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Calculus—2nd quarter

Summary of formulas

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1 Sequences

1.1 Basics and notation

Sequences come in multiple different notations. Here are the most important ones:

$$\{a_1, a_2, a_3, \dots, \} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

with $n \in \mathbb{N}$.

It is possible to plot sequences. For that, take the x-axis for n and the y-axis for the resultant values of a_n.

If you want to calculate the limit of a sequence, the following theorem holds.

$$\lim_{x\to\infty} f(x) = L, \quad f(n) = a_n, \quad \Rightarrow \lim_{n\to\infty} a_n = L$$

again, $n \in \mathbb{N}$. In words: The limt of a sequence equals the limit of the corresponding function.

1.2 Limit laws

Calculus is all about limits. Derivatives are limits, Integrals are limits, limits are limits. So, figure that limits are important. You should know the following laws. For convinience's sake, from now on, by lim it is actualy referred to $\lim_{n\to\infty}$.

$$\begin{aligned}
&\lim(a_n + b_n) = \lim a_n + \lim b_n \\
&\lim(a_n - c_n) = \lim a_n - \lim b_n \\
&\lim ca_n = c \lim a_n \\
&\lim(a_n b_n) = \lim a_n \cdot \lim b_n \\
&\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}, \lim b_n \neq 0 \\
&\lim a_n^p = [\lim a_n]^p, p > 0, a_n > 0 \\
&\lim c = c \\
&\lim |a_n| = L, \quad \text{then} \quad \lim a_n = L
\end{aligned}$$

1.3 More about sequences

Another more confusing theorem is, that if you have a term with variables inside of a function, it is possible to solve only for the inner part of the function.

If $\lim a_n = L$ and f is continous at L, then $\lim f(a_n) = f(L)$.

An example will make this clear:

$$\lim\left(\sin\left(\frac{\pi}{n}\right)\right) = \sin\left(\lim\left(\frac{\pi}{n}\right)\right) = \sin 0 = 0$$

For solving limits, the following trick can very often be applied. Pull out the highest exponential out of the equation and simply solve the limit.

A special limit that you have to know is $\lim\{r^n\}$.

This limit is convergent for $-1 < r \leq 1$, else, it is divergent.

Check

$r > 1$	r will constantly increase
	E.g. $1.05^2 + 1.05^3$ etc.
$-1 < r < 1$	will turn out to be zero.
	E.g. $-0.5^2 - 0.5^3$ etc.
$r = 1$	convergent, since $\lim 1^n = 1$
$r = -1$	divergent, since $\lim(-1)^n = \emptyset$.

Some more terms concerning sequence. $\{a_n\}$ is called *increasing* if $a_n < a_{n+1}$ for all $n \geq 1$, *decreasing* if $a_n > a_{n+1}$ for all $n \geq 1$.

If one of these conditions is fulfilled, the sequence is called *monotonic*.

For several checks, these terms are of utmost importance. So, to prove if a sequence is monotonic, simply take the derivative of its correlating function.

Example:

$$a_n = \frac{n}{n^2+1} \quad \Leftrightarrow \quad f(x) = \frac{x}{x^2+1} \tag{1}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \tag{2}$$

for all $x^2 >$, so it will always become smaller and is therefore decreasing, monotonic.

One last term for sequences:

A sequence is *bounded*, from above if $a_n \leq M$ for all $n \geq 1$ and bounded from below when $m \leq a_n$ for all $n \geq 1$.

However, these terms have to be treated with care! Always remember that not every bounded sequence is convergent. Also, not every monotonic sequence is convergent. *But*, when both is the case, it must be convergent.

2 Series

Now that we have set the base, we can talk about series.

2.1 Basics and notation

Sequences are nothing else but the sum of all elements a_n up to n, where $n \in \mathbb{N}$. So, when adding all terms of a sequence $\{a_n\}_{n=1}^{\infty}$, we get an *infinite series*

$$\sum_{n=1}^{\infty} a_n \ .$$

For example

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \ .$$

The sum of this series can be written as

$$\begin{aligned} s_2 &= a_1 + a_2 \\ &[\ldots] \\ s_n &= a_1 + \ldots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

For calculating limits of series, we agree upon the following notation:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

From now on, as in the previous section, the notation $\sum_{n=1}^{\infty}$ will for convience's sake be meant by $\sum$.

2.2 Geometric series

Geometric series are a special case of series. In fact, besides of another type of series, they are the only series we can calculate the sum of rationally.

Geometric series are series of the form like

$$\sum ar^{n-1} = \frac{a}{1-r} \quad |r| < 1 \ ,$$

$$a + ar^1 + ar^2 + \ldots$$

which is convergent.

If $|r| \geq 1$ the series would be convergent.

Geometric series occur quite often. In many cases, you have to rearrange a series and such a way that a geometric series is resulting. An example:

$$\begin{aligned} \sum 2^{2n} 3^{1-n} &= (2^2)^n 3^{-n-1} \\ &= \frac{4^n}{3^{n-1}} = 4\left(\frac{4}{3}\right)^{n-1} \end{aligned}$$

Because $r = 4/3 > 1$ the series is divergent. Don't get confused, also series like $\sum x^n$ are geometric series because $a = 1$, $r = x$.

Also, the series $\sum 1/x^n$ is a geometric series with $a = 1/x$ and $r = 1/x$, because

$$\frac{1}{x}\frac{1}{x^{n-1}} = \frac{1}{x}\frac{x}{x^n} = \frac{1}{x^n}$$

3 Convergence Tests

3.1 Test for divergence

From here on, some test for the convergence or divergence of series will be listed.

$$\text{If} \quad \sum a_n \quad \text{convergent, then} \quad \lim a_n = 0 \,.$$

This theorem might be a little bit confusing, because it is *not* true in general! From the side of $\lim a_n = 0$ we can concluding *absolutelty nothing* about the convergence or divergence of a series.

However:

$$\text{If} \quad \lim a_n \quad \text{does not exist or} \quad \lim a_n \neq 0,$$

$$\text{then} \sum a_n \quad \text{is divergent.}$$

3.2 Integral test

Necessary conditions: f must be continuous, positive, and decreasing in $[1, \infty)$.

$$a_n = f(n), \quad \sum_{n=1}^{\infty} a_n \quad \text{is convergent if} \quad \int_1^{\infty} f(x)\mathrm{d}x$$

and it yields a finite value.

Note that for the integral test as the lower integrating boundary you have to take the start value of the series, e.g. for $\sum_{n=2}^{\infty}$, take the integral from $\int_2^{\infty}$.

3.3 P-series

This is the same as we have learnt for integrals in the first quater.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent for $p > 1$ and divergent for $p \leq 1$.

3.4 Comparison test

Once you have grasped the concept of the comparison test, it is quite nice.

You are given a function $g(x)$. Now consider a function $f(x)$ that is larger than $g(x)$. If this function $g(x)$ converges, then also $f(x)$ converges. If it should turnout that this test indicates that the function might be divergent, you may not stop yet, but rather compare it to a smaller function, because if a smaller function diverges, then the larger function also diverges.

Summarized:

- If a function $g(x)$ is smaller than a function $f(x)$ which it is compared to, and that function is convergent, then $g(x)$ also is convergent.
- If a function $g(x)$ is larger than a function $f(x)$ which it is compared to, and that function is divergent, then $g(x)$ is also divergent.

Thus, when trying to prove that a function is convergent, try to replace the function in the numerator such that it gets bigger or do the opposite in the denominator. For instance, if there is a function that is always smaller than 1 (e.g. $\cos x$), you can replace it by 1.

3.5 Comparison test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

In general this means that if you have to comparable series, you only have to test one series!

3.6 Ratio test

A series $\sum a_n$ is called

- absolutely convergent if $\sum |a_n|$ is convergent,
- conditionally convergent if it is convergent but not absolutely convergent.

This fact can be exploited for some functions like $\sum \cos n/n^2$ as follows

$$\left|\frac{\cos n}{n^2}\right| = \sum \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

Thus, the comparison test can be applied. The series must be convergent since this is a p-series.

This is the base for the ratio test:

$$\boxed{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L}$$

$$\text{If}\quad \begin{array}{ll} L<1 & \text{– absolutely convergent} \to \text{convergent} \\ L=1 & \text{– divergent} \\ L>1 & \text{– inconclusive, choose other test} \end{array}$$

The ratio test should be applied if there are factorials $n!$ or functions raised to the power of n, e.g. 2^n in the series.

3.7 Root test

When there is a series with an exponential, the root test can be applied:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

$$\text{If}\quad \begin{array}{ll} L<1 & \text{– absolutely convergent} \to \text{convergent} \\ L=1 & \text{– inconlusive} \\ L>1 \text{ or } L=\infty & \text{– divergent} \end{array}$$

3.8 Alternating series test

The alternating series test is the coolest of all. It can't get easier!

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 \ldots \quad b_>0$$

satisfies

$$\begin{array}{ll} \text{(i)} & b_{n+1} \le b_n \quad \text{for all } n \\ \text{(ii)} & \lim_{n\to\infty} b_n = 0 \end{array}$$

then the series is convergent.

That's it!

4 Strategy for testing series

- If it is obvious that $\lim_{n\to\infty} a_n \neq 0$, test for divergence.
- Is the series a p-series $(1/n^p)$?
  If yes, $\begin{array}{ll} p>1 & \to \text{convergent} \\ p\le 1 & \to \text{divergent} \end{array}$
- Is the series a geometric series $(a \cdot r^{n-1})$?
  If yes, $\begin{array}{ll} |r|<1 & \to \text{convergent} \\ |r|\ge 1 & \to \text{divergent} \end{array}$
- If the series is similar to a geometric- or p-series, then choose the comparison test. Especially if it involves an algebraic function of n (involving roots or polynomials). For the comparing series, choose only the highest power of n as the p in the p series $(1/n^p)$. If there are positive and negative terms, only apply the test to $\sum |a_n|$.
  For example

  $$\frac{\sqrt{n^3+2n^2+n}}{3n^3+6n}$$

  should be compared to

  $$\frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

  which is a p-series.
- If the series includes a term like $(-1)^{n-1}$, obvioulsy use the alternating series test.
- In case a series includes factorials or products involving a constant raised to the power of n, then the ratio test should be useful.
- Should there be something of the form like $(b_n)^n$, then apply the root test.
- If $a_n = f(n)$, where $\int_1^{\infty} f(n)\mathrm{d}n$ can be easily evaluated, then the integral test should be applied.

5 Power series

Power series have the form:

$$\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

Where x is a variable and all cs are coefficients. The domain of a power series is all the x for which it converges.

There are only three possibilities for a power series:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

1. converges only when $x = a$
2. converges for all x
3. there is a positive number R such that the the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

This number R is called the radius of convergence.

Always take the ratio or root test to find the radius of convergence. If there is an endpoint, the test will always fail; in that case, take another test.

Example: When you find the series to be convergent for $|x-5| < 1$ and divergent for $|x-5| > 1$, the interval of convergence is from 4 to 6, the radius of convergence therefore is 1. Now, plug in the numbers into the function and check how the series behaves. If it the test diverges the bracket is (, for convergence [, e.g. $(4, 5]$

5.1 Representation of functions as power series

A function of the form $f(x) = 1/(1-x)$ can be expressed as a power series:

$$f(x) = \frac{1}{1-x} = \sum 1 + x + x^2 + \ldots + x^n = \sum_{n=0}^{\infty} x^n$$

Note that this is a geometric series with $a = 1$, $r = x$.

✔ Example . . .

$$\frac{x^3}{x+2} = x^3 \frac{1}{2} \frac{1}{1+\frac{x}{2}} = x^3 \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{2^{n+1}}$$

It is possible to integrate/differentiate such a function. For this, simply integrate/differentiate every single element of the series. This makes it possible to integrate complicated functions.

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$

$$= \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $\int f(x) \mathrm{d}x =$

$$c + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \ldots$$

$$= c + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{(n+1)}}{n+1}$$

..

✔ Example . . .

$$\ln(1-x) = \int \frac{1}{1-x} \mathrm{d}x = \int (1 + x + x^2 + \ldots) \mathrm{d}x$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + c = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + c$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n} + c$$

To obtain a value for c, put $x = 0$. $\ln 1 = c = 0$. The same works for $\arctan(x)$.

6 Taylor and McLaurin series

With Taylor and McLaurin series, it is possible to express a function in terms of a series. This makes it possible to approximate functions up to a wanted degree of accuracy. To obtain a Taylor series, the following formula should be used:

$$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{\text{This is the coefficient!}} (x-a)^n$$

For instance

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

A McLaurin series is nothing else but a Taylor series centred at 0.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n$$

If a series is centred at a position other than 0, the approximation of the values will be more accurate around that point.

✔ Example . . .

Finding Taylor series is extremely important. That is why this long example is devoted to it.

Find the McLaurin series for $f(x) = \sin(x)$

$$f'(x) = \cos x$$

$$f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = -1$$

$$f''''(0) = 0 \quad f'''''(0) = 1$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

$$= 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 - \frac{1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$$

Note that when trying to find such series, it is still often useful to use the "integration trick". For instance, when trying to find the series for $\cos x$, it is smart to seize that fact that $\cos x$ is $(\sin x)'$, because

$$\cos x = \left(\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} \right)'$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

Taylor series can also be used for approximating integrals.

✔ Example . . .

To approximate the integral, after writing down the series as a binomial series, you only have to integrate that series:

$$\int_0^{1/2} \sqrt{x^3 + 1}\,\mathrm{d}x = \left[x + \frac{1}{8} x^4 + \frac{1}{56} x^7 + \dots \right]_0^{1/2}$$

7 Binomial series

Binomal series are just a special case of taylor series.

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

8 Vectors

8.1 Dot product

"Multiplying" two vectors leads to a scalar. If the result is 0, the vectors are perpendicular to each other.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

It can also be used to evaluate the angle between two vectors.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

The scalar projection of a vector $\vec{b}$ onto another vecot $\vec{b}$ is:

$$\text{comp}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

The vector projection is the scalar projection times the unit vector of $\vec{a}$.

$$\text{proj}_a \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \cdot \frac{\vec{a}}{|\vec{a}|}$$

8.2 Cross product

Sometimes also referred to as "vector product", this product gives you a vector perpendicular to both other vectors.

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b3 \\ a_1 b_3 - a_3 b_1 \end{pmatrix}$$

It can also be used to find the sine of two vectors.

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

If $\vec{a} \| \vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$

The length of the cross product is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$. In the same manner, half of it is equal to the the are of the triangle.

$$A_\Delta = \frac{1}{2}|\vec{a}\times\vec{b}|$$

8.3 Tripple products

These kind of products can be used to calculate volumes. The volume of a parallel piped is:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

Note

$$\vec{a}\times\vec{b} \neq \vec{b}\times\vec{b}$$
$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

9 Vector functions and space curves

9.1 Definitions for vector functions

A vector function can be written in the form of

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

To compute the limit of such a vector function, one simply has to take the limit of the separate functions f, g, h.

A vector function is continuous at a if

$$\lim \mathbf{r}(t) = \mathbf{r}(a)$$

That means, if all functions of the vector function are continous at a.

The set of points

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

is called a “space curve”.

✔ Example . . .

$$\mathbf{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle$$

Note that this is the equation of a line. The following notation might be more familiar for you:

$$\vec{r}(t) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$$

9.2 Types of curves

A circle/spiral can be expressed as in the following example:

$$\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$$

This is a spiral because

$$x = \cos t, \quad y = \sin t$$

$$x^2 + y^2 = \sin^2 + \cos^2 = 1$$

So that all points lie on a circular cylinder with a radius of 1.

A sphere is a set of points $P(x,y,z)$ whose distance from C is r. If P is on the sphere, then $|PC| = r$.

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

where h, k, l, are the coordinates of the centre.

To find cutting points or functions of curves that cut each other, simply plug the functions into each other.

9.3 Derivatives and Integrals of vector functions

Simply take the integral/derivative of all the functions.

A tangent vector is the derivative of a vector function. Often, it is looked for a unit tangent vector.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

9.4 Arc length and curvature

The length of a space curve in an interval a, b is defined as:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \mathrm{d}t$$

Which can shortly be written as:

$$L = \int_a^b \left|\mathbf{r}'(t)\right| \mathrm{d}t$$

A curve is called "smooth" if $\mathbf{r}'$ is continous and $\mathbf{r}'(t) \neq 0$. The curvatured is

$$\kappa(t) = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$$

Sometimes, the following formula is more convinient.

$$\kappa(t) = \frac{|\mathbf{r}(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

A Some tricks

Algebraic relation:

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{(x+1)}$$

To Solve integrals like $\int 1/(n^2+4)$, substitue $x = 2t$, $\mathrm{d}x = 2\mathrm{d}t$,

$$\frac{2\mathrm{d}t}{4t^2+4} = \frac{1}{2}\int \frac{\mathrm{d}t}{t^2+1} = \frac{1}{2}\arctan x$$

Factorials:

$$(n+1)! = n!(n+1)$$

$$\frac{x}{x!} = \frac{1}{(x-1)!} \text{ e.g. } \frac{4}{4!} = \frac{1}{3!}$$

Expression of e:

$$\left(1 + \frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

Useful substitutions

The following substitution works well if there is a logarithmic function in the integral, because an e function can pretty well be integrated using integration by parts.

$$\ln x = u \to e^u = x$$

For square roots:

$$\sqrt{x} = u \to x = u^2$$

And never forget to differentiate the substitution!

In the exam, when they give you a "hint", it is very likely that you have to differentiate or integrate a series to obtain a result.