n. of motion of an Euler- Bernoulli beam :

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] + m(x)\frac{\partial^2 y}{\partial t^2} = f(x,t) \qquad (1)$$

Modal expansion theorem $\qquad y(x,t) = \sum_{r=1}^{\infty} \varphi_r(x)\,\eta_r(t) \qquad (2)$

Substituting eqn. (2) into eqn (1), we get,

$$\frac{d^2}{dx^2}\left[EI(x)\sum \varphi_r''(x)\,\eta_r(t)\right] + m(x)\sum \varphi_r(x)\,\ddot{\eta}_r(t) = f(x,t)$$

Multiplying both sides by $\varphi_s(x)$ and integrating, we get

$$\int_0^L \varphi_s(x)\frac{d^2}{dx^2}\left[EI(x)\sum \varphi_r''(x)\,\eta_r(t)\right]dx + \int_0^L m(x)\,\varphi_s(x)\underbrace{\sum \varphi_r(x)\,\ddot{\eta}_r(t)\,dx}_{=\ \ddot{\eta}_r(t)\ \text{for } s=r}$$

$$= \int_0^L \varphi_s(x) f(x,t)\,dx$$

$$\underbrace{\cancel{\varphi_s(x)\frac{d}{dx}\left[EI(x)\sum \varphi_r''(x)\,\eta_r(t)\right]_0^L}}_{=0\ (\text{for all BCs})} - \int_0^L \frac{d\varphi_s}{dx}\cdot\frac{d}{dx}\left[EI(x)\frac{d^2\varphi_r}{dx^2}\,\eta_r(t)\right]dx$$

$$+\ \ddot{\eta}_r(t) = \int_0^L \varphi_r(x) f(x,t)\,dx$$

$$\Rightarrow\ \underbrace{\cancel{-\frac{d\varphi_s}{dx}\cdot\left[EI(x)\frac{d^2\varphi_r}{dx^2}\right]\ddot{\eta}_r(t)}}_{=0\ (\text{for all BCs})} + \underbrace{\int_0^L \frac{d^2\varphi_s}{dx^2}\left[EI(x)\frac{d^2\varphi_r}{dx^2}\,\eta_r(t)\,dx\right]}_{=\ \omega_r^2\eta_r(t)\ (\text{for } s=r)\ =0\ \text{for all other } r\neq s)} + \ddot{\eta}_r(t) = \int_0^L \varphi_r(x) f(x,t)\,dx$$

$$\Rightarrow\ \boxed{\omega_r^2\eta_r + \ddot{\eta}_r = \int_0^L \varphi_r(x) f(x,t)\,dx = N_r(t)}$$

Similar to the SDOF eqn & can be solved accordingly.

A simply-supported beam is subjected to initial displacement given as,

$$y(x,0) = A\left(\frac{x}{L} - \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right)$$

al expansion theorem:

$$y(x,t) = \sum \varphi_r(x)\,\eta_r(t)$$

$$y(x,0) = \sum \varphi_r(x)\,\eta_r(0)$$

Multiplying both sides of the above expression with $\varphi_s(x)$ & m and integrating, we get

$$\int_0^L \varphi_s(x)\, m\, y(x,0)\,dx = \sum_r \underbrace{\int_0^L \varphi_s(x)\,\varphi_r(x)\, m\,dx\ \eta_r(0)}_{=\ \eta_r(0)\text{ for } r=s;\ =0 \text{ for } r\neq s}$$

$$\Rightarrow \eta_r(0) = \int_0^L \varphi_r(x)\, m\, y(x,0)\,dx$$

where, $\varphi_r(x)$ (for simply-supported Euler-Bernoulli beam) $= \sqrt{\frac{2}{mL}}\sin\frac{\pi r x}{L}$

$$\therefore \eta_r(0) = \int_0^L m\sqrt{\frac{2}{mL}}\sin\frac{\pi r x}{L}\cdot A\left(\frac{x}{L} - \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right)dx \quad \text{—— (1)}$$

Eqn. of motion of an Euler-Bernoulli beam $\quad EI\frac{\partial^4 y}{\partial x^4} + \cancel{f(x,t)}\; m\frac{\partial^2 y}{\partial t^2}$

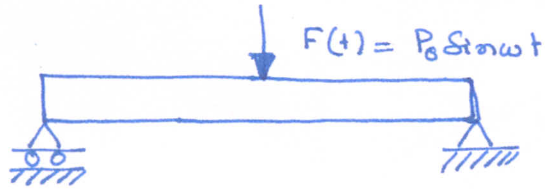
After modal analysis $\quad \ddot{\eta}_r + \omega_r^2\eta_r = 0$

$$\Rightarrow \eta_r(t) = \eta_r(0)\cos\omega_r t + \frac{\cancel{\dot{\eta}_r(0)}^{\,=0}}{\omega_r}\sin\omega_r t$$

$$= \underbrace{\eta_r(0)}_{\text{given by (1)}}\cos\left(\frac{\pi r}{L}\right)^2\sqrt{\frac{EI}{m}}\,t$$

$$\therefore y(x,t) = \sum_{r=1}^{\infty}\sqrt{\frac{2}{mL}}\sin\frac{\pi r x}{L}\cdot\eta_r(0)\cos\left(\frac{\pi r}{L}\right)^2\sqrt{\frac{EI}{m}}\,t \longleftarrow \underline{\underline{\text{Ans}}}$$

Example



Eqn. of motion

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} = f(x,t) = P_0\,\delta\left(x-\frac{L}{2}\right)\sin\omega t$$

For a simply-supported Euler–Bernoulli beam:

Mass normalized mode shapes $\varphi_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{\lambda r x}{L}$

Natural frequency $\omega_r = \left(\frac{\lambda r}{L}\right)^2\sqrt{\frac{EI}{m}}$

After modal analysis:

$$\ddot{\eta}_r + \omega_r^2\eta_r = \int_0^L \varphi_r f(x,t)\,dx$$

$$= P_0\sqrt{\frac{2}{mL}}\int_0^L \sin\frac{\lambda r x}{L}\cdot\delta\left(x-\frac{L}{2}\right)dx\,\sin\omega t$$

$$= P_0\sqrt{\frac{2}{mL}}\sin\omega t\,\sin\frac{\lambda r}{2} = A_0\sin\omega t$$

where $A_0 = P_0\sqrt{\frac{2}{mL}}\sin\frac{\lambda r}{2}$

$$\therefore\ \eta_r(t) = A_0 H(\omega)\cancel{\sin\omega t} = \frac{A_0\sin\omega t}{\omega_r^2-\omega^2} = \frac{A_0}{\omega_r^2-\omega^2}\sin\omega t$$

$$\therefore\ y(x,t) = \sqrt{\frac{2}{mL}}\cdot P_0\cdot\sqrt{\frac{2}{mL}}\sum \sin\frac{\lambda r}{2}\cdot\frac{1}{\omega_r^2-\omega^2}\sin\omega t$$

$$= \cancel{\frac{2P_0}{mL}}\ \frac{2P_0}{mL}\sum\frac{1}{\omega_r^2-\omega^2}\sin\omega t \qquad r = 1, 5, \ldots$$

$$= -\frac{2P_0}{mL}\sum\frac{1}{\omega_r^2-\omega^2}\sin\omega t \qquad r = 3, 7, \ldots$$