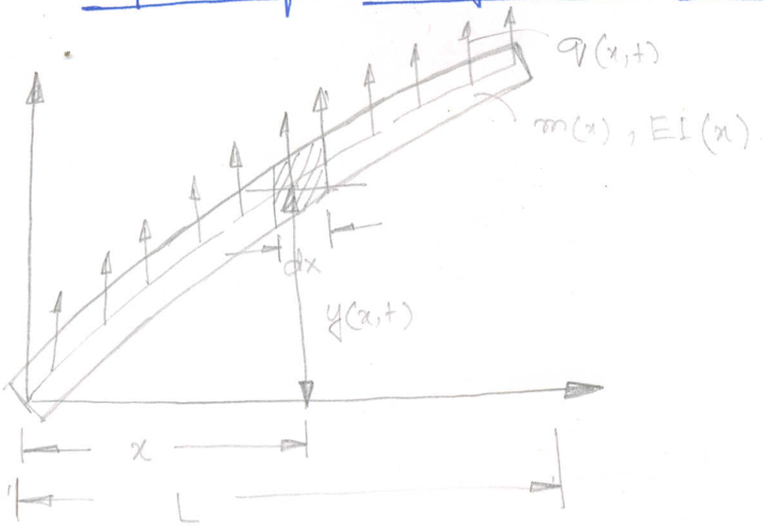
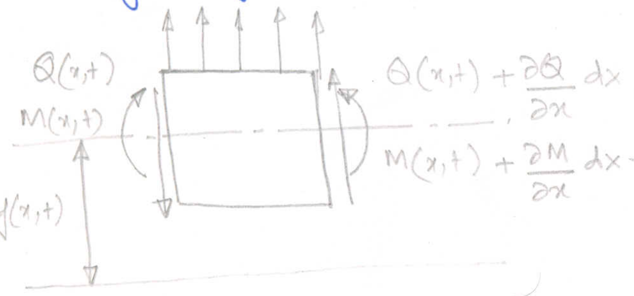
Lecture 14

Equation of motion for Euler-Bernoulli beam



Free body diagram



Force balance

$$\left[Q(x,t) + \frac{\partial Q}{\partial x}dx\right] - Q(x,t) + q(x,t)\,dx = m(x)\,dx\,\ddot{y}(x,t)$$

$$\Rightarrow \quad \frac{\partial Q}{\partial x} + q(x,t) = m(x)\ddot{y}(x,t). \quad \text{——— (1)}$$

Moment balance

$$\left[M(x,t) + \frac{\partial M}{\partial x}dx\right] - M(x,t) + \left[Q(x,t) + \frac{\partial Q}{\partial x}dx\right]dx + q(x,t)\,dx\,\frac{dx}{2} = 0$$

$$\Rightarrow \quad \frac{\partial M}{\partial x} + Q(x,t) = 0 \quad \text{——— (2).}$$

Substituting eqn (2) into eqn (1), we get,

$$-\frac{\partial^2 M}{\partial x^2} + q(x,t) = m(x)\,\ddot{y}(x,t).$$

For Euler-Bernoulli beam, $M(x,t) = EI(x)\dfrac{\partial^2 y}{\partial x^2}$.

$$\therefore \quad \boxed{m(x)\,\ddot{y}(x,t) + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] = q(x,t)} \quad \text{equation of motion}$$

Eqn. of motion under free vibration condition

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] + m(x)\frac{\partial^2 y}{\partial t^2} = 0 \quad \text{——— (1)}$$

Under normal mode of vibration, the deformation shape will not change with time,

$$\frac{y(x_1,t)}{y(x_2,t)} \neq f(t) = \frac{\varphi(x_1)\,T(t)}{\varphi(x_2)\,T(t)}$$

$\therefore$ Under normal mode of vibration, $y(x,t)$ is separable in space and time

$$y(x,t) = \varphi(x)\,T(t) \quad \text{——— (2)}$$

Substituting, eqn. (2) into eqn (1), we get,

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi}{dx^2}\right]T(t) + m(x)\varphi(x)\ddot{T}(t) = 0$$

$$\Rightarrow \quad -\frac{1}{m(x)\varphi(x)}\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi}{dx^2}\right] = \frac{\ddot{T}(t)}{T(t)} = -\omega^2 \text{ (constant)}$$

$$\therefore \quad \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi}{dx^2}\right] - \omega^2 m(x)\varphi(x) = 0 \quad \text{——— (3)}$$

Assuming uniform c/s beam, eqn. (3) reduces to

$$EI\frac{d^4\varphi}{dx^4} - \omega^2 m\,\varphi(x) = 0$$

$$\Rightarrow \quad \frac{d^4\varphi}{dx^4} - \beta^4\varphi(x) = 0 \quad \text{——— (4)} \qquad \boxed{\beta^4 = \frac{\omega^2 m}{EI}}$$

$$\therefore \quad \boxed{\varphi(x) = C_1\sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x}$$

Simply - supported beam :

Boundary Conditions :

@ $x=0$, $y(0,t)=0 \Rightarrow \varphi(0)T(t)=0 \Rightarrow \varphi(0)=0$.

@ $x=0$, $M = EI\dfrac{d^2y}{dx^2} = EI\varphi''(0)T(t)=0 \Rightarrow \varphi''(0)=0$.

and

@ $x=L$ $y(L,t) = \varphi(L)T(t)=0 \Rightarrow \varphi(L)=0$

@ $x=L$ $EI\dfrac{d^2y}{dx^2}=M \Rightarrow \varphi''(L)T(t)=0 \Rightarrow \varphi''(L)=0$.

$$\varphi(x) = C_1\sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x.$$

$\varphi(0) = C_2+C_4=0$

$\varphi''(0) = -C_2+C_4=0$

$\Rightarrow C_2=C_4=0 \therefore \boxed{\varphi(x) = C_1\sin\beta x + C_3\sinh\beta x}$

$\varphi(L) = C_1\sin\beta L + C_3\sinh\beta L = 0$

$\varphi''(L) = -\beta^2C_1\sin\beta L + C_3\beta^2\sinh\beta L=0$

$\Rightarrow C_1\sin\beta L=0 \Rightarrow \sin\beta L=0$

$C_3\sinh\beta L=0 \Rightarrow C_3=0$

$\Rightarrow \boxed{\beta L = \pi r}$

$$\beta = \frac{\pi r}{L}$$

$$\boxed{\omega = \frac{\pi^2 r^2}{L^2}\sqrt{\frac{EI}{m}}}$$ natural frequencies

$$\boxed{\varphi_r(x) = C_r\sin\frac{\pi r}{L}x.}$$ mode shapes.

as normalized mode shape

$$\int_0^L \varphi_r(x)\, m\, \varphi_r(x)\,dx = 1 \Rightarrow C_r = \sqrt{\frac{2}{mL}} \quad \therefore \boxed{\varphi_r = \sqrt{\frac{2}{mL}}\sin\frac{\pi r}{L}x}$$

mass normalized mode shape.

$$\frac{C_r^2 m}{2}\int_0^L 2\sin^2\beta_r x\,dx = 1$$

$$\frac{C_r^2 m}{2}\int_0^L (1-\cos 2\beta_r x)\,dx = 1$$

$$\frac{C_r^2 m}{2}\left[x - \frac{1}{2\beta_r}\sin 2\beta_r x\right]_0^L = \frac{C_r^2 mL}{2} = 1 \Rightarrow C_r = \sqrt{\frac{2}{mL}}.$$