AAE 251 Formulas

Compiled Fall 2016
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Standard Atmosphere

$p_0 = 1.01325 \times 10^5$ Pascals
$\rho_0 = 1.225 \, \frac{\text{kg}}{\text{m}^3}$
$R = 287 \, \frac{\text{J}}{\text{kg} \cdot \text{K}}$
$\gamma = 1.4$ (for air)
$p = \rho RT$; Equation of State
$h = \dfrac{r_e}{r_e + h_G} \, h_G$

Gradient Layer

$T = T_1 + a(h - h_1)$
$\dfrac{p}{p_1} = \left(\dfrac{T}{T_1}\right)^{-\dfrac{g_0}{aR}}$
$\dfrac{\rho}{\rho_1} = \left(\dfrac{T}{T_1}\right)^{-\left[\dfrac{g_0}{aR}+1\right]}$

Isothermal Layer

$\dfrac{p}{p_1} = e^{\left[\frac{-g_0}{RT}(h-h_1)\right]}$
$\dfrac{\rho}{\rho_1} = \dfrac{p}{p_1}$

Aerodynamics

Continuity Equation: $\dot{m} = \text{constant} \rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2$
Bernoulli's Equation: $p_0 = p + \frac{1}{2}\rho v^2;$ total (stagnation) p = static p + dynamic p

Aerodynamic Coefficients

$$\begin{aligned} L &= N\cos\alpha - A\sin\alpha \\ D &= N\sin\alpha + A\cos\alpha \\ N &= L\cos\alpha + D\sin\alpha \\ A &= -L\sin\alpha - D\cos\alpha \end{aligned}$$

$c_l = \dfrac{L}{q_\infty S}; \quad c_M = \dfrac{M}{q_\infty S c}; \quad c_p = \dfrac{p - p_\infty}{q_\infty}$
$a = \sqrt{\gamma R T};$ Mach Number $M = \dfrac{v}{a}$
Prandtl-Glauert Correction: $C_c = \dfrac{C_0}{\sqrt{1 - M_\infty^2}}$

Wave Drag

Mach Angle: $\mu = \sin^{-1}\left(\dfrac{1}{M}\right)$ Thicker objects $\rightarrow$ shockwaves form at angle greater than μ
$c_l = \dfrac{4\alpha}{\sqrt{M_\infty^2 - 1}}; \quad c_d = \dfrac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$

Induced Drag

$\alpha_{\text{eff}} = \alpha - \alpha_i \quad D_i = L\sin\alpha_i \approx L\alpha_i$ (for small α)

$C_{D_i} = \dfrac{C_L^2}{\pi e AR}; \quad \alpha_i = \dfrac{C_L}{\pi e AR}$ elliptical lift distribution $\rightarrow e = 1$, otherwise, $e < 1$

$\dfrac{dc_l}{d\alpha} = \dfrac{dC_L}{d\alpha_{\text{eff}}} = a_0$ (lift curve slope for an infinite wing)

$$\frac{dC_L}{d\alpha_{\text{eff}}} = \frac{dC_L}{d(\alpha - \alpha_i)} \rightarrow \int \rightarrow C_L = a_o(\alpha - \alpha_i) + C$$

Flaps

$$V_\infty = \sqrt{\frac{2L}{\rho_\infty S\, C_L}}, \quad V_{stall} = \sqrt{\frac{2W}{\rho_\infty S\, C_{L,max}}}$$

Aircraft Performance

Thrust

$$\frac{T}{W} = \frac{C_D}{C_L}; \quad T_R = \frac{W}{\frac{C_L}{C_D}}; \quad \left(\frac{C_L}{C_D}\right)_{max} = \frac{1}{2}\sqrt{\frac{\pi e AR}{C_{D,0}}}$$

$$T_{A,\,alt} = \frac{\rho}{\rho_0}\,T_{A,\,0}$$

Power

$$P = \vec{F}\cdot\vec{v}$$

$$V_\infty = \sqrt{\frac{2W}{\rho_\infty S\,C_L}} = \sqrt{\frac{2W C_D^2}{\rho_\infty S\,C_L^3}}; \quad P_R = T_R V_\infty; \quad P_A = \eta P, \text{ where } \eta < 1$$

$$V_{alt} = V_0\sqrt{\frac{\rho_0}{\rho}}; \quad P_{R,\,alt} = P_{R,\,0}\sqrt{\frac{\rho_0}{\rho}}; \quad P_{A,\,alt} = \frac{\rho}{\rho_0}\,P_{A,\,0}$$

$$R.C. = \frac{P_A - P_R}{W} = v\sin\theta$$

Range and Endurance

$$R_{prop} = \frac{\eta}{c}\cdot\frac{C_L}{C_D}\cdot\ln\left(\frac{W_0}{W_1}\right), \quad E_{prop} = \frac{\eta}{c}\cdot\frac{C_L^{3/2}}{C_D}\cdot\left(W_1^{-1/2} - W_0^{-1/2}\right), \qquad \max\left[\frac{C_L^{3/2}}{C_D}\right] = \frac{(3C_{D,\,0}\,\pi AR)^{\frac{3}{4}}}{4C_{D,\,0}}$$

$$E_{jet} = \frac{1}{c_t}\cdot\frac{C_L}{C_D}\cdot\ln\left(\frac{W_0}{W_1}\right), \quad R_{jet} = \sqrt{\frac{8}{\rho_\infty\,S}}\cdot\frac{1}{c_t}\cdot\frac{C_L^{1/2}}{C_D}\cdot\left(W_0^{1/2} - W_1^{1/2}\right), \quad \max\left[\frac{C_L^{1/2}}{C_D}\right] = \frac{(C_{D,\,0}\,\pi AR)^{\frac{1}{4}}}{4C_{D,\,0}}$$

W_0 = total gross weight, W_1 = ending weight

Takeoff

$$V = \frac{F}{m}\,t, \quad s = \frac{V^2\,m}{2F}$$

$V_{LO} = 1.2\sqrt{\dfrac{2W}{\rho_\infty S\,C_{L,\,max}}}$. For the average, $V_\infty = 0.7\,V_{LO}$.

$s_{LO} = \dfrac{1.2^2\,W^2}{g\rho_\infty S\,C_{L,\,max}\left(T - [D + \mu_r(W-L)]_{avg}\right)}$. $\quad \mu_r$ ranges from 0.02 (smooth surface) to 0.10 (grass)

Landing

$V_T = 1.3\sqrt{\dfrac{2W}{\rho_\infty S\,C_{L,\,max}}}$. For the average, $V_\infty = 0.7\,V_T$.

$s_T = \dfrac{1.3^2\,W^2}{g\rho_\infty S\,C_{L,\,max}\,[D + \mu(W-L)]_{avg}}$ $\quad \mu \approx 0.40$ (paved surface; surface friction plus brakes).

Propulsion

Air-Breathing

$\phi = \beta - \alpha$, where β is the pitch angle of the propeller and α is the angle of attack
$\omega = 2\pi n$, where n is number of revolutions per second the propeller spins.

$\tan\phi = \dfrac{V_\infty}{\omega r}$, where r is the radius at a given point of the propeller

$J = \dfrac{V_\infty}{nd}$; Advance Ratio, where d is the diameter of the propeller

$T_{turbojet} = (\dot{m}_{air} + \dot{m}_{fuel})V_e - \dot{m}_{air}V_\infty + (p_e - p_\infty)A_e$, where typically $\dot{m}_{fuel} = 0.05\,\dot{m}_{air}$

$T_{fan} = \dot{m}_F(V_e' - V_\infty) + (p_e' - p_\infty)A_e'$

$T_{propeller} = \dfrac{\eta P}{V_\infty}$, where P is the power of the engine driving the shaft

$\beta = \dfrac{\dot{m}_F}{\dot{m}_c}$; Bypass Ratio; $\dot{m}_F$ is the fan air mass flow rate and $\dot{m}_c$ is the core air mass flow rate

Rocket

$T = \dot{m}V_e + (p_e - p_\infty)A_e$

$I_{sp} \equiv \dfrac{T}{\dot{W}} = \dfrac{T}{g_0\dot{m}}$, where g_0 is acceleration at sea-level on EARTH

If nozzle is designed such that $p_e = p_\infty$, then $V_e = g_0 I_{sp}$

Assuming constant V_e, $\ln\left(\dfrac{m_f}{m_0}\right) = -\frac{\Delta V}{g_0 I_{sp}} \implies m_0 = m_f\, e^{\left[\frac{\Delta V}{g_0 I_{sp}}\right]}$ (Rocket Equation)

m_0 = initial rocket mass = $m_{payload} + m_{inert} + m_{propellant}$; m_f = final rocket mass = $m_{payload} + m_{inert}$

Inert Mass Fraction: $f_{inert} = \dfrac{m_{inert}}{m_{prop} + m_{inert}}$; Propellant Mass Fraction = PMF = $\zeta = \dfrac{m_{prop}}{m_0}$

$$m_{prop} = \frac{m_{pay}(e^{\frac{\Delta V}{g_0 I_{sp}}} - 1)(1 - f_{inert})}{1 - f_{inert}\, e^{\frac{\Delta V}{g_0 I_{sp}}}};\ \ m_{inert} = \frac{m_{pay} \cdot f_{inert}(e^{\frac{\Delta V}{g_0 I_{sp}}} - 1)}{1 - f_{inert}\, e^{\frac{\Delta V}{g_0 I_{sp}}}};\ \ m_0 = \frac{m_{pay}\, e^{\frac{\Delta V}{g_0 I_{sp}}}(1 - f_{inert})}{1 - f_{inert}\, e^{\frac{\Delta V}{g_0 I_{sp}}}}$$

Orbital Mechanics

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

$$\ddot{\vec{r}} = \frac{-G(M+m)\hat{r}}{r^2} \implies r = \frac{p}{1+e\cos\theta}$$

When $M \gg m$, $G(M+m) \approx GM = \mu$

$$PE = -\frac{GMm}{r} = \frac{-\mu m}{r} \ (\leq 0)$$

Specific Energy: $\varepsilon = \dfrac{PE + KE}{m} \quad \rightarrow \quad \varepsilon = \dfrac{v^2}{2} - \dfrac{\mu}{r}$ (Vis-Viva Equation)

$r_{SOI} = D\left(\dfrac{m_1}{M_2}\right)^{2/5}$, where $m_1 < M_2$, and D is the distance between celestial bodies

Conic Sections

$2p \equiv$ width of each conic at the focus (latus rectum)
$2a \equiv$ length of the chord that passes through the two foci (major axis)
$2c \equiv$ distance between foci

$e = \dfrac{c}{a} = \dfrac{2c}{2a} = \dfrac{r_a - r_p}{r_a + r_p}$
$p = a(1-e^2)$
$r_p = a(1-e)$ (periapsis)
$r_a = a(1+e)$ (apoapsis)
$r_a + r_p = 2a;\ r_a - r_p = 2c$
$\varepsilon = -\dfrac{\mu}{2a}$

Elliptical Orbits

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

Circular Orbits

$e = 0;\ r = \text{constant} = a = b = p$
$T = \dfrac{2\pi r^{3/2}}{\sqrt{\mu}}$
$v_c = \sqrt{\dfrac{\mu}{r}}$

Parabolic Orbits

$e = 1;\ r_p = \dfrac{p}{1+e} = \dfrac{p}{2};\quad r_a = \dfrac{p}{1-e} = \infty$

$v_{esc} = \sqrt{\dfrac{2\mu}{r}} = \sqrt{2}\, v_c$

Hyperbolic Orbits

δ is the **turn angle** or **bend angle**
v_∞ is the **hyperbolic excess speed**

$\sin\left(\dfrac{\delta}{2}\right) = \dfrac{a}{c} = \dfrac{1}{e}$

$\varepsilon = \dfrac{v_\infty^2}{2} \longrightarrow v_\infty = \sqrt{v^2 - \dfrac{2\mu}{r}} = \sqrt{v^2 - v_{esc}^2}$

Ground Tracks

$$\begin{array}{ll} \text{Posigrade Orbit:} & 0° \leq i < 90° \\ \text{Polar Orbit:} & i = 90° \\ \text{Retrograde Orbit:} & 90° < i < 180° \end{array}$$

$\Delta = \dfrac{T}{R}(360°) = T\omega$; T is the period of orbit, R is the rotational period, and ω is the rotational rate

Effects of Launch Sites

$\cos i = \sin\beta\cos\lambda$, where i is orbit inclination, β is azimuth, and λ is latitude of the launch site
$\Delta V_{\text{Earth's Rotation}} = \omega_E\, r_E\, \cos\lambda\sin\beta$

Plane Changes

$\Delta V = 2V_i\sin\left(\dfrac{\Delta i}{2}\right)$

Losses During Launch

$\Delta V_{\text{effective}} = \Delta V_{ideal} + \Delta V_{drag} + \Delta V_{gravity} + \Delta V_{steering}$

$\Delta V_{drag} = \displaystyle\int \frac{dV_{drag}}{dt}\, dt = \int \frac{D}{m}\, dt$

$\Delta V_{gravity} = \displaystyle\int g\sin\theta\, dt$