Aerodynamics B Summary

1. Basic Concepts

1.1 Flow types

If there is friction, thermal conduction or diffusion in a flow, it is termed **viscous**. If none of these things is present, the flow is **inviscid**. Inviscid flows do not appear in nature, but some flows are almost inviscid.

A flow in which the density ρ is constant, is termed **incompressible**. If the density is variable, the flow is **compressible**.

The Mach number M is defined as V/a, where V is the airflow velocity and a is the speed of sound. If $M < 1$, the flow is called **subsonic**. If $M = 1$, the flow is called **sonic**. If $M > 1$ the flow is called **supersonic**.

The **flow field variables** p, ρ, T and $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ represent the **flow field**. All these variables are functions of x, y, z and t (they differ per position and time). However, for a **steady flow**, the flow field variables are constant in time. The flow is steady if

$$\frac{dp}{dt} = 0, \qquad \frac{d\rho}{dt} = 0, \qquad \frac{dT}{dt} = 0, \qquad \frac{du}{dt} = 0, \qquad \frac{dv}{dt} = 0 \quad \text{and} \quad \frac{dw}{dt} = 0. \tag{1.1.1}$$

Otherwise the flow is unsteady.

1.2 Gradient, divergence and curl

Consider the vector

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{1.2.1}$$

The **gradient** of a scalar field $p(x, y, z)$ is defined as

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}. \tag{1.2.2}$$

The **divergence** of a vector field $\mathbf{A}(x, y, z) = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ is defined as

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{1.2.3}$$

The **curl** of a vector field $\mathbf{A}(x, y, z) = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ is defined as

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}. \tag{1.2.4}$$

Note that ∇p gives a vector field, $\nabla \cdot \mathbf{A}$ gives a scalar field and $\nabla \times \mathbf{A}$ gives a vector field.

These functions can also be derived for cylindrical coordinates (where $p = p(r, \theta, z)$ and $\mathbf{A} = \mathbf{A}(r, \theta, z)$) and for spherical coordinates (where $p = p(r, \theta, \phi)$ and $\mathbf{A} = \mathbf{A}(r, \theta, \phi)$), but those equations do not have to be known by heart.

1.3 Integrals

Given a closed curve C, the line integral is given by

$$\oint_C \mathbf{A} \cdot \mathbf{ds}. \tag{1.3.1}$$

where the counterclockwise direction around C is considered positive.

Now consider a closed surface S, or a surface S bounded by a closed curve C. The possible surface integrals that can be taken are

$$\iint_S p\,\mathbf{dS}, \qquad \iint_S \mathbf{A} \cdot \mathbf{dS} \quad \text{and} \quad \iint_S \mathbf{A} \times \mathbf{dS}. \tag{1.3.2}$$

where $\mathbf{dS} = \mathbf{n}\,dS$ with $\mathbf{n}$ being the unit normal vector. For closed surface S, $\mathbf{n}$ points outward.

Consider a volume ν. Possible volume integrals are

$$\iiint_\nu p\,d\nu \quad \text{and} \quad \iiint_\nu \mathbf{A}\,d\nu. \tag{1.3.3}$$

1.4 Integral Theorems

There are several theorems using the integral described in the previous paragraph. If S is the surface bounded by the closed curve C, **Stokes' theorem** states that

$$\oint_C \mathbf{A} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS}. \tag{1.4.1}$$

If ν is the volume closed by the closed surface S, **Gauss' divergence theorem** states that

$$\iint_S \mathbf{A} \cdot \mathbf{dS} = \iiint_\nu (\nabla \cdot \mathbf{A}) d\nu. \tag{1.4.2}$$

Analogous to this equation is the **gradient theorem**, which states that

$$\iint_S p\,\mathbf{dS} = \iiint_\nu \nabla p\,d\nu. \tag{1.4.3}$$

2. Navier-Stokes Equations

2.1 Continuity equation

The **continuity equation** is based on conservation of mass. Let's look at a volume ν with surface S, which is fixed in space. The mass flow out of this volume B is equal to the decrease of mass inside the volume C.

The mass flow through a certain area $\mathbf{dS}$ is $\rho\mathbf{V}\cdot\mathbf{dS}$. Since $\mathbf{dS}$ points outward, we're looking at the mass flowing outward. To find the total mass flowing outward, we just integrate over the surface S, to find that

$$B = \iint_S \rho\mathbf{V}\cdot\mathbf{dS}. \tag{2.1.1}$$

Now let's find C. The mass in a small volume $d\nu$ is $\rho\, d\nu$. The total mass in the volume ν can be found by a triple integral. But we're not looking for the total mass, but for the rate of mass decrease. So we simply take a time derivative of the mass. This gives

$$C = -\frac{\partial}{\partial t}\iiint_\nu \rho\, d\nu. \tag{2.1.2}$$

Note that the minus is there, because we're looking for the rate of mass decrease. (Not increase!) Using $B = C$ we can find the continuity equation

$$\frac{\partial}{\partial t}\iiint_\nu \rho\, d\nu + \iint_S \rho\mathbf{V}\cdot\mathbf{dS} = 0. \tag{2.1.3}$$

Since the control volume is fixed, we can pull $\frac{\partial}{\partial t}$ within the integral. And by using Gauss' divergence theorem, we can rewrite this to

$$\iiint_\nu \frac{\partial\rho}{\partial t}\, d\nu + \iiint_\nu \nabla\cdot(\rho\mathbf{V})\, d\nu = \iiint_\nu \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V})\right) d\nu = 0. \tag{2.1.4}$$

Now it may be assumed that, for every small volume $d\nu$ in the volume ν, the integrand is zero:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0. \tag{2.1.5}$$

Note that in the case of a steady flow $\frac{\partial\rho}{\partial t} = 0$, so also $\nabla\cdot(\rho\mathbf{V}) = 0$. And if the flow is also incompressible, then $\nabla\cdot\mathbf{V} = 0$. The value $\nabla\cdot\mathbf{V}$ occurs relatively often in equations and will be discussed later.

2.2 Momentum equation

The **momentum equation** is based on the principle "Sum of forces = Time rate of change of momentum". Let's look once more at a fixed volume in space ν with boundary surface S. First we'll examine the forces acting on it. Then we'll examine the change in momentum.

Two types of forces can act on our volume ν. Body forces, such as gravity, and surface forces, such as pressure and shear stress. First let's look at the body forces. Suppose $\mathbf{f}$ represents the net body force per unit mass exerted on the fluid inside ν. On a small volume $d\nu$, the body force is $\rho\mathbf{f}\, d\nu$. So the total body force is

$$\iiint_\nu \rho\mathbf{f}\, d\nu. \tag{2.2.1}$$

Now let's examine the surface forces. On a small surface $\mathbf{dS}$ acts a pressure p, directed inward. But $\mathbf{dS}$ is directed outward, so the actual force vector caused by the pressure is $-p\,\mathbf{dS}$. The total pressure force therefore is

$$-\iint_S p\,\mathbf{dS}. \tag{2.2.2}$$

The shear stresses on the volume, caused by viscous forces, may be complicated. So let's just define $\mathbf{F_{viscous}}$ as the sum of all the viscous stresses. This makes the total force acting on our volume ν

$$\mathbf{F} = \iiint_\nu \rho\mathbf{f}\,d\nu - \iint_S p\,\mathbf{dS} + \mathbf{F_{viscous}} \tag{2.2.3}$$

Now let's look at the rate of change of momentum in ν. This consists of two parts. First, particles leave ν, taking momentum with them. From the previous paragraph, we know that the mass flow leaving ν through $\mathbf{dS}$ is $\rho\mathbf{V}\cdot\mathbf{dS}$. So the flow of momentum that leaves ν through $\mathbf{dS}$ is $(\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V}$. The total momentum leaving ν therefore is

$$\iint_S (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} \tag{2.2.4}$$

Second, unsteady fluctuations of flow properties inside ν can also cause a change in momentum. The momentum of a small volume $d\nu$ is the mass times the velocity, being $(\rho\,d\nu)\mathbf{V}$. The total momentum of ν can be obtained by integrating. But we don't want the total momentum, but the time rate of change of momentum. So just like in the last paragraph, we put $\frac{\partial}{\partial t}$ in front of it to get

$$\frac{\partial}{\partial t}\iiint_\nu \rho\mathbf{V}\,d\nu \tag{2.2.5}$$

We now have calculated both the sum of the forces, and the change in momentum. It's time to put it all together in one equation

$$\iiint_\nu \rho\mathbf{f}\,d\nu - \iint_S p\,\mathbf{dS} + \mathbf{F_{viscous}} = \iint_S (\rho\mathbf{V}\cdot\mathbf{dS})\mathbf{V} + \frac{\partial}{\partial t}\iiint_\nu \rho\mathbf{V}\,d\nu \tag{2.2.6}$$

Just like we did in the previous paragraph, we can use the gradient theorem to bring the entire equation under one integral. Let's define $\Im_{\mathbf{viscous}}$ as the part of $\mathbf{F_{viscous}}$ acting on a small volume $d\nu$. If we simplify the equation and split it up in components, we find

$$\rho f_x - \frac{\partial p}{\partial x} + \Im_{x_{viscous}} = \frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\mathbf{V}), \tag{2.2.7}$$

$$\rho f_y - \frac{\partial p}{\partial y} + \Im_{y_{viscous}} = \frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\mathbf{V}), \tag{2.2.8}$$

$$\rho f_z - \frac{\partial p}{\partial z} + \Im_{z_{viscous}} = \frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w\mathbf{V}). \tag{2.2.9}$$

If the flow is steady ($\frac{\partial}{\partial t} = 0$), inviscid ($\mathbf{F_{viscous}} = 0$) and if there are no body forces ($\mathbf{f} = 0$), these equations reduce to

$$-\frac{\partial p}{\partial x} = \nabla\cdot(\rho u\mathbf{V}) \tag{2.2.10}$$

$$-\frac{\partial p}{\partial y} = \nabla\cdot(\rho v\mathbf{V}) \tag{2.2.11}$$

$$-\frac{\partial p}{\partial z} = \nabla\cdot(\rho w\mathbf{V}) \tag{2.2.12}$$

If the flow is incompressible (ρ is constant), we have four equations (the momentum equation has three components) and four unknowns, being p, u, v and w. It can be solved. But if ρ is not constant, we need an additional equation.

2.3 Energy equation

The **energy equation** is based on the principle that energy can be neither created nor destroyed. Let's once more take a fixed volume ν with boundary surface S. We will be looking at the time rate of change of energy. But first we make a few definitions. B_1 is the rate of heat added to ν. B_2 is the rate of work done on ν. B_3 is the rate of change of energy in ν. So all values are rates of changes and therefore have unit J/s. Putting it all together gives something similar to the first law of thermodynamics. The relation between B_1, B_2 and B_3 is

$$B_1 + B_2 = B_3. \tag{2.3.1}$$

First let's look at B_1. The heat can increase by volumetric heating (for example due to radiation). Let's denote the volumetric rate of heat addition per unit mass be denoted by $\dot{q}[J/kg\,s]$. The heating of a small volume $d\nu$ is $\dot{q}\rho\,d\nu$.

In addition, if the flow is viscous, heat can be transferred across the surface, for example by thermal conduction. This is a complicated thing, so let's just denote the rate of heat addition due to viscous effects by $\dot{Q}_{viscous}$. Now we know that B_1 is

$$B_1 = \iiint_\nu \dot{q}\rho\,d\nu + \dot{Q}_{viscous}. \tag{2.3.2}$$

Now let's look at B_2. The rate of work done on a body is $\mathbf{F}\cdot\mathbf{V}$. Just like in the previous paragraph, three forces are acting on a small volume $d\nu$. Body forces ($\rho\mathbf{F}\,d\nu$), pressure forces ($-p\,\mathbf{dS}$) and viscous forces. Let's denote the contribution of the friction forces to the work done by $\dot{W}_{viscous}$. Putting it all together gives

$$B_2 = \iiint_\nu \rho(\mathbf{f}\cdot\mathbf{V})\,d\nu - \iint_S p\mathbf{V}\cdot\mathbf{dS} + \dot{W}_{viscous} \tag{2.3.3}$$

To find B_3, we look at the energy in ν. The internal energy in ν is denoted by e, while the kinetic energy per unit mass if $\frac{V^2}{2}$. The total energy per unit mass is simply $E = e + \frac{V^2}{2}$.

The particles leaving ν through the surface S take energy with them. The mass flow leaving through a surface $\mathbf{dS}$ is still $\rho\mathbf{V}\cdot\mathbf{dS}$. Multiply this by the energy per unit mass gives $\rho E(\mathbf{V}\cdot\mathbf{dS})$, being the rate of energy leaving ν through $\mathbf{dS}$. To find the total rate of energy leaving, simply integrate over the surface S.

In addition, if the flow is unsteady, the energy inside ν can also change due to transient fluctuations. The energy of a small volume $d\nu$ is $\rho E\,d\nu$. The total energy can be obtained by integrating over the volume ν. But we don't want the total energy, we want the time rate of change of energy. So, just like in the last two paragraphs, we use $\frac{\partial}{\partial t}$. Now we have enough data to find B_3, which is

$$B_3 = \iint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} + \frac{\partial}{\partial t}\iiint_\nu \rho\left(e + \frac{V^2}{2}\right)\,d\nu. \tag{2.3.4}$$

Putting everything together gives us the energy equation

$$\iiint_\nu \rho\dot{q}\,d\nu + \dot{Q}_{viscous} + \iiint_\nu \rho(\mathbf{f}\cdot\mathbf{V})\,d\nu - \iint_S p\mathbf{V}\cdot\mathbf{dS} + \dot{W}_{viscous} = \iint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} + \frac{\partial}{\partial t}\iiint_\nu \rho\left(e + \frac{V^2}{2}\right)\,d\nu. \tag{2.3.5}$$

Just like in the previous paragraphs, we can follow steps to remove the triple integral. Doing this results in

$$\rho\dot{q} + \rho(\mathbf{f}\cdot\mathbf{V}) - \nabla\cdot(p\mathbf{V}) + \dot{Q}'_{viscous} + \dot{W}'_{viscous} = \nabla\cdot\left(\rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\right) + \frac{\partial}{\partial t}\left(\rho\left(e + \frac{V^2}{2}\right)\right), \tag{2.3.6}$$

where $\dot{Q}'_{viscous}$ and $\dot{W}'_{viscous}$ represent the proper forms of the viscous terms after being put inside the triple integral.

If the flow is steady ($\frac{\partial}{\partial t} = 0$), inviscid ($\dot{Q}_{viscous} = 0$ and $\dot{W}_{viscous} = 0$), adiabatic (no heat addition, $\dot{q} = 0$) and without body forces ($\mathbf{f} = 0$), the energy equation reduces to

$$\nabla \cdot \left(\rho \left(e + \frac{V^2}{2} \right) \mathbf{V} \right) = -\nabla \cdot (p\mathbf{V}). \tag{2.3.7}$$

2.4 Equation of state

Now we have five equations, but six unknowns, being p, ρ, u, v, w and e. To solve it, we need more equations. If the gas is perfect, then

$$e = c_v T, \tag{2.4.1}$$

where c_v is the specific gas constant for constant volume and T is the temperature. But this gives us yet another unknown variable, being the temperature. To complete the system, we can make use of the equation of state

$$p = \rho R T. \tag{2.4.2}$$

We now have seven unknowns and seven equations, which means the system can be solved.

2.5 Substantial derivative

Suppose we look at a very small point in space (from a stationary reference frame). The density changes according to $\frac{\partial \rho}{\partial t}$. But now let's look at a very small volume in space (from a co-moving reference frame). The time rate of change of this volume is defined as the **substantial derivative** $\frac{D\rho}{Dt}$. It can be shown that this derivative is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\nabla \cdot \mathbf{V}) \qquad \Leftrightarrow \qquad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\nabla \cdot \mathbf{V})\, \rho. \tag{2.5.1}$$

Of course the ρ can be replaced by other variables. The first $\frac{\partial}{\partial t}$ is called the **local derivative** and the second part $(\mathbf{V} \cdot \nabla)$ is called the **convective derivative**.

The substantial derivative can be used to write the Navier-Stokes equations in a simpler form. To do that, we make use of a vector relation, which is rather similar to the chain rule, being

$$\nabla \cdot (\rho \mathbf{V}) = \rho(\nabla \cdot \mathbf{V}) + (\nabla \rho) \cdot \mathbf{V} = \rho \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla \rho. \tag{2.5.2}$$

Applying this relation and the substantial derivative to the continuity equation (equation 2.1.5) gives

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0. \tag{2.5.3}$$

Using the same tricks, the momentum equation (equation 2.2.7 to 2.2.9) can be rewritten as

$$\rho f_x - \frac{\partial p}{\partial x} + \Im_{x_{viscous}} = \rho \frac{Du}{Dt} \tag{2.5.4}$$

$$\rho f_y - \frac{\partial p}{\partial y} + \Im_{y_{viscous}} = \rho \frac{Dv}{Dt} \tag{2.5.5}$$

$$\rho f_z - \frac{\partial p}{\partial z} + \Im_{z_{viscous}} = \rho \frac{Dw}{Dt} \tag{2.5.6}$$

If the flow is steady ($\frac{\partial}{\partial t} = 0$) and inviscid ($\mathbf{F_{viscous}} = 0$), these equations can be simplified even more.

Now let's look at the energy equation (equation 2.3.6). In the same way as the above equations, it can be rewritten. The outcome is

$$\dot{q}\rho + \rho(\mathbf{f} \cdot \mathbf{V}) - \nabla \cdot (p\mathbf{V}) + \dot{Q}'_{viscous} + \dot{W}'_{viscous} = \rho \frac{D\left(e + \frac{V^2}{2}\right)}{Dt}. \qquad (2.5.7)$$

It is conventional to call the earlier forms of the equations (equations 2.1.5, 2.2.7 to 2.2.9 and 2.3.6) the **conservation form** (or sometimes the **divergence form**), while the equations of this paragraph are called the **non-conservation form**. In most cases, there is no particular reason to choose one form over the other.

2.6 Divergence of velocity

The quantity $\nabla \cdot \mathbf{V}$ occurs frequently in equations. Let's consider an amount of air ν from a co-moving reference frame. As the air moves, the volume of ν can change. We will take a look at that change now.

Let's consider a small bit of surface $\mathbf{dS}$ of ν. This surface moves. The change in volume that this piece of surface causes is $\mathbf{V} \cdot \mathbf{dS}$. So the total change in volume per unit time can be found, using an integral over the surface, giving

$$\frac{D\nu}{Dt} = \iint_S \mathbf{V} \cdot \mathbf{dS} = \iiint_\nu (\nabla \cdot \mathbf{V}) d\nu. \qquad (2.6.1)$$

The latter part is known due to the divergence theorem. Note that we have used the substantial derivative $\frac{D\nu}{dt}$ instead of $\frac{d\nu}{dt}$ since we are considering a moving volume of air, instead of air passing through a fixed volume in space.

If the volume ν is small enough, such that $\nabla \cdot \mathbf{V}$ is the same everywhere in ν, then we can find that

$$\nabla \cdot \mathbf{V} = \frac{1}{\nu} \frac{D\nu}{Dt}. \qquad (2.6.2)$$

This equation states that $\nabla \cdot \mathbf{V}$ is the time rate of change of the volume of a moving fluid element per unit volume. This sounds complicated, but an example will illustrate this fact. If $\nabla \cdot \mathbf{V} = -0.8s^{-1}$, then the volume ν will decrease by 80% every second (the minus sign indicates a decrease). If $\nabla \cdot \mathbf{V} = 1s^{-1}$, then the volume ν will double in size every second (that is, as long $\nabla \cdot \mathbf{V}$ remains $1s^{-1}$).

3. Aerodynamics Lines and Equations

3.1 Pathlines, streamlines and streaklines

A **pathline** is a curve in space traced out by a certain particle in time. A **streamline** is a line where the flow is tangential. A **streakline** is the line formed by all the particles that previously passed through a certain point. However, for steady flows, pathlines, streamlines and streaklines simply coincide.

Of these three lines, streamlines are the lines most used in aerodynamics. But how can we find the equation for a streamline? Since the velocity vector is tangential to the streamline at that point, we know that

$$\mathbf{V} \times \mathbf{ds} = \mathbf{0}. \tag{3.1.1}$$

Looking at the components of the vectors also shows that

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{3.1.2}$$

Note that for 2-dimensional situations this equation reduces to

$$v\,dx = u\,dy. \tag{3.1.3}$$

3.2 Vorticity

If ω is the angular velocity of a small volume in space, then the **vorticity** ξ is defined as

$$\xi = 2\omega = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{V}. \tag{3.2.1}$$

So in a velocity field, the curl of the velocity is equal to the vorticity. If $\xi = 0$ at every point in a flow, the flow is called **irrotational**. The motion of fluid elements is then without rotation - there is only pure translation. If $\xi \neq 0$ for some point, then the flow is called **rotational**.

Note that for 2-dimensional flows the vorticity is given by $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. So if the flow is irrotational, then

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \tag{3.2.2}$$

This equation is the **condition of irrotationality for two-dimensional flow** and will be used quite frequently.

3.3 Circulation

Consider a closed curve C in a flow field. The **circulation** Γ is defined as

$$\Gamma = -\oint_C \mathbf{V} \cdot \mathbf{ds}. \tag{3.3.1}$$

By definition the integral along C has counter-clockwise as positive direction, but by definition the circulation has clockwise as positive direction. Therefore the minus sign is present in the equation.

If the flow is irrotational everywhere in the surface bounded by C, then $\Gamma = 0$.

3.4 Stream functions

Let's consider a 2-dimensional flow for now. If the velocity distribution of the flow is known, equation 3.1.3 can be integrated to find the equation for a streamline $\bar{\psi}(x, y) = c$. The function $\bar{\psi}$ is called the **stream function**. Different values of c result in different streamlines.

If a stream function ψ is known, then the product ρV at a certain point in the flow can be found, using

$$\rho u = \frac{\partial \bar{\psi}}{\partial y}, \qquad \rho v = -\frac{\partial \bar{\psi}}{\partial x}. \tag{3.4.1}$$

Now suppose we're dealing with incompressible flows, and thus $\rho = \text{constant}$. Let's define a new stream function $\psi = \bar{\psi}/\rho$. (Note that ψ has unit $[m^3/(s\,m) = m^2/s]$.) Then equation 3.4.1 becomes

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{3.4.2}$$

In polar coordinates this becomes

$$V_r = \frac{1}{r}\frac{d\psi}{d\theta}, \qquad V_\theta = -\frac{d\psi}{dr}. \tag{3.4.3}$$

3.5 Velocity potential

For an irrotational flow, it is known that $\xi = \nabla \times \mathbf{V} = 0$. There is also a vector identity, stating that $\nabla \times (\nabla \phi) = 0$. Combining these equations, we see that there is a scalar function ϕ such that.

$$\mathbf{V} = \nabla \phi. \tag{3.5.1}$$

The function ϕ is called the **velocity potential**. If the velocity potential is known, then the velocity at every point can be determined, using

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z}. \tag{3.5.2}$$

In polar coordinates, this is

$$V_r = \frac{d\phi}{dr}, \qquad V_\theta = \frac{1}{r}\frac{d\phi}{d\theta}. \tag{3.5.3}$$

Since irrotational flows can be described by a velocity potential ϕ, such flows are also called **potential flows**.

3.6 Stream function versus velocity potential

The stream function and the velocity potential have important similarities and differences. Keep in mind that the velocity potential is defined for irrotational flow only, while the stream function can be used for both rotational and irrotational flows. On the contrary, the velocity potential applies for three-dimensional flows, while the stream function is defined for two-dimensional flows only.

There is another interesting relation between the stream function and the velocity potential. Suppose we plot lines for constant values of the stream function $\psi = \text{constant}$. The streamlines do not intersect other stream lines. Now we also plot lines for constant values of the velocity potential $\phi = \text{constant}$, being so-called **equipotential lines**. The equipotential lines do not intersect other equipotential lines either. However, the streamlines and the equipotential lines do intersect. The peculiar thing is that they always intersect perpendicular. This, in fact, can be mathematically proven. So streamlines and equipotential lines are orthogonal.

4. Basics of Inviscid Incompressible Flows

4.1 Bernoulli's equation

An **incompressible flow** is a flow where the density ρ is constant. Let's assume we're dealing with an incompressible flow. From the momentum equation and the streamline condition, we can derive that

$$dp = -\rho V \, dV. \tag{4.1.1}$$

This equation is called **Euler's equation**. Since the streamline condition was used in the derivation, it is only valid along a streamline. Integrating the Euler equation between point 1 and point 2 gives

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2. \tag{4.1.2}$$

In other words, $p + \frac{1}{2}\rho V^2$ is constant along a streamline.

An **inviscid** flow is a flow without friction, thermal conduction or diffusion. It can be shown that inviscid flows are irrotational flows. For irrotational flows $p + \frac{1}{2}\rho V^2$ is constant, even for different streamlines.

4.2 Continuity equation

In a low-speed wind tunnel the flow field variables can be assumed to be a function of x only, so $A = A(x)$, $V = V(x)$, $p = p(x)$, etcetera. Such a flow is called a **quasi-one-dimensional flow**. From the continuity equation can be derived that

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2, \tag{4.2.1}$$

for two points in the tunnel. This applies to both compressible and incompressible flows. If the flow becomes incompressible, then $\rho_1 = \rho_2$. The equation then reduces to $A_1 V_1 = A_2 V_2$. If we combine this with Bernoulli's equation, we find

$$V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left(\frac{A_1}{A_2} - 1\right)}}. \tag{4.2.2}$$

4.3 Dynamic pressure

The **dynamic pressure** is defined as

$$q = \frac{1}{2}\rho V^2. \tag{4.3.1}$$

Let's suppose that the velocity at some point 0 is zero ($V_0 = 0$). If the flow is incompressible, it follows that

$$p_1 + \frac{1}{2}\rho V_1^2 = p_0 \qquad \Rightarrow \qquad q_1 = p_0 - p_1. \tag{4.3.2}$$

Note that this follows from Bernoulli's equation. If the flow is compressible, Bernoulli's equation is not valid and thus $p_0 - p_1 \neq q_1$.

4.4 Pressure coefficient

The **pressure coefficient** C_p is defined as

$$C_p = \frac{p - p_\infty}{q_\infty}, \tag{4.4.1}$$

where $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$. The ∞ subscript denotes that the values are measured in the free stream, as if being infinitely far away from the examined object. For incompressible flows, C_p can also be written as

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2. \qquad (4.4.2)$$

4.5 Laplace's equation

If the flow is incompressible, it follows from the continuity equation that

$$\nabla \cdot \mathbf{V} = 0. \qquad (4.5.1)$$

If the flow is also inviscid, and thus irrotational, it follows that $\nabla \times \mathbf{V} = 0$. It also implicates that there is a velocity potential ϕ such that $\mathbf{V} = \nabla\phi$. Combining this with equation 4.5.1 gives

$$\nabla \cdot (\nabla\phi) = \nabla^2\phi = 0. \qquad (4.5.2)$$

This simple but important relation is called **Laplace's equation**. It seems that the velocity potential satisfies Laplace's equation. But what about the stream function? We can recall from the previous chapter that

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \qquad (4.5.3)$$

We can also remember the irrotationality condition, stating that $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$. Inserting 4.5.3 in this condition gives

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad \Rightarrow \qquad \nabla^2\psi = 0. \qquad (4.5.4)$$

So the stream function ψ also satisfies Laplace's equation, just like the velocity potential function ϕ.

4.6 Applying Laplace's equation

Note that the Laplace equation is a linear partial differential equation. So if we find multiple solutions $\phi_1, \ldots, \phi_n$ for it, then any linear combination $\phi = c_1\phi_1 + \ldots + c_n\phi_n$ is also a solution. So if we find a couple of basic solutions to Laplace's equation, and if we add them up in just the right way, we can display any inviscid incompressible flow.

But how do we know how to put the independent solutions together? We have to make use of **boundary conditions**. First, there are the **boundary conditions on velocity at infinity**, stating that, at infinity,

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = V_\infty, \qquad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = 0. \qquad (4.6.1)$$

There are also the **wall boundary conditions**. The flow can not penetrate an airfoil. So the velocity at the airfoil edge is directed tangentially. This can be expressed in many ways. If $\mathbf{n}$ is the normal vector at the airfoil surface, then $\mathbf{V} \cdot \mathbf{n} = (\nabla\phi) \cdot \mathbf{n} = 0$. This is called the **flow tangency condition**. But since the airfoil edge is a streamline itself, also $\psi_{surface} = \text{constant}$.

If we are dealing with neither ϕ or ψ, but rather with u and v themselves, things are different. If the shape of the airfoil is given by $y_b(x)$, then

$$\frac{dy_b}{dx} = \left(\frac{v}{u}\right)_{surface}. \qquad (4.6.2)$$

With those boundary conditions, we can put the elementary solutions to Laplace's equation together to represent, for example, the flow over a cylinder or over an airfoil. All that is left now, is to find those elementary solutions. That is the subject of the next chapter.

5. Elementary Flows

5.1 Uniform flows

A **uniform flow** (oriented in the positive x-direction) is a flow with velocity components $u = V_\infty$ and $v = 0$ everywhere. Such a flow is irrotational. It therefore has a velocity potential ϕ, which can be shown to be

$$\phi = V_\infty x. \tag{5.1.1}$$

Also the stream function can be determined to be

$$\psi = V_\infty y. \tag{5.1.2}$$

Note that these two functions both satisfy Laplace's equation.

In a 3-dimensional world, the velocity potential is the same, and therefore $w = 0$.

5.2 Source flows

A **source flow** is a flow where all the streamlines are straight lines emanating from a central point O, and where the velocity varies inversely with the distance from O. In formula this is

$$V_r = \frac{\Lambda}{2\pi r}, \qquad V_\theta = 0, \tag{5.2.1}$$

where Λ is the source stength. Such a flow is incompressible (except at point O itself) and irrotational. If $\Lambda > 0$, we are dealing with a source flow. If $\Lambda < 0$, we are looking at a so-called **sink flow**, where the velocity vectors point inward.

The velocity potential of the flow can be found using the above velocity relations. The result will be

$$\phi = \frac{\Lambda}{2\pi} \ln r. \tag{5.2.2}$$

Note that this function is not defined for $r = 0$, since the flow is not incompressible there. Identically, the stream function can be shown to be

$$\psi = \frac{\Lambda}{2\pi} \theta. \tag{5.2.3}$$

Now let's look at 3-dimensional sources. 3-Dimensional source flows are similar to 2-dimensional ones. Let's define λ as the volume flow originating from the source. The velocity is now, in spherical coordinates,

$$V_r = \frac{\lambda}{4\pi r^2}, \qquad V_\theta = 0, \qquad V_\phi = 0. \tag{5.2.4}$$

The stream function is not defined for 3-dimensional situations. The velocity potential is

$$\phi = -\frac{\lambda}{4\pi r}. \tag{5.2.5}$$

5.3 Doublets

Suppose we have a source of strength Λ at coordinates $(-\frac{1}{2}l, 0)$ and a source of strength $-\Lambda$ (thus being a sink) at coordinates $(\frac{1}{2}l, 0)$. If $l \to 0$, we obtain a flow pattern called a **doublet**. The **strength** of the doublet is defined as $\kappa = l\Lambda$. So as $l \to 0$ also $\Lambda \to \infty$. The velocity potential now is

$$\phi = \frac{\kappa}{2\pi} \frac{\cos\theta}{r}. \tag{5.3.1}$$

Also, the stream function is

$$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}. \tag{5.3.2}$$

The streamlines of a doublet are therefore given by

$$\psi = c \qquad \Rightarrow \qquad r = -\frac{\kappa}{2\pi c}\sin\theta. \tag{5.3.3}$$

It can mathematically be shown that these are circles with diameter $d = \frac{\kappa}{2\pi c}$ and with their centers positioned at coordinates $(0, \pm\frac{1}{2}d)$.

Now let's look at 3-dimensional doublets. Just like in a 2-dimensional doublet, a 3-dimensional doublet has a 3-dimensional source and sink at a very small distance from each other. The 3-dimensional doublet strength is defined as $\mu = \Lambda l$. The velocity potential then is

$$\phi = -\frac{\mu}{4\pi}\frac{\cos\theta}{r^2}. \tag{5.3.4}$$

5.4 Vortex flows

A **vortex flow** is a flow in which the stream lines form concentric circles about a given point. Such a flow is described by

$$V_r = 0, \qquad V_\theta = -\frac{\Gamma}{2\pi r}, \tag{5.4.1}$$

where Γ is the circulation. In this case Γ is also called the **strength** of the vortex. A positive strength corresponds to a clockwise vortex, while a counterclockwise vortex indicates a negative strength.

Vortex flow is irrotational everywhere except at $r = 0$, where the vorticity is infinite. The velocity potential is given by

$$\phi = -\frac{\Gamma}{2\pi}\theta. \tag{5.4.2}$$

Also, the stream function is

$$\psi = \frac{\Gamma}{2\pi}\ln r. \tag{5.4.3}$$

There are no 3-dimensional vortex flows. The only way in which three-dimensional vortex flows can occur is if multiple 2-dimensional vortex flows are stacked on top of each other. This is then, in fact, still a 2-dimensional problem and can be solved with the above equations.

5.5 Elementary flow overview

Flow type	**Velocity**	**Velocity potential**	**Stream function**
Uniform flow in x-direction	$u = V_\infty$ $v = 0$	$\phi = V_\infty x$	$\psi = V_\infty y$
Source/Sink	$V_r = \frac{\Lambda}{2\pi r}$ $V_\theta = 0$	$\phi = \frac{\Lambda}{2\pi}\ln r$	$\psi = \frac{\Lambda}{2\pi}\theta$
Doublet	$V_r = -\frac{\kappa}{2\pi}\frac{\cos\theta}{r^2}$ $V_\theta = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r^2}$	$\phi = \frac{\kappa}{2\pi}\frac{\cos\theta}{r}$	$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$
Vortex	$V_r = 0$ $V_\theta = -\frac{\Gamma}{2\pi r}$	$-\frac{\Gamma}{2\pi}\theta$	$\psi = \frac{\Gamma}{2\pi}\ln r$

6. Basic Applications of Elementary Flows

6.1 Nonlifting flow over a cylinder

If we combine a uniform flow with a doublet, we get the stream function

$$\psi = V_\infty r \sin\theta - \frac{\kappa}{2\pi}\frac{\sin\theta}{r} = V_\infty r \sin\theta \left(1 - \frac{\kappa}{2\pi V_\infty r^2}\right) = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right), \tag{6.1.1}$$

where $R^2 = \frac{\kappa}{2\pi V_\infty}$. This is also the stream function for a flow over a cylinder/circle with radius

$$R = \sqrt{\frac{\kappa}{2\pi V_\infty}}. \tag{6.1.2}$$

The velocity field can be found by using the stream function, and is given by

$$V_r = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right), \qquad V_\theta = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right). \tag{6.1.3}$$

Note that if $r = R$, then $V_r = 0$, satisfying the wall boundary condition. At the wall also $V_\theta = -2V_\infty \sin\theta$. This means that the pressure coefficient over the cylinder is given by

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - 4\sin^2\theta. \tag{6.1.4}$$

6.2 Nonlifting flow over a sphere

Let's combine a uniform 3-dimensional flow with a 3-dimensional doublet. Let's define R as

$$R = \sqrt[3]{\frac{\mu}{2\pi V_\infty}}. \tag{6.2.1}$$

Using the combined stream function, it can be shown that the velocity field is given by

$$V_r = -V_\infty \cos\theta \left(1 - \frac{R^3}{r^3}\right), \qquad V_\theta = V_\infty \sin\theta \left(1 + \frac{R^3}{r^3}\right), \qquad V_\phi = 0. \tag{6.2.2}$$

At the wall, the velocity is $V_\theta = \frac{3}{2} V_\infty \sin\theta$. This means that the pressure coefficient over the sphere is given by

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 = 1 - \frac{9}{4}\sin^2\theta. \tag{6.2.3}$$

6.3 Lifting flow over a cylinder

Let's combine a nonlifting flow over a cylinder with a vortex of strength Γ. This results in a lifting flow over a cylinder. The resulting stream function is

$$\psi = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{r}{R}. \tag{6.3.1}$$

From the stream function we can derive the velocity field, which is given by

$$V_r = V_\infty \cos\theta \left(1 - \frac{R^2}{r^2}\right), \qquad V_\theta = -V_\infty \sin\theta \left(1 + \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi r}. \tag{6.3.2}$$

To find the stagnation points, we simply have to set V_r and V_θ to 0. If $\frac{\Gamma}{4\pi V_\infty R} \leq 1$, then the solution is given by

$$r = R, \qquad \theta = \arcsin\left(-\frac{\Gamma}{4\pi V_\infty R}\right). \tag{6.3.3}$$

However, if $\frac{\Gamma}{4\pi V_\infty R} \geq 1$, then the solution is given by

$$r = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{\left(\frac{\Gamma}{4\pi V_\infty}\right)^2 - R^2}, \qquad \theta = -\frac{1}{2}\pi. \tag{6.3.4}$$

At the surface of the cylinder (where $r = R$) is the velocity $V = V_\theta$. Using this, the pressure coefficient can be calculated. The result is

$$C_p = 1 - \left(2\sin\theta + \frac{\Gamma}{2\pi R V_\infty}\right)^2. \tag{6.3.5}$$

Using this pressure coefficient, the drag coefficient can be found to be $c_d = 0$. So there is no drag. Also, the lift coefficient is

$$c_l = \frac{\Gamma}{R V_\infty}, \tag{6.3.6}$$

where $R = \frac{1}{2}c$. Now the lift per unit span L' can be obtained from

$$L' = c_l q_\infty c = \frac{\Gamma}{R V_\infty}\frac{1}{2}\rho_\infty V_\infty^2 2R = \rho_\infty V_\infty \gamma. \tag{6.3.7}$$

This equation is called the **Kutta-Joukowski Theorem**. It states that the lift per unit span is directly proportional to the circulation. It also works for shapes other than cylinders. However, for other shapes a complex distribution of sources and vortices may be necessary, as is the subject of the following paragraph.

6.4 Source Panel Method

The **source panel technique** is a numerical method to use elementary flows. Let's put a lot of sources along a curve with source strength per unit length $\lambda = \lambda(s)$. Such a source distribution is called a **source sheet**. Note that λ can be positive at some points and negative in other points.

Now look at an infinitely small part of the source sheet. The source strength of this part is $\lambda\, ds$. So for any point P, the contribution of this small source sheet part to the velocity potential is

$$d\phi = \frac{\lambda\, ds}{2\pi} \ln r, \tag{6.4.1}$$

where r is the distance between the source sheet part and point P. The entire velocity potential can be obtained by integrating, which simply gives

$$\phi = \int_a^b \frac{\lambda\, ds}{2\pi} \ln r. \tag{6.4.2}$$

In the source panel method, usually an airfoil (or an other shape) is split up in a number of small straight lines for which the velocity potential is separately calculated and the boundary conditions are separately applied.

7. Two-Dimensional Airfoils

7.1 Definitions

There are various ways to describe an airfoil. The NACA-terminology is a well-known standard, which defines the following airfoil properties. The **mean camber line** is the line formed by the points halfway between the upper and lower surfaces of the airfoil. The most forward and rearward points of the airfoil are the **leading edge** and the **trailing edge**, respectively. The straight line connecting the leading and trailing edges is the **chord line**.

The length of the chord line is defined as the **chord** c. The maximum distance between the chord line and the camber line is called the **camber**. If the camber is 0, then the airfoil is called **symmetric**. And finally, the **thickness** is the distance between the upper and lower surfaces of the airfoil.

In this chapter we will be looking at 2-dimensional airfoils. We're interested in finding c_l, the **lift coefficient per unit length**. At low **angles of attack** α, the value of c_l varies linearly with α. The **lift slope** a_0 is the ratio of them, so $a_0 = \frac{dc_l}{d\alpha}$.

If α gets too high, this relation doesn't hold, since **stall** will occur. The maximum value of c_l is denoted by $c_{l,max}$. This value determines the minimum velocity of an aircraft. The value of α when $c_l = 0$ is called the **zero-lift angle of attack** and is denoted by $\alpha_{L=0}$.

7.2 Vortex sheets

In the last chapter we treated the source panel method. We put a lot of sources on a sheet. We can also put a lot of vortices on a curve s. Let's define $\gamma = \gamma(s)$ as the strength of the vortex sheet per unit length along s. The velocity potential at some point P can then be determined, using

$$d\phi = -\frac{\gamma\, ds}{2\pi}\theta \qquad \Rightarrow \qquad \phi = -\frac{1}{2\pi}\int_a^b \theta\gamma\, ds. \tag{7.2.1}$$

Here θ is the angle between point P and the point on the vortex sheet we're at that moment looking at. Also a and b are the begin and the end of the vortex sheet.

The circulation of the vortex sheet can be determined to be

$$\Gamma = \int_a^b \gamma\, ds. \tag{7.2.2}$$

If the circulation is known, the resulting lift can be calculated using the Kutta-Joukowski theorem

$$L' = \rho_\infty V_\infty \Gamma. \tag{7.2.3}$$

7.3 Kutta condition

We can put a vortex sheet on the camber line of an airfoil. We can then use boundary conditions and numerical computation to find the vortex strength γ at every point. But it turns out that there are multiple solutions. To get one solution, we can use the **Kutta condition**, which states that the flows leaves the trailing edge smoothly.

What can we derive from this? For now, let's call φ the angle of the trailing edge. Also let's call V_1 the velocity on top of the airfoil at the trailing edge and V_2 the velocity at the bottom of the airfoil at the same point. If φ is finite, then it can be shown that $V_1 = V_2 = 0$. However, if $\varphi \to 0$ (the trailing edge is cusped), then only $V_1 = V_2$. Nevertheless, we can derive the same rule from both situations. Namely, that the vortex strength at the trailing edge is

$$\gamma(TE) = 0. \tag{7.3.1}$$

7.4 Thin airfoil theory

Suppose we want to calculate the flow over a very thin airfoil by using a vortex sheet in a free stream flow. We can put vortices on the camber. But the camber line doesn't differ much from the chord line, so to keep things simple we place vortices on the chord line.

Since the airfoil is thin, it is by itself a streamline of the flow. So the velocity perpendicular to the camber line is 0. Let's define $z(x)$ to be the distance between the mean camber line and the chord line, where x is the distance from the leading edge. The velocity perpendicular to the camber line, caused by the free stream flow, at position x, can be shown to be

$$V_{\infty,n} = V_\infty \left(\alpha - \frac{dz}{dx}\right), \tag{7.4.1}$$

where α is in radians. The velocity perpendicular to the mean camber line, due to the vortices, is approximately equal to the velocity perpendicular to the chord. It can be shown that this velocity component on a small part $d\varepsilon$, with distance x from the airfoil leading edge, is

$$dw = -\frac{\gamma(\varepsilon)\, d\varepsilon}{2\pi(x-\varepsilon)}. \tag{7.4.2}$$

Integrating along the chord gives the total velocity perpendicular to the chord at position x due to the vortex sheet, being

$$w(x) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\varepsilon)\, d\varepsilon}{x-\varepsilon}. \tag{7.4.3}$$

We have already derived that the velocity perpendicular to the airfoil is zero. So $V_{\infty,n} + w = 0$, which results in

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\varepsilon)\, d\varepsilon}{x-\varepsilon} = V_\infty \left(\alpha - \frac{dz}{dx}\right). \tag{7.4.4}$$

This is the **fundamental equation of thin airfoil theory**.

7.5 Vortex distributions of symmetric airfoils

If we have a symmetric airfoil, then there is no camber, so $dz/dx = 0$ everywhere on the airfoil. This simplifies equation 7.4.4 and we might actually try to solve it now. If we make the change of variable $\varepsilon = \frac{1}{2}c(1-\cos\theta)$ and also define $x = \frac{1}{2}c(1-\cos\theta_0)$, we get

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\, d\theta}{\cos\theta - \cos\theta_0} = V_\infty \alpha. \tag{7.5.1}$$

This is a complicated integral, but it can be solved. The solution will be

$$\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}. \tag{7.5.2}$$

We might want to take a closer look on the change of variable we have made. How can we visualize this change of variable? Imagine the airfoil being the diameter of a circle. Now imagine we are moving over the top half of the circle, from the leading edge to the trailing edge. The angle θ we make with respect to the center of the airfoil corresponds to the point on the airfoil directly below it, as is shown in figure 1.

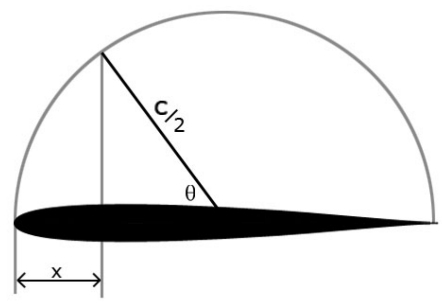

Figure 1: Clarification of the change of variable.

7.6 Lift coefficients of symmetric airfoils

In the last paragraph, we found the vortex strength of a thin symmetric airfoil. Using the vortex strength, we can find the circulation, which will turn out to be

$$\Gamma = \pi\alpha c V_\infty. \tag{7.6.1}$$

Using the Kutta-Joukowski theorem, we can calculate the lift per unit span on the airfoil, which is

$$L' = \rho_\infty V_\infty \Gamma = \pi\alpha c \rho_\infty V_\infty^2. \tag{7.6.2}$$

The lift coefficient now is

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \frac{\pi\alpha c\rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_\infty^2 c} = 2\pi\alpha. \tag{7.6.3}$$

So we know have the important conclusion that for thin symmetric airfoils, the lift slope is $a_0 = 2\pi$.

7.7 Moment coefficients of symmetric airfoils

We can use this theory as well to calculate the moment per unit span exerted on the airfoil around, for example, the leading edge. Let's call M' the moment per unit span around the leading edge. Moment is force times distance, so $dM' = -\varepsilon dL'$. The minus sign is there due to sign convention. We know that the lift per unit span is $L' = \rho_\infty V_\infty \Gamma$, so we find that $dL' = \rho_\infty V_\infty d\Gamma$. We also know that $d\Gamma = \gamma(\varepsilon) d\varepsilon$. Combining this all gives

$$M'_{LE} = -\int_0^c \varepsilon \, dL' = -\rho_\infty V_\infty \int_0^c \varepsilon\gamma(\varepsilon)\, d\varepsilon. \tag{7.7.1}$$

Using the familiar change of variable and integrating gives

$$M'_{LE} = -q_\infty \left(\frac{c}{2}\right)^2 2\pi\alpha = -c_l q_\infty \left(\frac{c}{2}\right)^2. \tag{7.7.2}$$

The **moment coefficient about the leading edge** now is

$$c_{m,le} = \frac{M'_{LE}}{q_\infty c^2} = -\frac{c_l q_\infty \left(\frac{c}{2}\right)^2}{q_\infty c^2} = -\frac{1}{4} c_l. \tag{7.7.3}$$

The **quarter-chord point** is the point at distance $\frac{1}{4}c$ from the leading edge. Taking sum of the moments about the quarter-chord point gives the **moment coefficient about the quarter-chord point**

$$c_{m,c/4} = c_{m,le} + \frac{1}{4} c_l = 0. \tag{7.7.4}$$

The **center of pressure** is the point around which there is no moment. So the center of pressure is equal to the quarter-chord position. The **aerodynamic center** is the point around which the moment coefficient is independent of α. Since $c_{m,c/4} = 0$ for every α, the quarter point position is also the aerodynamic center. So the center of pressure and the aerodynamic center are both located at the quarter-chord point.

7.8 Vortex distributions of cambered airfoils

For cambered airfoils, it is a lot more difficult to solve equation 7.4.4, since $\frac{dz}{dx} \neq 0$. Mathematicians have found the solution to be

$$\gamma(\theta) = 2V_\infty \left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right). \tag{7.8.1}$$

We will not show the derivation, since that will be too complicated. You will just have to accept the equations.

The values A_n depend on $\frac{dz}{dx}$ and A_0 depends on both $\frac{dz}{dx}$ and α. In fact, using even more complicated mathematics, it can be shown that

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx} d\theta_0, \qquad A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos n\theta_0\, d\theta_0. \tag{7.8.2}$$

Note that $\frac{dz}{dx}$ is the derivative of $z(x)$, taken at point x. So the value of $\frac{dz}{dx}$ depends on x. And x also depends on θ_0, since $x = \frac{1}{2}c(1-\cos\theta)$.

7.9 Lift coefficients of cambered airfoils

Let's take a loot at the lift coefficient of the airfoil. The circulation can be found using

$$\Gamma = cV_\infty \left(\pi A_0 + \frac{\pi}{2} A_1\right). \tag{7.9.1}$$

The lift per unit span now is

$$L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 c\pi \left(A_0 + \frac{1}{2}A_1\right). \tag{7.9.2}$$

The lift coefficient can be shown to be

$$c_l = \frac{L'}{\frac{1}{2}\rho_\infty V_\infty^2 c} = \pi(2A_0 + A_1) = 2\pi\left(\alpha + \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)d\theta_0\right). \tag{7.9.3}$$

We now see that the lift slope is once more $a_0 = \frac{dc_l}{d\alpha} = 2\pi$. So camber does not change the lift slope. However, it does change the zero-lift angle of attack, which will be

$$\alpha_{L=0} = 2\pi\alpha - c_l = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta_0 - 1)d\theta_0. \tag{7.9.4}$$

7.10 Moment coefficients of cambered airfoils

Just like we did for symmetric airfoils, we can calculate the moment coefficient. The result will be

$$c_{m,le} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right) = -\frac{c_l}{4} + \frac{\pi}{4}\left(A_2 - A_1\right). \tag{7.10.1}$$

We can once more derive the moment coefficient with respect to the quarter-chord point. It will not be 0 this time, but

$$c_{m,c/4} = \frac{\pi}{4}\left(A_2 - A_1\right). \tag{7.10.2}$$

The value of $c_{m,c/4}$ is independent of α, so the quarter-chord point is the aerodynamic center. However, the moment coefficient is not zero, so this point is not the center of pressure. The position of the center of pressure can be calculated to be

$$x_{cp} = -\frac{M'_{LE}}{L'} = -\frac{c_{m,le}c}{c_l} = \frac{c}{4}\left(1 + \frac{\pi}{c_l}(A_1 - A_2)\right) \tag{7.10.3}$$

7.11 Designing a camber line

We used the camber line (described by $\frac{dz}{dx}$) to find the coefficients $A_0, A_1, \ldots$. We can also use the coefficients to find the camber line. We then have several boundary conditions. Of course $z(0) = 0$ and $z(c) = 0$.

First we need to think of suitable coefficients for our design. What these coefficients will be depends on what properties we want to give our airfoil. For example, if we want to have $c_{m,c/4} = 0$, then we should take $A_1 = A_2$. If we have determined our coefficients, we can find our camber line by using

$$\frac{dz}{dx} = \alpha - A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta_0. \tag{7.11.1}$$

7.12 Design lift coefficient

Thin airfoils do have a disadvantage. For most angles of attack, the airflow separates at the leading edge (and reattaches afterward for low velocities). This reduces lift. For one angle of attack, the flow smoothly attaches to the leading edge. This is the so-called **ideal** or **optimal angle of attack** α_{opt}.

Theoretical calculations can show that this only occurs if the vortex at the leading edge is zero, so $\gamma_{LE} = 0$. Combining this fact with equation 7.8.1 gives $A_0 = 0$. Inserting this in equation 7.8.2 results in

$$\alpha_{opt} = \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}\theta_0. \tag{7.12.1}$$

The lift coefficient at the optimal angle of attack is called the **design lift coefficient**. Thanks to equation 7.9.3, we can calculate it, using

$$(c_l)_{design} = \pi A_1 = 2\int_0^{\pi} \frac{dz}{dx}\cos\theta_0 \, d\theta_0. \tag{7.12.2}$$

8. Three-Dimensional Wings

8.1 Induced drag

So far we have looked at two-dimensional (infinite) wings. Now let's look at three-dimensional wings. Lift is created by a high pressure on the bottom of the wing and a low pressure on top of the wing. At the wing edges, air tries to go from the bottom to the top of the wing. This causes **vortices**.

These vortices cause a small velocity component in the downward direction at the wing, called **downwash**. So the airfoil "sees" a different flow direction than the free stream flow. Even though α is the **geometric angle of attack** (with respect to the free stream flow), the **effective angle of attack** α_{eff}, which actually contributes to the lift, is different. This is such that

$$\alpha_{eff} = \alpha - \alpha_i, \tag{8.1.1}$$

where α_i is the change of the direction of the air flow close to the airfoil. α_i is called the **induced angle of attack**.

But the decrease in lift is only small. The real disadvantage is that the lift factor is tilted backward by an angle α_i. So part of the "lift" is pointing in the direction of the free stream flow, so it is actually drag. This drag is called **induced drag**.

8.2 Coefficients, lift and drag

In the last chapter we have dealt with the lift coefficient per unit span c_l. Now we will deal with the actual lift coefficient C_L of the entire wing. Identically, the lift per unit span L' becomes the total lift L. Also c_d becomes C_D and D' becomes D. The same goes for moment coefficients.

In real life, the drag consists of three parts. There is **skin friction drag** D_f, **pressure drag** D_p and induced drag D_i. The first two are caused by viscous effect, and together form the **profile drag**. If $C_{D,p}$ is the **profile drag coefficient**, then

$$C_{D,p} = \frac{D_f + D_p}{q_\infty S}. \tag{8.2.1}$$

The **induced drag coefficient** $C_{D,i}$ is

$$C_{D,i} = \frac{D_i}{q_\infty S}. \tag{8.2.2}$$

Together they form the total drag coefficient, being

$$C_D = \frac{D_f + D_p + D_i}{q_\infty S} = C_{D,p} + C_{D,i}. \tag{8.2.3}$$

8.3 Vortex filaments

In the last chapter, we considered 2-dimensional vortices. We can put a lot of them in a three-dimensional curve, being a so-called **vortex filament**. A vortex filament has a strength Γ. If we now look at any part of the curve $\mathbf{dl}$, then the velocity $\mathbf{dV}$ at some point P, caused by this part, is

$$\mathbf{dV} = \frac{\Gamma}{4\pi} \frac{\mathbf{dl} \times \mathbf{r}}{|\mathbf{r}|^3}, \tag{8.3.1}$$

where $\mathbf{r}$ is the vector from the part $\mathbf{dl}$ to point P. This important relation is called the **Biot-Savart law**.

There are a few important rules concerning vortex filaments. These are called **Helmholtz's vortex theorems**.

- The strength of a vortex filament is constant along its entire length.
- A vortex filament can not end. It is either a closed curve or it is infinitely long.

8.4 Horseshoe vortices

We can model three-dimensional wings using vortex filaments. Let's take a wing with wing span b, and put it in a coordinate system such that the tips are positioned at $y = -\frac{b}{2}$ and $y = \frac{b}{2}$. Now we can let a vortex filament run from one tip to the other. But a vortex filament may not end. So from the tip, we let the filaments (both ends) run to infinity in the direction of the free stream flow (which is defined as the positive x-direction). The part of the vortex filament on the wing is called the **bound vortex**. The two infinite parts are the **trailing vortices**. The entire vortex filament is called a **horseshoe vortex**, since it has the shape of a horseshoe (except for the fact that horses don't have infinite feet).

Using the horseshoe vortex, we can already, more or less, model the wing. But if we go to the wing tips, the induced velocity will go to infinity, which isn't what happens in real life. So we need to change our model. Instead of having one horseshoe vortex, running between $-\frac{b}{2}$ and $\frac{b}{2}$, we put infinitely many, running between $-y$ and y, where $0 \leq y \leq \frac{b}{2}$. We now have a vortex distribution $\Gamma(y)$ along the wing and a vortex sheet with strength $d\Gamma(y)$ behind the wing.

8.5 Induced angle of attack

Now look at a point on the wing with y-coordinate y_0. The velocity induced by the semi-infinite trailing vortex at position y can be found using the Biot-Savart law. The result will be

$$dw = -\frac{\left(\frac{d\Gamma}{dy}\right) dy}{4\pi(y_0 - y)}. \tag{8.5.1}$$

So if we want to find the entire induced velocity at point y_0, we need to integrate along the entire wing, giving

$$w(y_0) = -\frac{1}{4\pi} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d\Gamma}{dy}\right) dy}{(y_0 - y)}. \tag{8.5.2}$$

Using the induced velocity, we can find the induced angle of attack to be

$$\alpha_i(y_0) = \tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right) \approx -\frac{w(y_0)}{V_\infty} = \frac{1}{4\pi V_\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d\Gamma}{dy}\right) dy}{(y_0 - y)}. \tag{8.5.3}$$

Note that w is defined positive upward, but the induced angle of attack was defined to be positive directed downward. Therefore a minus sign is present. It is also assumed that w is small with respect to V_∞, so the small angle approximation can be used.

8.6 Finding the vortex distribution

Now let's derive some more expressions. From the previous chapter, we know that the lift coefficient per unit span at the point y_0 is

$$c_l = a_0(\alpha_{eff}(y_0) - \alpha_{L=0}), \tag{8.6.1}$$

where $a_0 = 2\pi$ for thin wings. But the lift coefficient can also be found using

$$L' = \frac{1}{2}\rho_\infty V_\infty^2 c(y_0) c_l = \rho_\infty V_\infty \Gamma(y_0) \qquad \Rightarrow \qquad c_l = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}. \tag{8.6.2}$$

Combining these equations and solving for α_{eff} gives

$$\alpha_{eff} = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}. \tag{8.6.3}$$

If we put everything together, the angle of attack can be calculated. The result is

$$\alpha = \alpha_{eff} + \alpha_i = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0} + \frac{1}{4\pi V_\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\left(\frac{d\Gamma}{dy}\right) dy}{(y_0 - y)}. \tag{8.6.4}$$

This equation is the **fundamental equation of Prandtl's lifting-line theory**. For a wing with a given design, all values are known except Γ. So this is in fact a differential equation with which Γ can be found.

If Γ is found, we can find the lift distribution using the Kutta-Joukowski theorem ($L'(y) = \rho_\infty V_\infty \Gamma$). Also the induced drag distribution can be found by using

$$D_i'(y) = L' \sin \alpha_i \approx L' \alpha_i = \rho_\infty V_\infty \Gamma \alpha_i. \tag{8.6.5}$$

From the lift and drag distribution, the total lift and drag can be found, by integrating over the wing (from $-\frac{b}{2}$ to $\frac{b}{2}$).

8.7 Elliptical lift distribution

Suppose we have a wing with a circulation distribution given by

$$\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}, \tag{8.7.1}$$

where Γ_0 is (per definition) the circulation at $y = 0$. It can now be shown that the induced velocity and induced angle of attack are

$$w = -\frac{\Gamma_0}{2b} \qquad \Rightarrow \qquad \alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_0}{2bV_\infty}. \tag{8.7.2}$$

Now let's define the **aspect ratio** as

$$A = \frac{b^2}{S}. \tag{8.7.3}$$

If we first express Γ as a function of C_L, fill it in in equation 8.7.2 and use the definition of the aspect ratio, then we can derive that

$$\alpha_i = \frac{C_L}{\pi A} \qquad \Rightarrow \qquad C_{D,i} = \frac{C_L^2}{\pi A}. \tag{8.7.4}$$

So the induced drag only depends on the lift coefficient and the aspect ratio. Long slender wings thus give low induced drag.

8.8 General lift distribution

Let's suppose we don't know Γ. If we make the change-of-variable $y = -\frac{b}{2}\cos\theta$, we can use a lot of complicated mathematics to transform equation 8.6.4 to

$$\alpha(\theta) = \frac{2b}{\pi c(\theta)} \sum_{1}^{N} A_n \sin n\theta + \alpha_{L=0}(\theta) + \sum_{1}^{N} nA_n \frac{\sin n\theta}{\sin\theta}. \tag{8.8.1}$$

In this equation, the coefficients $A_1, \ldots, A_n$ are the unknown coefficients that need to be determined. If N is higher (so if there are more coefficients), the result will be more precise. Do not mix up the coefficients A_i and the aspect ratio A.

To find $A_1, \ldots, A_N$, you have to apply the equation at N points on the wing. Then you have N equations and N unknowns, which can be solved. You can take any N points on the wing, except for the tips, since $\Gamma = 0$ at those positions.

We can also derive the lift coefficient to be

$$C_L = A_1 \pi A. \tag{8.8.2}$$

If we work things out a lot more, we get an expression for the drag coefficient, which appears very familiar. The result is

$$C_{D,i} = \frac{C_L^2}{\pi A e}. \tag{8.8.3}$$

The number e is called **oswald's factor** and is defined as

$$e = A_1^2 \left(\sum_{1}^{N} nA_n^2 \right)^{-1} = \frac{A_1^2}{A_1^2 + 2A_2^2 + \ldots + nA_n^2}. \tag{8.8.4}$$

It is clear that $e \leq 1$ (with $e = 1$ only if $A_i = 0$ for $i \geq 2$). For an elliptical lift distribution $e = 1$, so this lift distribution is the distribution with the lowest induced drag. However, to minimize induced drag, it is often more important to worry about the aspect ratio, then about the lift distribution.