Approximate Methods

Collocation Methods

Euler – Bernoulli beam equation:

Under normal mode of vibration

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] + m(x)\frac{\partial^2 y}{\partial t^2} = 0 \leftarrow y(x,t) = \varphi(x)T(t).$$

$$\Rightarrow \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi}{dx^2}\right]T(t) + m(x)\varphi(x)\ddot{T}(t) = 0$$

Approximate mode shape

$$\Rightarrow \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi}{dx^2}\right] - \omega^2 m(x)\varphi(x) = 0 \quad \Rightarrow \varphi(x) = \sum_{j=1}^{N} c_j\psi_j(x)$$

Types of collocation methods

Boundary methods: ψ_i satisfies the differential equation and not the BCs.

Interior Methods: ψ_i satisfies the BCs, but not the differential eqns. (most conventional)

Mixed methods ψ_i satisfies neither the BCs nor the differential eqn.

Interior Collocation Method: We need to select ψ_i which satisfies all the BCs. Station points s_i also need to be chosen, and the $\varphi(x) = \sum c_j\psi_j(x)$ satisfies the differential equation only at the station points.

$$\frac{d^2}{dx^2}\left[EI(x)\sum_{j=1}^{N} c_j\psi_j''(x)\right]_{\text{at } x = x_i (\sim s_i) \text{ station point}} - \omega^2\left[m(x)\sum c_j\psi_j(x)\right]_{x=x_i} = 0.$$

The above written for all station points (N) can be written in matrix form as,

$$\left(\underset{\text{non-symmetric}}{[K]} - \omega^2\overset{\text{non-symmetric.}}{[M]}\right)\{c_j\} = 0$$

$$K_{ij} = \frac{d^2}{dx^2}\left[EI(x)\psi_j''(x)\right]_{x=x_i}$$

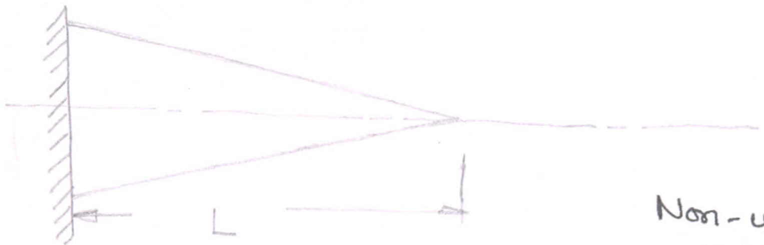
$$K_{ji} = \frac{d^2}{dx^2}\left[EI(x)\psi_i''(x)\right]_{x=x_j}$$

Similarly

$$M_{ij} = \left[m(x)\psi_j(x)\right]_{x=x_i}$$

$$M_{ji} = \left[m(x)\psi_i(x)\right]_{x=x_j}$$

Example of collocation method



Non-uniform fixed-free rod under axial vibration

The mass distribution is given as, $m(x) = 2m\left(1-\frac{x}{L}\right)$

The stiffness ~~distrib~~ distribution is given as $EA(x) = 2EA\left(1-\frac{x}{L}\right)$

Let the trial functions be,

$$\psi_1 = \sin\frac{\pi x}{2L} \quad \text{and} \quad \psi_2 = \sin\frac{3\pi x}{2L}$$ | satisfy all the BCs.

Let the station points be,

$$S_1 = \frac{L}{4} \quad \text{and} \quad S_2 = \frac{3L}{4}$$

At the ~~sta~~ stations points, S_1 & S_2, the following eqn. needs to be satisfied,

$$-\frac{d}{dx}\left[EA(x)\sum_{j=1}^{2} c_j \psi_j'(x)\right] - \omega^2 m(x) \sum_{j=1} c_j \psi_j(x) = 0 \qquad \text{——(1)}$$

~~For~~ Eqn. (1) can be expanded as,

$$\cancel{-\frac{d}{dx}\left[EA(x)\psi_1'(x)\right]_{S_1} - \omega^2 [m(x)}$$

$$-\frac{d}{dx}\left[EA(x)\psi_1'(x)\right]C_1 - \frac{d}{dx}\left[EA(x)\psi_2'(x)\right]C_2 - \omega^2 m(x)\psi_1(x)C_1 - \omega^2 m(x)\psi_2(x)C_2 = 0$$

at $x = S_1$ $\quad K_{11}C_1 + K_{12}C_2 - \omega^2 M_{11}C_1 - \omega^2 M_{12}C_2 = 0$

and at $x = S_2$ $\quad K_{21}C_1 + K_{22}C_2 - \omega^2 M_{21}C_1 - \omega^2 M_{22}C_2 = 0$

$$K_{11} = -\frac{d}{dx}\left[EA(x)\psi_1'(x)\right]_{S_1} = -\frac{d}{dx}\left[2EA\left(1-\frac{x}{L}\right)\left(\frac{\pi}{2L}\right)\cos\frac{\pi x}{2L}\right]_{S_1}$$

$$= -\frac{\pi EA}{L}\left[-\frac{1}{L}\cos\frac{\pi x}{2L} + \left(1-\frac{x}{L}\right)\left(-\frac{\pi}{2L}\right)\sin\frac{\pi x}{2L}\right]_{S_1}$$

$$= \frac{\pi EA}{L^2}\left[\cos\frac{\pi x}{2L} + \frac{\pi}{2}\left(1-\frac{x}{L}\right)\sin\frac{\pi x}{2L}\right]_{S=L/4}$$

$$= \frac{\pi EA}{L^2}\left[\cos\frac{\pi}{8} + \frac{\pi}{2}\left(1-\frac{1}{4}\right)\sin\frac{\pi}{8}\right] = \frac{\pi EA}{L^2}\left[\cos\frac{\pi}{8} + \frac{3\pi}{8}\sin\frac{\pi}{8}\right]$$

$$K_{12} = -\frac{d}{dx}\left[EA(x)\,\psi_2'(x)\right]_{S_1}$$

$$= -\frac{d}{dx}\left[2EA\left(1-\frac{x}{L}\right)\left(\frac{3\pi}{2L}\right)\cos\frac{3\pi x}{2L}\right]_{S_1}$$

$$= -\frac{3\pi EA}{L}\left[-\frac{1}{L}\cos\frac{3\pi x}{2L} - \frac{3\pi}{2L}\left(1-\frac{x}{L}\right)\sin\frac{3\pi x}{2L}\right]_{S_1}$$

$$= \frac{3\pi EA}{L^2}\left[\cos\frac{3\pi x}{2L} + \frac{3\pi}{2}\left(1-\frac{x}{L}\right)\sin\frac{3\pi x}{2L}\right]_{S_1 = L/4}$$

$$= \frac{3\pi EA}{L^2}\left[\cos\frac{3\pi}{8} + \frac{9\pi}{8}\sin\frac{3\pi}{8}\right]$$

$$K_{21} = -\frac{d}{dx}\left[EA(x)\,\psi_1'(x)\right]_{S_2} = -\frac{d}{dx}\left[2EA\left(1-\frac{x}{L}\right)\left(\frac{\pi}{2L}\right)\cos\frac{\pi x}{2L}\right]_{S_2}$$

$$= \frac{\pi EA}{L^2}\left[\cos\frac{\pi x}{2L} + \frac{\pi}{2}\left(1-\frac{x}{L}\right)\sin\frac{\pi x}{2L}\right]_{S_2 = \frac{3L}{4}}$$

$$= \frac{\pi EA}{L^2}\left[\cos\frac{3\pi}{8} + \frac{\pi}{8}\sin\frac{3\pi}{8}\right]$$

$$K_{22} = -\frac{d}{dx}\left[EA(x)\,\psi_2'(x)\right]_{S_2} = -\frac{d}{dx}\left[2EA\left(1-\frac{x}{L}\right)\left(\frac{3\pi}{2L}\right)\cos\frac{3\pi x}{2L}\right]_{S_2}$$

$$= \frac{3\pi EA}{L^2}\left[\cos\frac{3\pi x}{2L} + \frac{3\pi}{2}\left(1-\frac{x}{L}\right)\sin\frac{3\pi x}{2L}\right]_{S_2 = \frac{3L}{4}}$$

$$= \frac{3\pi EA}{L^2}\left[\cos\frac{9\pi}{8} + \frac{3\pi}{2}\cdot\frac{1}{4}\sin\frac{9\pi}{8}\right]$$

Similarly the M_{ij} coefficients can be obtained as follows,

$$M_{11} = \left[m(x)\,\psi_1(x)\right]_{S_1} = \left[2m\left(1-\frac{x}{L}\right)\sin\frac{\pi x}{2L}\right]_{S_1 = \frac{L}{4}}$$

$$= \left[2m\cdot\frac{3}{4}\cdot\sin\frac{\pi}{8}\right]$$

$$= \frac{3m}{2}\sin\frac{\pi}{8}$$

$$M_{12} = \left[m(x)\,\psi_2(x)\right]_{S_1} = \left[2m\left(1-\frac{x}{L}\right)\sin\frac{3\pi x}{2L}\right]_{S_1 = L/4}$$

$$= 2m\cdot\frac{3}{4}\sin\frac{3\pi}{8} = \frac{3m}{2}\sin\frac{3\pi}{8}$$

$$M_{21} = \left[m(x)\,\psi_1(x)\right]_{s_2} = \left[2m\left(1-\frac{x}{L}\right)\sin\frac{\pi x}{2L}\right]_{s_2=\frac{3L}{4}}$$

$$= 2m\cdot\frac{1}{4}\sin\frac{3\pi}{8} = \frac{m}{2}\sin\frac{3\pi}{8}$$

$$M_{22} = \left[m(x)\,\psi_2(x)\right]_{s_2} = \left[2m\left(1-\frac{x}{L}\right)\sin\frac{3\pi x}{2L}\right]_{s_2=\frac{3L}{4}}$$

$$= 2m\cdot\frac{1}{4}\sin\frac{9\pi}{8}$$

$$= \frac{m}{2}\sin\frac{9\pi}{8}$$

~~The~~ The above equations can written in matrix form as,

$$\left(\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\right)\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0.$$

~~The~~ ~~The~~ Solution of the above equation gives the natural frequencies ω and $\{C_1 \;\; C_2\}^T$.