sin\frac{\bar{\lambda} r x}{L}}$

Rod with a lumped mass at tip



Boundary conditions

@ $x=0$, $\varphi(0)=0 \Rightarrow C_2 = 0$

@ $x=L$, $\quad EA\left.\frac{\partial u}{\partial x}\right|_{x=L} + M\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=L} = 0$

$$\Rightarrow EA\,\varphi'(L)\,T(t) + M\,\varphi(L)\,\ddot{T}(t) = 0$$

$$\Rightarrow EA\,\varphi'(L) - \omega^2 M\,\varphi(L) = 0$$

$$\left(\ddot{T}(t)/T(t) = -\omega^2\right)$$

$$\therefore \quad EA\,C_1\beta\cos\beta L - \omega^2 M C_1 \sin\beta L = 0$$

$$\Rightarrow \quad 1 - \frac{\omega^2 M}{EA\beta}\tan\beta L = 0$$

$$\Rightarrow \quad 1 - \frac{\beta^2 EA}{m}\cdot\frac{M}{EA\beta}\tan\beta L = 0$$

$$\Rightarrow \quad \boxed{\frac{\beta M L}{mL}\tan\beta L = 1}$$ $\longrightarrow$ solution gives expression for β/ω (natural frequencies).

Orthogonality of mode shapes

$$\frac{d}{dx}\left[EA(x)\frac{d\varphi_r}{dx}\right] = -\omega_r^2 m(x)\varphi_r(x) \quad \text{——— (1)}$$

and

$$\frac{d}{dx}\left[EA(x)\frac{d\varphi_s}{dx}\right] = -\omega_s^2 m(x)\varphi_s(x) \quad \text{——— (2)}$$

Multiplying both sides of eqn. (1) with $\varphi_s(x)$ and integrating,

$$\int_0^L \varphi_s(x)\frac{d}{dx}\left[EA(x)\frac{d\varphi_s}{dx}\right]dx = -\omega_r^2\int_0^L \varphi_s(x)\, m(x)\,\varphi_r(x)\,dx$$

$$\Rightarrow \quad \underbrace{\cancel{\varphi_s EA(x)\frac{d\varphi_s}{dx}\Big|_0^L}}_{=0 \text{ (for fixed, free BCs)}} - \int_0^L \frac{d\varphi_s}{dx}EA(x)\frac{d\varphi_r}{dx}dx = -\omega_r^2\int_0^L \varphi_s(x)\, m(x)\,\varphi_r(x)\,dx$$

$$\Rightarrow \quad -\int_0^L \frac{d\varphi_s}{dx}EA(x)\frac{d\varphi_r}{dx}dx = -\omega_r^2\int_0^L \varphi_s(x)\, m(x)\,\varphi_r(x)\,dx \quad \text{——— (3)}$$

Similarly, premultiplying both sides of eqn. (2) with $\varphi_r(x)$ and integrating, we get

$$-\int_0^L \frac{d\varphi_r}{dx}EA(x)\frac{d\varphi_s}{dx}dx = -\omega_s^2\int_0^L \varphi_r(x)\, m(x)\,\varphi_s(x)\,dx \quad \text{——— (4)}$$

Subtracting eqn. (4) from (3),

$$-(\omega_r^2-\omega_s^2)\int_0^L \varphi_s(x)\, m(x)\,\varphi_r(x)\,dx = 0$$

For $r \neq s$, $\omega_r^2 \neq \omega_s^2 \Rightarrow$

$$\int_0^L \varphi_s(x) m(x)\varphi_r(x)dx = 0$$

and

$$\int_0^L \frac{d\varphi_s}{dx}EA(x)\frac{d\varphi_r}{dx}dx = 0$$

$\forall\ r \neq s$ Orthogonality condition.