Calculus - Period 2

Testing Series

Convergence/Divergence:
Suppose a is a series of numbers $a_1, a_2, \ldots$, and $s_n = \sum_{k=1}^{n} a_k$. A series s_n converges if $\lim_{n\to\infty} s_n = s$ exists as a real number. The limit s is then called the sum of series a. If s doesn't exist as a finite number, the series is divergent. Be careful not to confuse the series a_n with the series $\sum a_n = s$.

Monotonic Sequence Theorem
If a sequence is either increasing ($a_{n+1} > a_n$ for all $n \geq 1$) or decreasing ($a_{n+1} < a_n$ for all $n \geq 1$), it is called a monotonic sequence. If there are c_1 and c_2 such that $c_1 < a_n < c_2$ for all $n \geq 1$, it is called bounded. Every bounded monotonic sequence is convergent.

Test for divergence:
If $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} \neq 0$, then the series s_n is divergent.

Integral test:
If f is a continuous positive decreasing function on $[1, \infty)$ and $a_n = f(n)$ for integer n, then the series s_n is convergent if, and only if, the integral $\int_1^\infty f(x)dx$ is convergent.

Comparison test:
Suppose a_n and b_n are series with positive terms and $a_n \leq b_n$ for all n, then:

- If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.
- If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Limit comparison test:
Suppose a_n and b_n are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $0 < c \neq \infty$, then either both series are convergent or divergent.

Alternating series test:
If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \ldots$$

satisfies $a_{n+1} \leq a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, then the series is convergent.

Absolute convergence:
A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent. A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Ratio test:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series $\sum a_n$ is divergent.

Root test:

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then the series $\sum a_n$ is divergent.

Power Series

Radius of convergence:
Power series are written as

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{1}$$

where x is a variable and the c_n's are constant coefficients of the series. When tested for converges, there are only three possibilities:

- The series converges only if $x = a$. ($R = 0$)
- The series converges for all x. ($R = \infty$)
- The series converges for $|x - a| < R$ and diverges for $|x - a| > R$. For $|x - a| = R$ other means must point out whether convergence or divergence occurs.

The number R is called the radius of convergence, and can often be found using the ratio test.

Differentiation and integration:
Differentiation and integration of power functions is possible in the interval $(a - R, a + R)$, where the function does not diverge. It goes as follows:

$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \tag{2}$$

$$\int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \tag{3}$$

Representation of functions as power series: The first way to represent functions as power series is simple, but doesn't always work. To find the representation of $f(x)$, first find a function $g(x)$ such that $f(x) = ax^b \frac{1}{1-g(x)}$, where a and b are constants. The power series is then equal to:

$$f(x) = \sum_{n=0}^{\infty} a \cdot g(x)^{n+b} \tag{4}$$

The second way to represent functions as power series goes as follows. Let $f^{(n)}(x)$ be the n'th derivative of $f(x)$. Supposing the function $f(x)$ has a power series (this sometimes still has to be proven), the following function must be true:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{5}$$

This representation is called the Taylor series of $f(x)$ at a. For the special case that $a = 0$, it is called the Maclaurin series.

Binomial series:
If k is any real number and $|x| < 1$, the power function representation of $(1+x)^k$ is:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \tag{6}$$

$$\text{where} \quad \binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \tag{7}$$

for $n \geq 1$, and $\binom{k}{0} = 1$.

Vectors

Notation:
A vector $\mathbf{a}$ is often written as:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{8}$$

Where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors.

Vector length:

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{9}$$

Vector addition and subtraction:

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k} \tag{10}$$

$$\mathbf{a} - \mathbf{b} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k} \tag{11}$$

Dot product:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \tag{12}$$

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \tag{13}$$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \tag{14}$$

Cross product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{15}$$

$$\begin{gathered} \mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + \\ +(a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \end{gathered} \tag{16}$$

Vector Functions:

Notation:

$$\mathbf{r}(\mathbf{t}) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \tag{17}$$

Differentiation and integration:

$$\mathbf{r}'(\mathbf{t}) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} \tag{18}$$

$$\mathbf{R}(\mathbf{t}) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k} + \mathbf{D} \tag{19}$$

Function dependant unit vectors:

$$\mathbf{T}(\mathbf{t}) = \frac{\mathbf{r}'(\mathbf{t})}{|\mathbf{r}'(\mathbf{t})|} \tag{20}$$

$$\mathbf{N}(\mathbf{t}) = \frac{\mathbf{T}'(\mathbf{t})}{|\mathbf{T}'(\mathbf{t})|} \tag{21}$$

$$\mathbf{B}(\mathbf{t}) = \mathbf{T}(\mathbf{t}) \times \mathbf{N}(\mathbf{t}) \tag{22}$$

Trajectory length:

$$d\mathbf{s}(\mathbf{t}) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = |\mathbf{r}'(\mathbf{t})|dt \tag{23}$$

$$s(t) = \int_a^t |\mathbf{r}'(\mathbf{t})|dt \tag{24}$$

Trajectory velocity and acceleration:

$$|\mathbf{v}(\mathbf{t})| = \frac{d\mathbf{s}(\mathbf{t})}{dt} = |\mathbf{r}'(\mathbf{t})| \tag{25}$$

$$\mathbf{a}(\mathbf{t}) = \mathbf{v}'(\mathbf{t}) = \mathbf{r}''(\mathbf{t}) \tag{26}$$

Trajectory curvature:

$$\kappa(t) = \left| \frac{d\mathbf{T(t)}}{d\mathbf{s(t)}} \right| == \frac{|\mathbf{T}'(\mathbf{t})|}{|\mathbf{r}'(\mathbf{t})|} \tag{27}$$

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \tag{28}$$

Expressing acceleration in unit vectors:

$$\mathbf{a(t)} = |\mathbf{v(t)}|'\mathbf{T(t)} + \kappa|\mathbf{v(t)}|^2\mathbf{N(t)} \tag{29}$$

$$\mathbf{a(t)} = \frac{\mathbf{r}'(\mathbf{t}) \cdot \mathbf{r}''(\mathbf{t})}{|\mathbf{r}'(\mathbf{t})|}\mathbf{T(t)} + \frac{|\mathbf{r}'(\mathbf{t}) \times \mathbf{r}''(\mathbf{t})|}{|\mathbf{r}'(\mathbf{t})|}\mathbf{N(t)} \tag{30}$$