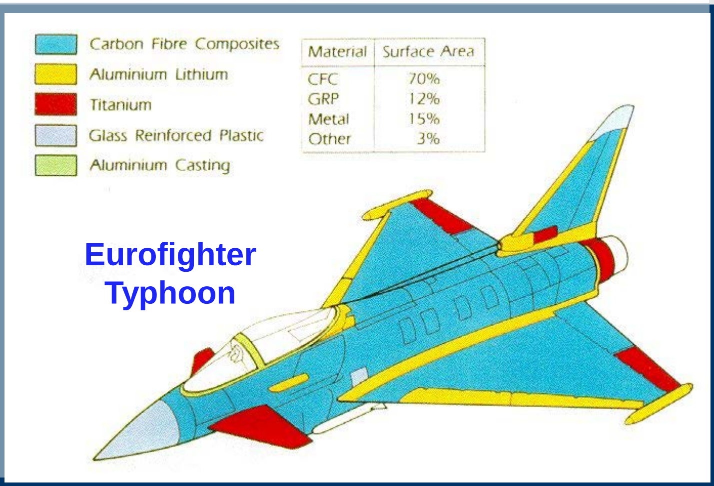
Lecture Note #3

IIT Kharagpur

2019 1st Semester

Short Course

Design of Aircraft Components using Composite Materials



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Lecture Note : Composites Structures

Analysis & Design of Composite Laminates (Part 3)

An introduction to the analysis and design of composite laminates using analytical and empirical methods. Examples are included to illustrate the methods and a laminate analysis program is provided to facilitate the analytical calculations.

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Analytical Analysis Method

Classical laminate Analysis

Empirical laminate Analysis methods

Netting Rule

Rule of Mixtures

Carpet Plotting

Laminate Analysis and Design Examples

Analytical laminate Analysis

“Classical laminate Theory”

The analysis is considered in the following sequence:

1. **Stress analysis**
   - Of a lamina : 1ply
   - Of a laminate
2. **Strength analysis**
   - Of a lamina
   - Of a laminate

A detailed list of this section is given below as a convenient reference

Contents:

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Compliance strain-stress relations
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- Force and moment resultants

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- Layer stresses and strains
  . Calculation of mid plane deformations
    . Calculation of of layer total strains
    . Transformation to 1-2 material axes
    . Calculation of layer total stresses

Laminate stress-strain analysis (Hygro-thermal loading)

Thermal analysis

- Single layer plate, one dimension
  . Unrestrained
    . Fully restrained
- Laminated plate, one dimension
  . Free thermal strains
    . Residual strains
    . Residual stresses

- Orthotropic laminate plate, two dimensions
  . Layer free thermal strains in 1-2 material axis directions
  . Layer free thermal strains x-y plate axis directions
  . Layer “free thermal stresses” in x-y plate axis directions
  . “Thermal forces and moments”
  . Laminate common strains
  . Layer residual stresses

Hygroscopic analysis

- Moisture diffusion

Lamina Strength analysis

Isotropic Materials

- Common failure criteria for isotropic materials
  . Maximum principal stress
  . Maximum principal strain
  . Maximum shear
  . Maximum shear strain energy
- Orthotropic materials
- Common failure criteria for orthotropic materials
  . Maximum stress failure criteria
  . Maximum strain failure criteria
  . Maximum “distortional energy” failure criteria
- Failure envelopes

Laminate Strength Analysis

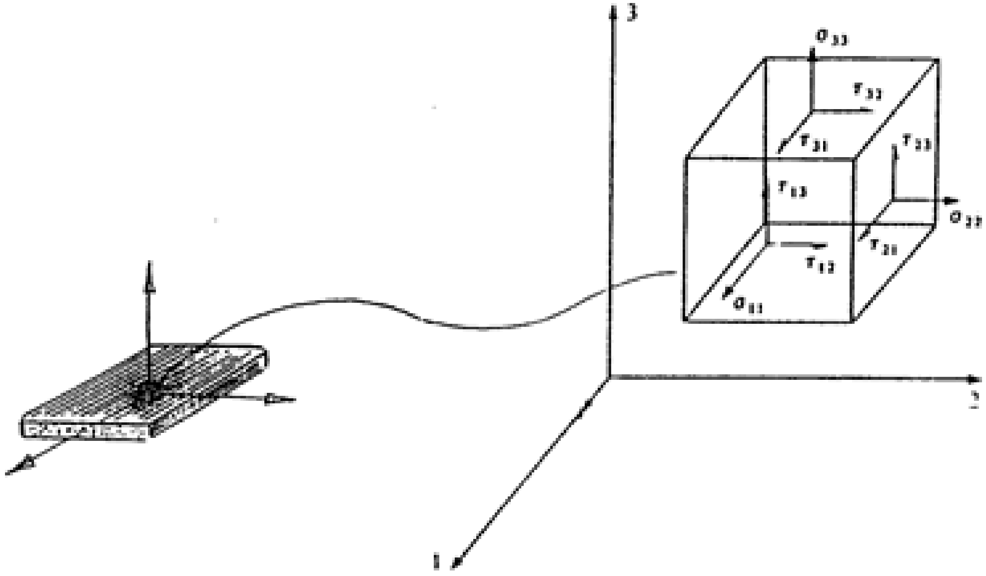
Laminate failure definition
Laminate failure analysis procedure
Other modes of failure

Lamina Stress-Strain Analysis

(Mechanical Loading)

Based on:

- **3D stress-strain system**
- **Material axes 1,2,3**
- **Macroscopic Scale**



9 stress-strain components are defined in the 3D system:

$$\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{32}, \tau_{13}, \tau_{12}, \tau_{21}, \tau_{31}$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{23}, \gamma_{32}, \gamma_{31}, \gamma_{13}, \gamma_{12}, \gamma_{21}$$

Relationships are defined between stress and strain assuming:

- Average macroscopic homogeneous linear elastic properties
- Expressed in contracted notation as:

$\{\sigma\} = [C]\{\varepsilon\}$ Constitutive relation $[C] = Stiffness\ \ matrix!$

$\{\varepsilon\} = [S]\{\sigma\}$ Compliance relation $[S] = Compliance\ \ matrix!$

"Constitutive" stress-strain relations

- **Fully anisotropic**

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}_{36imc}$$

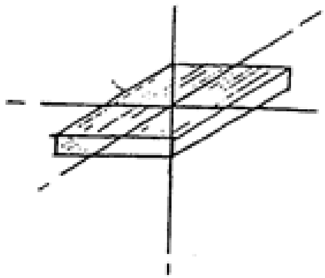
independent material constants

- **Independent of the order of loading (reciprocal behaviour)**

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix}_{21imc}$$

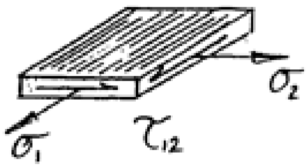
- **3 mutually perpendicular planes of symmetry (orthotropic)**

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix}_{9imc}$$



- **Plane stress**

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{4imc}$$



- **“Reduced stiffness matrix”**

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}_{4imc}$$

“Compliance” strain-stress relations

– **Anisotropic**

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix}_{36\, imc!}$$

– **Reciprocal**

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix}_{21\, imc!}$$

– **Orthotropic**

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix}_{9\, imc!}$$

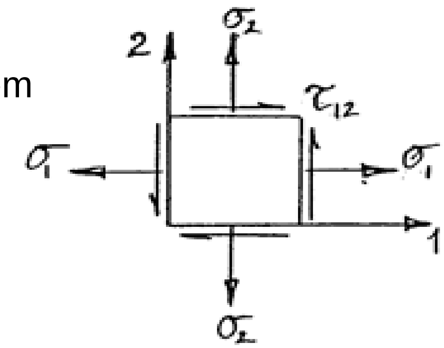
– **Plane stress, Orthotropic**

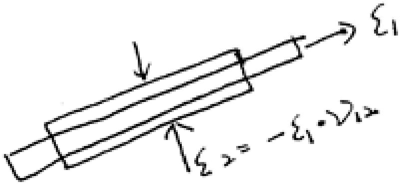
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_{4imc}$$

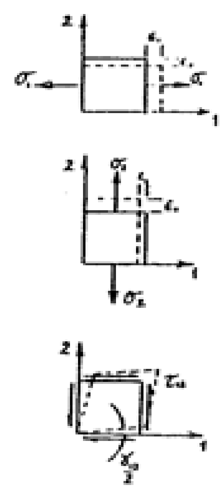
$$\{\varepsilon\} = [S]\{\sigma\}$$

In terms of Engineering Elastic Constants

– For Orthotropic material under a 2D plane stress system







– Consider Stress components separately:

Strains:

$$\varepsilon_1 = \quad \frac{\sigma_1}{E_1} \quad \frac{-\nu_{21}\sigma_2}{E_2} \quad 0$$

$$\varepsilon_2 = \frac{-\nu_{12}\sigma_1}{E_1} \quad \frac{\sigma_2}{E_2} \quad 0$$

$$\gamma_{12} = \quad 0 \quad 0 \quad \frac{\tau_{12}}{G_{12}}$$

★ i.e.: compliance relationship:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}$$

Stress:

★ Constitutive relationship:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{E_1}{(1-\nu_{12}\nu_{21})} & \frac{\nu_{21}E_1}{(1-\nu_{12}\nu_{21})} & 0 \\ \frac{\nu_{12}E_2}{(1-\nu_{12}\nu_{21})} & \frac{E_2}{(1-\nu_{12}\nu_{21})} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

*Note: 4imc: $E_1\ E_2\ v_{21}\ G_{12}$

*Note $v_{21} \neq v_{12}$ **But reciprocal relation** $\Rightarrow v_{12}E_2 = v_{21}E_1$

$$\left.\begin{aligned} v_{12} &= Major\ \ poisson\ \ ratio \\ v_{21} &= Minjor\ \ poisson\ \ ratio \end{aligned}\right\} Range\ of\ \ orthotropi\ \ poisson\ ratio:$$

- **For isotropic material:**

$$E_1 = E_2 = E$$

$$v_{21} = v_{12} = v$$

Range of isotropic poisson ratio:

$$G_{12} = G = \frac{E}{2(1+v)}$$

i.e. 2 imc: E, v

Summary of Elastic Constant Interrelations

For plane stress / orthotropic material

- **Engineering = Compliance = Stiffness**

$$E_1 = 1/S_{11} = \left(Q_{11}Q_{22} - Q^2{}_{12}\right)/Q_{22}$$
$$E_2 = 1/S_{22} = \left(Q_{11}Q_{22} - Q^2{}_{12}\right)/Q_{11}$$
$$v_{12} = -S_{12}/S_{11} = Q_{12}/Q_{22}$$
$$v_{21} = -S_{12}/S_{22} = Q_{12}/Q_{11}$$
$$G_{12} = 1/S_{66} = Q_{66}$$

- **Stiffness = Engineering = Compliance**

$$Q_{11} = E_1/(1-v_{12}v_{21}) = S_{22}/(S_{11}S_{22} - S^2{}_{12})$$
$$Q_{22} = E_2/(1-v_{12}v_{21}) = S_{11}/(S_{11}S_{22} - S^2{}_{12})$$
$$Q_{12} = v_{12}E_2/(1-v_{12}v_{21}) = -S_{12}/(S_{11}S_{22} - S^2{}_{12})$$
$$Q_{66} = G_{12} = 1/S_{66}$$

- **Compliance = Engineering = Stiffness**

$$S_{11} = 1/E_1 = Q_{22}/(Q_{11}Q_{22} - Q^2{}_{12})$$
$$S_{22} = 1/E_2 = Q_{11}/(Q_{11}Q_{22} - Q^2{}_{12})$$
$$S_{12} = -v_{12}/E_1 = Q_{12}/(Q_{11}Q_{22} - Q^2{}_{12})$$
$$S_{66} = 1/G_{12} = 1/Q_{66}$$

Transformation to general x-y axes

"Generally orthotropic"

Generally orthotropic constitutive relation

- Starting with:

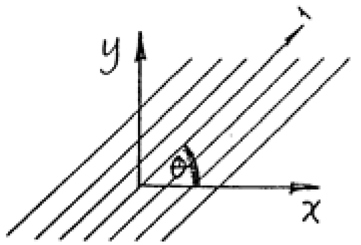
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

Where $[Q]$= Reduced stiffness matrix in 1-2 material axes

- Transform by trigonometric transformation to produce general structural axis relations at angle θ

i.e.:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$



where $[\overline{Q}]$ is the **"Transformed reduced stiffness matrix"**

– In general x-y plate axes

Calculated from $[\bar{Q}]=[T]^{-1}[Q][T]^{-T}$

where $[T]=\begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix}$ And $m=\cos\theta \quad n=\sin\theta$

i.e. simply geometric transformation

Generally orthotropic compliance relation

– Starting with

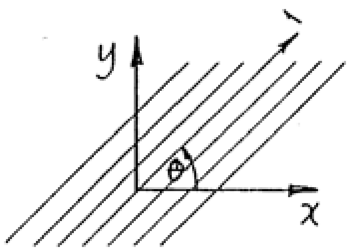
$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_{4imc}$$

where $[S]$= Reduced compliance matrix in 1-2 material axes

– Transform by trigonometric transformation to produce general structural axis relations at angle θ

i.e.:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{S}_{23} \\ \bar{S}_{31} & \bar{S}_{32} & \bar{S}_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{4imc}$$



where $[\bar{S}]$ is the **"Transformed reduced compliance matrix"**

– **In full:**

Transformed stiffness constants:

$$\overline{Q}_{11} = Q_{11}m^4 + 2(Q_{12} + Q_{66})n^2m^2 + Q_{22}n^4$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})n^2m^2 + Q_{12}(n^4 + m^4)$$

$$\overline{Q}_{22} = Q_{11}n^4 + 2(Q_{12} + 2Q_{66})n^2m^2 + Q_{22}m^4$$

$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})nm^3 + (Q_{12} - Q_{22} + 2Q_{66})n^3m$$

$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})n^3m + (Q_{12} - Q_{22} + 2Q_{66})nm^3$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})n^2m^2 + Q_{66}(n^4 + m^4)$$

Transformed compliance constants:

$$\overline{S}_{11} = S_{11}m^4 + (2S_{12} + S_{66})n^2m^2 + S_{22}n^4$$

$$\overline{S}_{12} = (S_{11} + S_{22} - S_{66})n^2m^2 + S_{12}(n^4 + m^4)$$

$$\overline{S}_{22} = S_{11}n^4 + (2S_{12} + S_{66})n^2m^2 + S_{22}m^4$$

$$\overline{S}_{16} = (2S_{11} - 2S_{12} - S_{66})nm^3 - (2S_{22} - 2S_{12} - S_{66})n^3m$$

$$\overline{S}_{26} = (2S_{11} - 2S_{12} - S_{66})n^3m - (2S_{22} - 2S_{12} - S_{66})nm^3$$

$$\overline{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})n^2m^2 + S_{66}(n^4 + m^4)$$

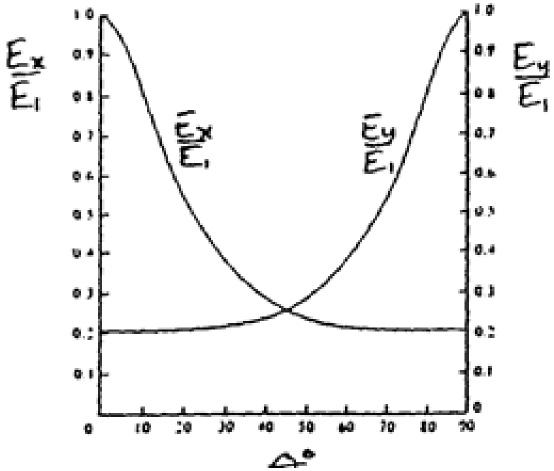
where $m = \cos\theta \quad n = \sin\theta$

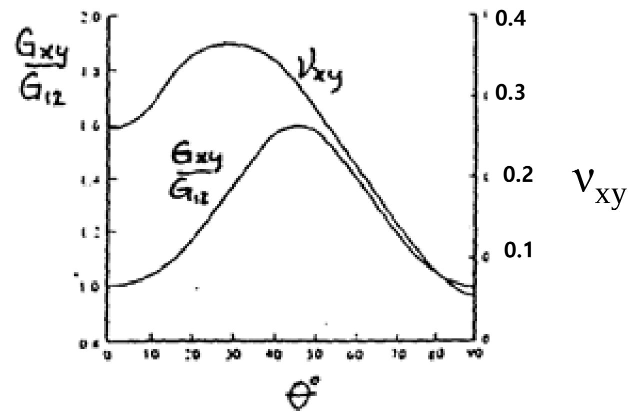
Transformed engineering constants

$$E_x = 1/\left[(1/E_1)m^4 + (1/G_{12} - 2\nu_{12}/E_1)n^2m^2 + (1/E_2)n^4\right]$$
$$E_y = 1/\left[(1/E_1)n^4 + (1/G_{12} - 2\nu_{12}/E_1)n^2m^2 + (1/E_2)m^4\right]$$
$$\nu_{xy} = E_x\left[(\nu_{12}/E_1)(n^4 + m^4) - (1/E_1 + 1/E_2 - 1/G_{12})n^2m^2\right]$$
$$\nu_{yx} = E_y\left[(\nu_{12}/E_1)(n^4 + m^4) - (1/E_1 + 1/E_2 - 1/G_{12})n^2m^2\right]$$
$$G_{xy} = 1/\left[(4/E_1 + 4/E_2 + 8\nu_{12}/E_1)\ n^2m^2 + (\frac{1}{G_{12}})(m^2 - n^2)^2 \quad)\right]$$
$$m_x = E_x\left[m^3n(1/G_{12} - 2\nu_{12}/E_1 - 2/E_1) - mn^3(1/G_{12} - 2\nu_{12}/E_1 - 2/E_2)\right]$$
$$m_y = E_y\left[mn^3(1/G_{12} - 2\nu_{12}/E_1 - 2/E_1) - m^3n(1/G_{12} - 2\nu_{12}/E_1 - 2/E_2)\right]$$

m_x, m_y : **"shear coupling coefficients";** *i.e. similar to poisson ratio deformation*

$$m_x = -\gamma_{xy}/\varepsilon_x \qquad \nu_{xy} = -\varepsilon_y/\varepsilon_x$$
$$m_y = -\gamma_{xy}/\varepsilon_y \qquad \nu_{yx} = -\varepsilon_x/\varepsilon_y$$





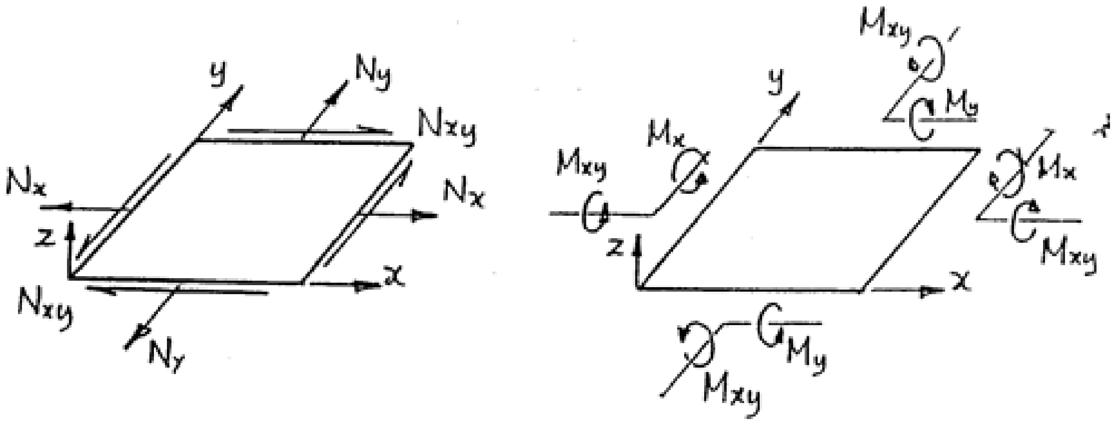
Laminate Stress-Strain Analysis

Mechanical Loading

Classical Lamination Theory

- Based on the constitutive relations for a lamina (as outlined above), i.e., linear elastic, generally orthotropic homogeneous material, assuming small deformation theory, i.e. plane sections remain plane

1. Plate under extensional and flexural loading:



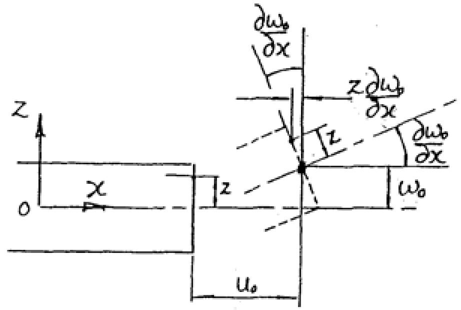
where:

N_x, N_y, N_{xy} = “loading intensities” i.e., forces per unit width of laminate

M_x, M_y, M_{xy} = “moment intensities” i.e., moments per unit width of laminate

*Note: M_x is defined as: “the moment which causes direct stresses in the x direction etc.”

2. Resulting deformation



Laminate mid plate strains, Laminate mid plate curvatures
Here, defined as:

$$\varepsilon^0{}_x = \frac{\partial u_0}{\partial x} \qquad \kappa_x = \frac{-\partial^2 w_0}{\partial x^2}$$

$$\varepsilon^0{}_y = \frac{\partial v_0}{\partial y} \qquad \kappa_y = \frac{-\partial^2 w_0}{\partial y^2}$$

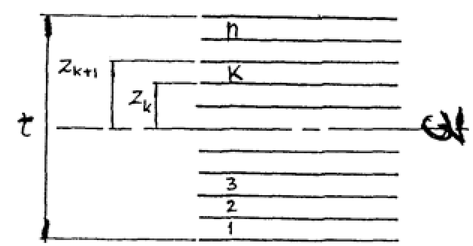
$$\gamma^0{}_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \qquad \kappa_{xy} = \frac{-2\partial^2 w_0}{\partial x \partial y}$$

Total layer strains:

$$\varepsilon_{x_k} = \varepsilon^0{}_x + z\kappa_x$$

$$\varepsilon_{y_k} = \varepsilon^0{}_y + z\kappa_y$$

$$\gamma_{xy_k} = \gamma^0{}_{xy} + z\kappa_{xy}$$



Reference!

1.2 Strain and Displacement

Assumption:

1) Plies are bonded perfectly.
2) Bonding thickness is neglected.
3) Ply thickness is very thin, so it can be considered as plane stress.
4) Laminate thicknesswise strain distribution is linear.

❖ In order to satisfy the assumption above, after deformation z-directional shear stress and normal stress must be neglected.

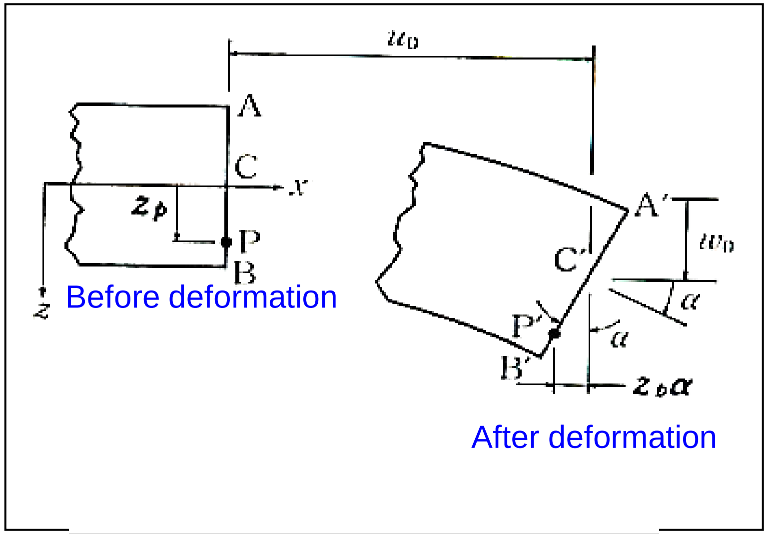


Fig. 2 x-z plane geometrical configuration

x, y, z axis strains : $\varepsilon_x, \varepsilon_y, \varepsilon_z$

x, y, z axis displacements: u, v, w

- As shown in Fig. 2 when point P moves to P' due to deformation, x directional displacement between C and C' is defined as u_0 , and let distance between mid plane axis x and P be z_p . Then x directional displacement of P is;

$$u_p = u_0 - z_p \sin\alpha \tag{4.5}$$

If α is very small angle, $\sin\alpha \approx \alpha$

$$u_p = u_0 - z_p\alpha \tag{4.6}$$

$$\alpha = \frac{\partial w_0}{\partial x}$$

- x directional displacement of z axis arbitrary point is;

$$u = u_0 - z\alpha$$

$$= u_0 - z\frac{\partial w_0}{\partial x} \tag{4.7}$$

- Similarly y directional displacement is;

$$v = v_0 - z\frac{\partial w_0}{\partial y} \tag{4.8}$$

assume $\varepsilon_z = 0, \gamma_{xz} = 0, \gamma_{yz} = 0$ (plane strain)

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4.9}$$

(Elastic strains in x - y plane)

- Substitute (4.7), (4.8) for (4.9),

ε_x becomes,

$$= \frac{\partial u_0}{\partial x} - \frac{\partial}{\partial x}\left(z \frac{\partial w_0}{\partial x}\right)$$

$$\varepsilon_y = \frac{\partial}{\partial y}\left(v_0 - z \frac{\partial w_0}{vy}\right)$$

$$= \frac{\partial v_0}{\partial y} - \frac{\partial}{\partial y}\left(z \frac{\partial w_0}{\partial y}\right)$$

$$\gamma_{xy} = \frac{\partial}{\partial y}\left(u_0 - z \frac{\partial w_0}{\partial x}\right) + \frac{\partial}{\partial x}\left(v_0 - z \frac{\partial w_0}{\partial y}\right) \qquad (4.10)$$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ (x, y are independent from z)

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w_0}{\partial x \partial y} \qquad (4.11)$$

$\frac{\partial u_0}{\partial x}, \frac{\partial v_0}{\partial y}, \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}$: strains at mid plane →express by $\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0$

$$\frac{\partial u_0}{\partial x} = \varepsilon_x{}^0$$

$$\frac{\partial v_0}{\partial y} = \varepsilon_y{}^0$$

$$\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = \gamma_{xy}{}^0 \qquad (4.12)$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x{}^0 \\ \varepsilon_y{}^0 \\ \gamma_{xy}{}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (4.13)$$

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \tag{4.14}$$

Where κ_x : x-directional curvature of neutral plane
κ_y : y-directional curvature of neutral plane
κ_{xy} : torsional curvature of neutral plane

i.e.:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \quad ①$$

Where

$z = z_k$ *for value at btm of layer thickness*

$z = z_{k+1}$ *for value at top of layer thickness*

$z = \dfrac{z_k + z_{k+1}}{2}$ *for value at middle of layer thickness*

Membrane strain

Bending strain

3. Force and Moment resultants

For equilibrium, applied forces and moments must be balanced by internal stresses

Equilibrium of forces:

$$\left.\begin{aligned} N_x &= \int_{-t/2}^{+t/2} \sigma_x dz \\ N_y &= \int_{-t/2}^{+t/2} \sigma_y dz \\ N_z &= \int_{-t/2}^{+t/2} \tau_{xy} dz \end{aligned}\right\} \qquad \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{+t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \sum \int_{Zk}^{Zk+1} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz \quad ②$$

Equilibrium of Moments:

$$\left.\begin{aligned} M_x &= \int_{-t/2}^{+t/2} z\sigma_x dz \\ M_y &= \int_{-t/2}^{+t/2} z\sigma_y dz \\ M_{xy} &= \int_{-t/2}^{+t/2} z\tau_{xy} dz \end{aligned}\right\} \quad \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{+t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} zdz = \sum \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k zdz \qquad ③$$

where summation $\sum$ applies from layer k=1 to n

4. Layer stresses

From the previous lamina analysis the x-y stress in layer k is given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \qquad ④$$

where $[\overline{Q}]$ is the reduced transformed stiffness matrix in the x-y axes

Substitute ① in ④ gives:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \end{Bmatrix}_k + \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad ⑤$$

Where $z = z_k$ *for value at btm of layer thicknes*
$z = z_{k+1}$ *for value at top of layer thickness*
$z = \frac{z_k + z_{k+1}}{2}$ *for value at middle of layer thick*

5. Force and Moment results

Equilibrium of force :

Substitute ⑤into ②

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum \left[\overline{Q}\right]_K \int_{Z_k}^{Z_{z+1}} dz \begin{Bmatrix} \varepsilon^{\circ}{}_x \\ \varepsilon^{\circ}{}_y \\ \gamma^{\circ}{}_{xy} \end{Bmatrix}_K + \sum \left[\overline{Q}\right]_k \int_{Z_K}^{Z_{k+1}} zdz \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}_k$$

$$= \sum \left[\overline{Q}\right]_k (Z_{k+1} - Z_k) \begin{Bmatrix} \varepsilon^{\circ}{}_x \\ \varepsilon^{\circ}{}_y \\ \gamma^{\circ}{}_{xy} \end{Bmatrix}_K + \frac{1}{2} \sum \left[\overline{Q}\right]_k (z^2{}_{k+1} - z^2{}_k) \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

[A] [B]

Equilibrium of moments :

Substitute ⑤into ③

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum \left[\overline{Q}\right]_k \int_{Z_k}^{Z_{k+1}} zdz \begin{Bmatrix} \varepsilon^{\circ}{}_x \\ \varepsilon^{\circ}{}_y \\ \gamma^{\circ}{}_{xy} \end{Bmatrix}_K + \sum \left[\overline{Q}\right]_k \int_{Z_k}^{Z_{k+1}} z^2 dz \begin{Bmatrix} \varepsilon^{\circ}{}_x \\ \varepsilon^{\circ}{}_y \\ \gamma^{\circ}{}_{xy} \end{Bmatrix}_K$$

$$= \frac{1}{2} \sum \left[\overline{Q}\right]_k (Z^2{}_{k+1} - Z^2{}_k) \begin{Bmatrix} \varepsilon^{\circ}{}_x \\ \varepsilon^{\circ}{}_y \\ \gamma^{\circ}{}_{xy} \end{Bmatrix}_K + \frac{1}{3} \sum \left[\overline{Q}\right]_k (z^3{}_{k+1} - z^3{}_k) \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}_K$$

[B] [D]

6. Laminate Constitutive Relation

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^\circ{}_x \\ \varepsilon^\circ{}_y \\ \gamma^\circ{}_{xy} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^\circ{}_x \\ \varepsilon^\circ{}_y \\ \gamma^\circ{}_{xy} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

i.e. :

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^\circ \\ k \end{Bmatrix}$$

wrt x, y plate axes

" Laminate stiffnesses"

where :

$$A = \sum \left[\overline{Q}\right]_k (Z_{k+1} - Z_k)$$

$$B = \frac{1}{2} \sum \left[\overline{Q}\right]_k (Z^2{}_{k+1} - Z^2{}_k)$$

$$D = \frac{1}{3} \sum \left[\overline{Q}\right]_k (Z^3{}_{k+1} - Z^3{}_k)$$

*Note :

Symmetric laminates: [B] = 0 → No extension-bending coupling

Extensional orthotropy: $A_{16} = A_{26} = 0$ → No direct-shear coupling

Flexural orthotropy : $D_{16} = D_{26} = 0$ → No bend-twist coupling

Special orthotropy : → Extensional orthotropy + Flexural orthotorpy

7. Laminate Compliance Relation
Solving for laminate strains

$$\begin{Bmatrix} \varepsilon^{\circ} \\ k \end{Bmatrix} = \begin{bmatrix} A^{1} & B^{1} \\ C^{1} & D^{1} \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}$$ wrt x ,y

Inverted stiffness matrix, i.e “Compliance matrix”

a) For symmetric laminates [B]=0

A and D matrixes can be inverted independently using standard matrix inversions :

$$\left[A^{1}\right] = [A]^{-1} = [a]$$

$$\left[D^{1}\right] = [\mathrm{D}]^{-1} = [d]$$

Then :

$$\varepsilon^{\circ} = [a]N$$

$$k = [d]N$$

i.e. :

$$\begin{Bmatrix} \varepsilon^{\circ} \\ k \end{Bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}$$ wrt x, y

b) For asymmetric laminate $[B] \neq 0$

i.e. with extension-bending coupling

. Coupled inverted matrixes A^1, B^1, C^1, D^1 must be derived

$$\begin{Bmatrix} \varepsilon \\ k \end{Bmatrix} = \begin{bmatrix} A^1 & B^1 \\ C^1 & D^1 \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}$$ wrt x, y

where

$$[A^1] = [A*] - [B*][D*^{-1}][C*]$$

$$[B^1] = [B*][D*^{-1}]$$

$$[C^1] = -[D*^{-1}][C*]$$

$$[D^1] = [D*^{-1}]$$

and

$$[A^*] = [A^{-1}]$$

$$[B^*] = -[A^{-1}][B]$$

$$[C^*] = [B][A^{-1}]$$

$$[D^*] = [D] - [B][A^{-1}][B]$$

Giving :

$$\begin{Bmatrix} \varepsilon^{\circ} \\ k \end{Bmatrix} = \begin{bmatrix} A^1 & B^1 \\ C^1 & D^1 \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix}$$ wrt x, y Usually $C^1 = B^1$

Standard matrix inversions:

Conversion of a full 3x3 matrix R_{ij} into its inverse r_{ij}

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \rightarrow \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

where,

$$r_{11} = (R_{22}R_{33} - R^2{}_{23}) / RR$$

$$r_{22} = (R_{11}R_{33} - R^2{}_{13}) / RR$$

$$r_{33} = (R_{11}R_{22} - R^2{}_{12}) / RR$$

$$r_{12} = (R_{13}R_{23} - R_{12}R_{33}) / RR$$

$$r_{13} = (R_{12}R_{23} - R_{22}R_{13}) / RR$$

$$r_{23} = (R_{12}R_{13} - R_{11}R_{23}) / RR$$

and

$$RR = R_{11}R_{22}R_{33} + 2R_{12}R_{23}R_{13} - R_{22}R^2{}_{13} - R_{33}R^2{}_{12} - R_{11}R^2{}_{23}$$

Conversion of a reduced 3x3 matrix R_{ij} into its inverse r_{ij}

$$\begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{12} & R_{22} & 0 \\ 0 & 0 & R_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{12} & r_{22} & 0 \\ 0 & 0 & r_{33} \end{bmatrix}$$

Where

$$r_{11} = R_{22} / RR$$

$$r_{22} = R_{11} / RR$$

$$r_{33} = 1 / RR$$

$$r_{12} = -R_{12} / RR$$

and

$$RR = R_{11} R_{22} + R^2{}_{12}$$

8. Equivalent Engineering Elastic Constants

For symmetric laminates, [B]=0

Using to work in familiar E, G, v terms in initial laminate design

★ **a) Membrane equivalent elastic constants**

Derived by considering average laminate stresses:

$$\sigma_x = N_x / t, \sigma_y = N_y / t, \tau_{xy} = N_{xy} / t$$

$$E_x = 1/(ta_{11})$$

$$E_y = 1/(ta_{22})$$

$$G_{xy} = 1/(ta_{66})$$

$$v_{xy} = -a_{12} / a_{11}$$

$$v_{yx} = -a_{12} / a_{22}$$

$$m_x = -a_{13} / a_{11}$$

$$m_y = -a_{23} / a_{22}$$

b) Bending equivalent elastic constants

Derived by considering general theory of bending: M/κ=EI

Where $I = t^3/12$ for laminate thickness t and unit width

$$E_x = 12/(t^3 d_{11})$$

$$E_y = 12/(t^3 d_{22})$$

$$G_{xy} = 12/(t^3 d_{66})$$

$$v_{xy} = -d_{12}/d_{11}$$

$$v_{yx} = -d_{12}/d_{22}$$

$$m_x = -d_{13}/d_{11}$$

$$m_y = -d_{23}/d_{22}$$

$$\boxed{\begin{aligned}\kappa &= dM = \frac{M}{EI} \\ E &= \frac{1}{dI} = \frac{12}{dt^3}\end{aligned}}$$

where a_{ij} and d_{ij} are compliance parameters from the inverse

laminate stiffness matrixes, $[A]^{-1}$ and $[D]^{-1}$ and t=laminate thickness

9. Layer Stresses and Strains

Once the laminate stiffness and inverse stiffness (compliance) matrixes have been found the layer strains and stresses can be calculated in x-y co-ordinates for use with chosen failure criteria.

a. Calculate mid plane deformations

$$\begin{Bmatrix} \varepsilon^\circ_x \\ \varepsilon^\circ_y \\ \gamma^\circ_{xy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{12} & a_{22} & a_{26} & b_{12} & b_{22} & b_{26} \\ a_{16} & a_{26} & a_{66} & b_{16} & b_{26} & b_{66} \\ b_{11} & b_{12} & b_{16} & d_{11} & d_{12} & d_{16} \\ b_{12} & b_{22} & b_{26} & b_{12} & b_{22} & b_{26} \\ b_{16} & b_{26} & b_{66} & b_{16} & b_{26} & b_{66} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

All b terms go to zero for symmetric laminates

b. Calculate layer total strains. e.g. for layer k:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_k = \begin{Bmatrix} \varepsilon^\circ_x + z_k k_x \\ \varepsilon^\circ_y + z_k k_y \\ \gamma^\circ_{xy} + z_k k_{xy} \end{Bmatrix}$$

Where $z=z_k$ *for value at btm of layer thickness*

$z=z_{k+1}$ *for value at top of layer thickness*

$z=\frac{z_k+z_{k+1}}{2}$ *for value at middle of layer thickness*

c. Transform to 1-2 material axes

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}_k = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_K$$

d. Calculate layer total 1-2 stresses

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}_K$$

Laminate Stress-Strain Analysis

Hygrothermal Loading

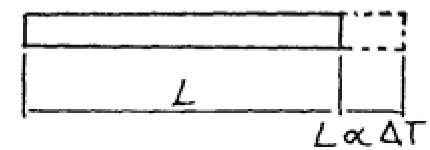
- Hygrothermal → Thermal and hygroscopic effects, i.e. temperature and moisture

Thermal Analysis

Assuming:

- Linear elastic response at elevated temperature
  i.e. "Thermo-elastic" analysis
- Constant thermal expansion coefficients, α
- Constant temperature distribution through thickness, ΔT

Single layer plate, one dimension

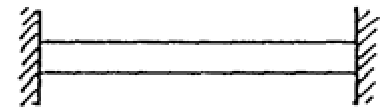


- Expansion coefficient α
- Temperature change ΔT assumed constant through thickness

Unrestrained

- Thermal strain $\varepsilon = \alpha\,\Delta T = \varepsilon^T$ "Free thermal strain"
- Thermal stress $\sigma = 0$ i.e. unrestrained

Fully restrained



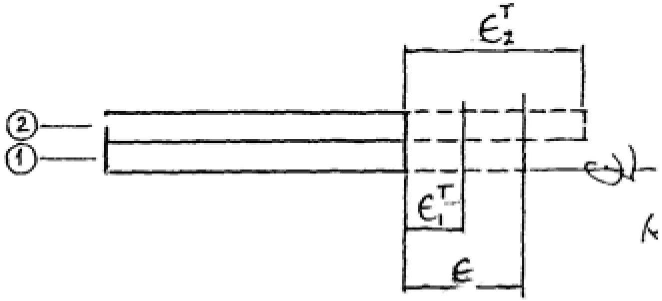
- Thermal strain $\varepsilon = \alpha\Delta T = \varepsilon^R$ "Residual strain"
- Thermal stress $\sigma = E\varepsilon^R = \sigma^R$ "Residual stress"

Laminated plate, one dimension

- Layer 1, expansion coefficient α_1
- Layer 2, expansion coefficient α_2
- Temperature change ΔT assumed constant for all layers

Free thermal strains : Layer 1 : $\varepsilon_1{}^T = \alpha_1 \Delta T$

Layer 2 : $\varepsilon_2{}^T = \alpha_2 \Delta T$



Residual strains : Layer 1 : $\varepsilon_1{}^R = \varepsilon - \varepsilon_1{}^T \Delta T$

Layer 2 : $\varepsilon_2{}^R = \varepsilon - \varepsilon_2{}^T \Delta T$

where ε =Laminate “common strain”

Residual stresses Layer 1 : $\sigma_1{}^R = E\varepsilon_1{}^R$

Layer 2 : $\sigma_2{}^R = E\varepsilon_2{}^R$

i.e.: Layer residual strain = laminate common strain less layer free thermal strain

$\varepsilon_k{}^R = \varepsilon - \varepsilon_K{}^T$ and $\sigma_k{}^R = E\varepsilon_K{}^R$

Orthotropic laminated plate, two dimensions

Layer expansion coefficients α_1 ,α_2 ,α_{12} in 1-2 material directions, temperature charge ΔT assumed constant for all layers

1. Layer free thermal strains in 1-2 material axis directions :

$$\begin{Bmatrix} \varepsilon_1{}^T \\ \varepsilon_2{}^T \\ \gamma_{12}{}^T \end{Bmatrix}_K = \begin{Bmatrix} \alpha_1 \Delta T \\ \alpha_2 \Delta T \\ \alpha_{12} \Delta T \end{Bmatrix}_K$$

2. Layer free thermal strains in x-y plate axis directions :

$$\begin{Bmatrix} \varepsilon_x{}^T \\ \varepsilon_y{}^T \\ \gamma_{xy}{}^T \end{Bmatrix}_K = [T] \begin{Bmatrix} \varepsilon_1{}^T \\ \varepsilon_2{}^T \\ \gamma_{12}{}^T \end{Bmatrix}_K$$

3. Layer “free thermal stresses” in x-y plate axis directions :

$$\begin{Bmatrix} \sigma_x{}^T \\ \sigma_y{}^T \\ \tau_{xy}{}^T \end{Bmatrix}_K = [\bar{Q}]_K \begin{Bmatrix} \varepsilon_x{}^T \\ \varepsilon_y{}^T \\ \gamma_{xy}{}^T \end{Bmatrix}_K$$

4. “Thermal forces and moments”

$$\begin{Bmatrix} N_x{}^T \\ N_y{}^T \\ N_{xy}{}^T \end{Bmatrix} = \sum \int_{Z_K}^{Z_{K+1}} \begin{Bmatrix} \sigma_x{}^T \\ \sigma_y{}^T \\ \tau_{xy}{}^T \end{Bmatrix}_K dz \qquad \begin{Bmatrix} M_x{}^T \\ M_y{}^T \\ M_{xy}{}^T \end{Bmatrix} = \sum \int_{Z_K}^{Z_{K+1}} \begin{Bmatrix} \sigma_x{}^T \\ \sigma_y{}^T \\ \tau_{xy}{}^T \end{Bmatrix}_K zdz$$

where summation $\sum$ applies from layer k=1 to n

5. Laminate common strains

- obtained by laminate analysis based on free thermal forces and moments

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_K = \begin{Bmatrix} \varepsilon^{\circ}{}_x \\ \varepsilon^{\circ}{}_y \\ \gamma^{\circ}{}_{xy} \end{Bmatrix}_k + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}_k \quad ①$$

Where, $z = z_k$ for value at btm of layer thickness

$z = z_{k+1}$ for value at top of layer thickness

$z = \dfrac{z_k + z_{z+1}}{2}$ for value at middle of layer thickness

6. Layer residual strains

$$\begin{Bmatrix} \varepsilon^R{}_x \\ \varepsilon^R{}_y \\ \gamma^R{}_{xy} \end{Bmatrix}_K = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_K - \begin{Bmatrix} \alpha_x \Delta T \\ \alpha_y \Delta T \\ \alpha_{xy} \Delta T \end{Bmatrix}_K$$

Layer residual strains	Layer common strains	Layer free thermal strains

7. Layer residual stresses

$$\begin{Bmatrix} \sigma^R{}_x \\ \sigma^R{}_y \\ \tau^R{}_{xy} \end{Bmatrix}_K = [\overline{Q}]_K \begin{Bmatrix} \varepsilon^R{}_x \\ \varepsilon^R{}_y \\ \gamma^R{}_{xy} \end{Bmatrix}_K$$

Hygroscopic analysis

Similar to thermal analysis

Using:

. Moisture coefficients of expansion β_1 , β_2 (analogous to α_1 , α_2)

. Moisture content m (analogous to ΔT)

. But note:

- Moisture contents m variation through laminate (ΔT assumed constant) $\rightarrow$ cannot take m outside integrals

Then calculate equivalent free hygroscopic loads and moments, etc.

$$\begin{Bmatrix} N^H{}_x \\ N^H{}_y \\ N^H{}_{xy} \end{Bmatrix}_k \qquad \begin{Bmatrix} M^H{}_x \\ M^H{}_y \\ M^H{}_{xy} \end{Bmatrix}_k$$

As for thermal analysis, etc.

Moisture diffusion

Ficke's law of diffusion;

$$\frac{\partial M}{\partial t} = \frac{D_x d}{dt}\left(\frac{dc}{dx}\right)$$

where $\frac{dc}{dx}$ = water concentration gradient

D_x =Diffusion coefficient in x direction

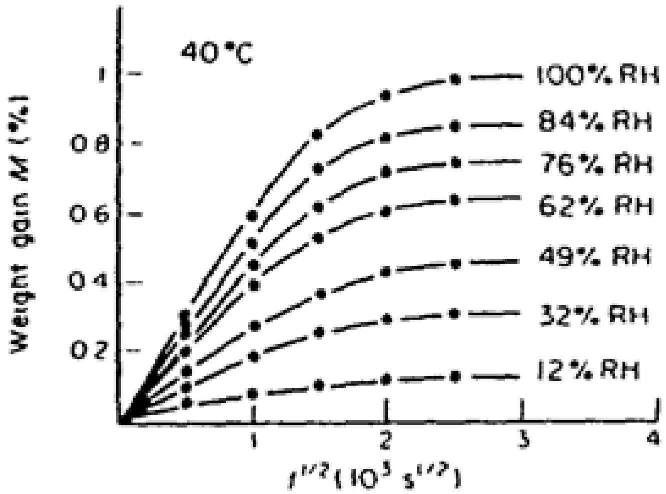
For an infinitely large plate

$$\frac{M_t}{M_\infty} = \frac{4}{h}\sqrt{\frac{D_x t}{\pi}}$$

where

M_t =Mass of water absorbed across unit surface area in time t

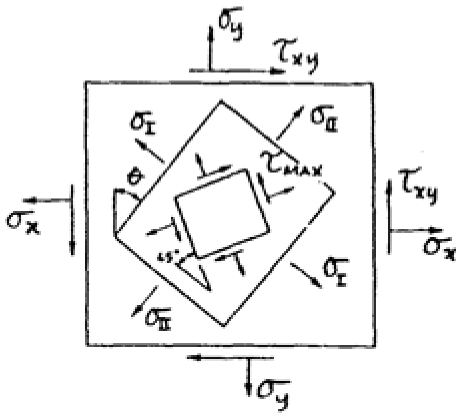
M_∞ =Mass of water absorbed at saturation



Lamina Strength Analysis

- First consider as **Isotropic Materials**
  - . Same strength all directions
- 3 failure stresses or strains defined at yield or ultimate failure condition

Failure Stress	Failure Strain	Mode
$\sigma_{T}{}^{*}$	$\varepsilon^{*}{}_{T}$	Tension
$\sigma_{c}{}^{*}$	$\varepsilon^{*}{}_{c}$	Compression
τ^{*}	γ^{*}	Shear



*Note strength values are the same in all directions

Common Failure Criteria for Isotropic Materials

- Failure is governed by principal stresses or strains

$$\sigma_1 = \sigma_{MAX} = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\sigma_{xy}{}^2}$$

$$\sigma_2 = \sigma_{MIN} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\sigma_{xy}{}^2}$$

$$\tau_{MAX} = \frac{\sigma_{MAX} + \sigma_{MIN}}{2}$$

$$\theta = \frac{1}{2}\tan^{-1}\{2\tau_{xy} / (\sigma_x - \sigma_y)\}$$

Giving the following failure criteria:

1. Maximum principal stress

Failure occurs when : $\sigma_{MAX} > \sigma_T *$

$\sigma_{MIN} > \sigma_C *$

2. Maximum principal strain(Rankine)

Failure occurs when : $\varepsilon_{MAX} > \varepsilon_T *$

$\varepsilon_{MIN} > \varepsilon_C *$

3. Maximum shear (Tresca)

Failure occurs when: $\tau_{MAX} > \tau^*$

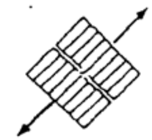
4. Maximum shear strain energy (von Mises)

Failure occurs when $\sqrt{\sigma_{MAX}{}^2 + \sigma_{MIN}{}^2 - \sigma_{MAX}\sigma_{MIN}} > \sigma_T *$

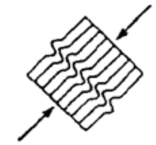
Orthotropic Materials

- Failure is governed by different strength in different directions

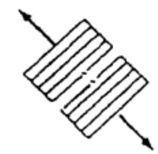
Failure Stress	**Failure Strain**	**Mode “ Intra-lamina”**
σ^*_{1T}	ε^*_{1T}	Longitudinal tension “1-1/T”
σ^*_{1C}	ε^*_{1C}	Longitudinal compression “1-1/C”
σ^*_{2T}	ε^*_{2T}	Transverse tension “2-2/T”
σ^*_{2C}	ε^*_{2C}	Transverse compression “2-2/C”
τ^*_{12}	γ^*_{12}	In-place shear “1-2”



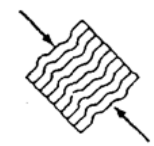
(Longitudinal tension)



(Longitudinal compression)



(Transverse tension)



(Transvese compression)

(In plane shear)

Intra –lamina failure criteria for orthotropic materials

– Failure is governed by directional material strengths rather than principal stresses or strains

1. Maximum stress failure criteria

Failure occurs when: $\sigma_{1T} > \sigma^*_{1T}$

$$\sigma_{1C} > \sigma^*_{1C}$$
$$\sigma_{2T} > \sigma^*_{2T}$$
$$\sigma_{2C} > \sigma^*_{2C}$$
$$\tau_{12} > \tau^*_{12}$$

2. Maximum strain failure criteria

Failure occurs when: $\varepsilon_{1T} > \varepsilon^*_{1T}$

$$\varepsilon_{1C} > \varepsilon^*_{1C}$$
$$\varepsilon_{2T} > \varepsilon^*_{2T}$$
$$\varepsilon_{2C} > \varepsilon^*_{2C}$$
$$\gamma_{12} > \gamma^*_{12}$$

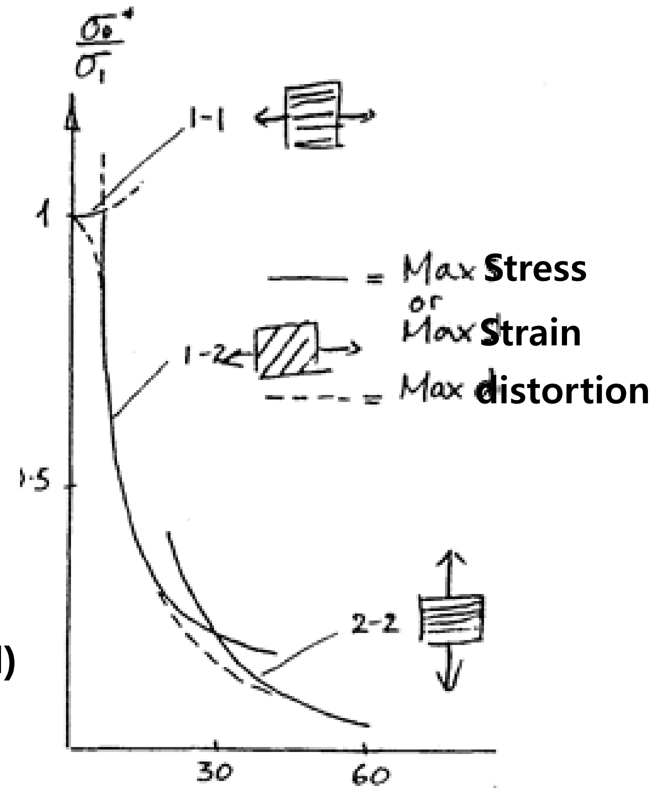
3. Maximum "distortional energy" failure criteria (Tsai-Hill)

Failure occurs when:

$$(\sigma_{1T}/\sigma^*_{1T})^2 + (\sigma_{2T}/\sigma^*_{2T})^2 + (\tau_{12}/\tau^*_{12})^2 + (\sigma_{1T}/\sigma^*_{1T})(\sigma_{2T}/\sigma^*_{2T}) > 1$$

i.e. **failure index sum >1**

Also : **Tsai-Wu , Hoffman** ,etc.



Failure envelopes in “2D stress or strain space”

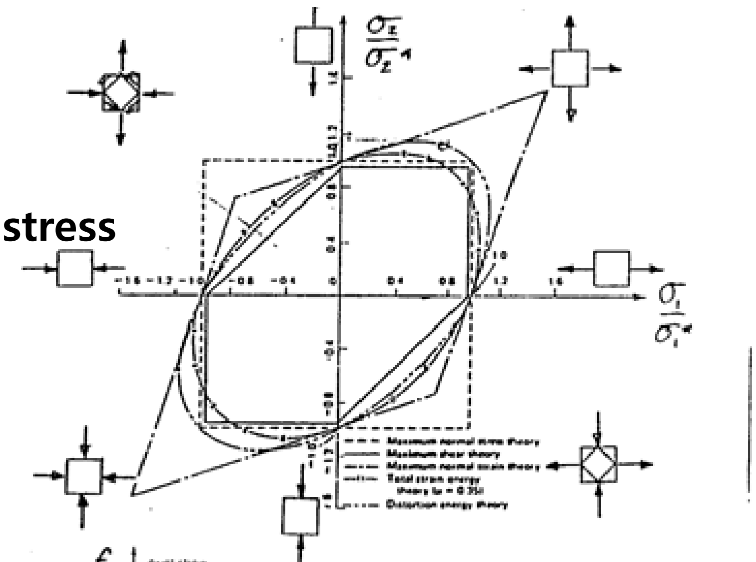
- Representing the combinations of stress that cause failure

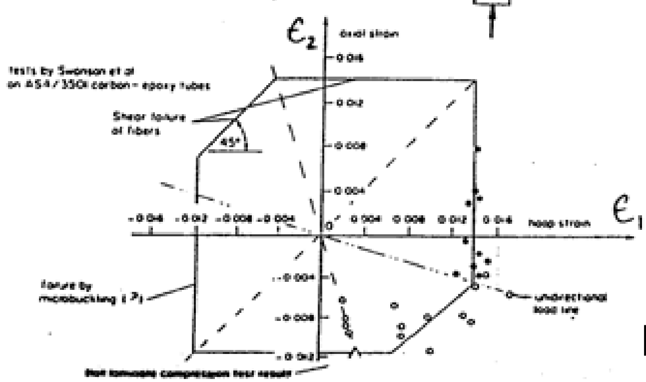
Max stress Non-interactive, linear

Max strain Non-interactive, linear

Max “distortional energy” Interactive, quadratic

Failure envelopes in 2D stress

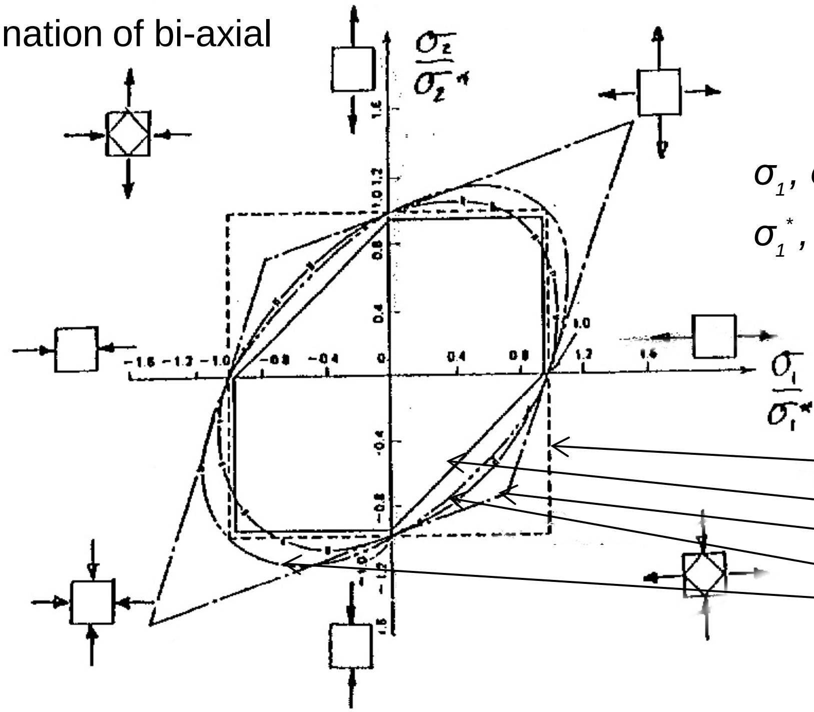




Failure envelopes in 2D strain

Failure Envelopes

- Locus of combination of bi-axial stress

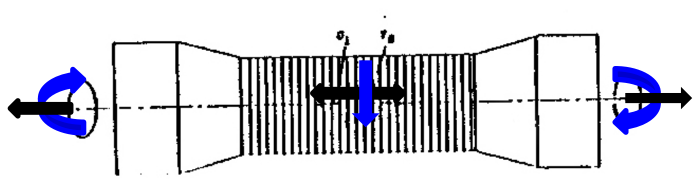


σ_1, σ_2 : *applied stresses in 1,2 directions*

σ_1^*, σ_2^* : *strengths in 1,2 directions*

Hoop specimen multi-axial test

- To validate failure criteria

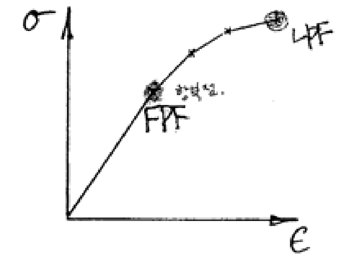


Laminate strength analysis

Laminate failure definition

Progressive failure

- **First ply failure**
  - . Usually matrix dominated
  - i.e. 2-2 or 1-2 modes
  - . Non-catastrophic
  - . Equivalent to yield failure condition
- **Last ply failure**
  - . Fibre dominated
  - i.e. 1-1 mode

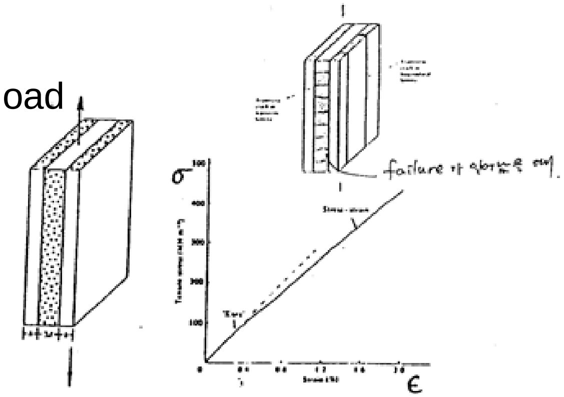


e.g. **Cross-ply Laminate**

FPF: transverse ply failure

LPF: load direction ply failure

At final failure: multiple transverse cracking, exponential load transfer



★ Laminate failure analysis procedure

Laminate analysis

↓

Stresses and strains in each layer in 1-2 material axes

↓

Check failure criteria for each mode of each layer

↓

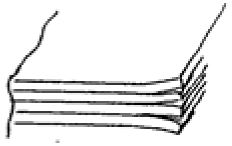
Assume failure for ply with lowest RF at RF x applied loads
If layer has failed in matrix dominated mode, i.e. 2-2 or 1-2
Then degrade (reduce) the layer properties and repeat analysis

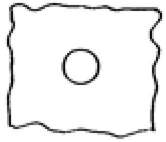
↓

If layer has failed in fibre dominated mode, i.e. 1-1
Then assume final failure and stop analysis

Inter-lamina modes of failure

- "Inter-lamina" failure, i.e. delamination is associated with
  - . Through thickness stresses at laminate edges
  - . Impact damage
  - . Hole and notch stress concentrations
- Delamination is not accounted for in classical laminate analysis and is usually covered by reduced allowables and more detailed analysis





Empirical Laminate Analysis

Approximate methods of laminate analysis are needed to compliment full computer laminate analysis for initial design and checks.

The following empirical laminate analysis methods are considered:

Netting (0%) Rule
10% Rule
Carpet plots

Rules of mixtures

Each is considered in turn below

Laminate 0% Rule of Mixtures

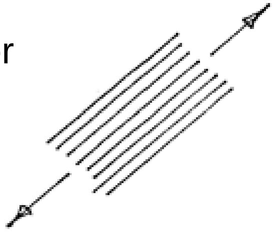
(**“Netting Rule”**)

Application:

- To provide a preliminary initial sizing of QI laminates for strength and stiffness

Assumptions:

- Loads are carried only in the fibre directions
- No contribution from off-axis layers



Data :

- Longitudinal strength and stiffness of basic layer

Limitations :

- Fibre failure criteria only
- Extensional “membrane” loading only
- Applicable to fibre dominated layups only
- Not applicable for prediction of matrix dominated properties

Method:

- The fibre direction properties are factored by a scaling factor and their thickness ratio which provides an estimate of the layer contribution according to the orientation and loading system. For the netting analysis method the scaling contribution factor is zero for off-axis plies

0%(Netting) Rule for laminate stiffness or strengths

Loading	Stiffness Strength	%ply contribution factor 0°	±45	90°	Apply to
	E_x	1.0	0	0	E_1
	E_y	0	0	1.0	E_1
	G_{xy}	No prediction of off-axis stiffness			
Uni-axial longitudinal	σ_x	1.0	0	0	σ_1 *
Uni-axial Transverse	σ_y	0	0	1.0	σ_1 *
Bi-axial Equal/opposite sign i.e. pure shear	τ_{xy}	0	1.0	0	σ_1 *

$\times$ RoM ply thickness fraction

$\frac{t_{0^\circ}}{t}$ $\frac{t_{45^\circ}}{t}$ $\frac{t_{90^\circ}}{t}$

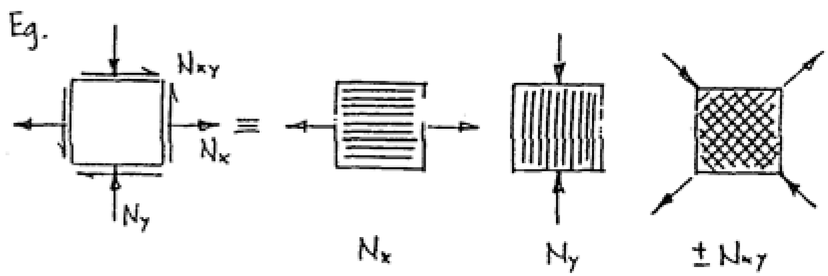
Note $t_{45^\circ} = t_{\pm 45^\circ} / 2$

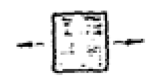
Example Application

Laminate initial sizing using netting analysis

Given plate loading N_x, N_y, N_{xy} and sizing for strength:

- Consider N_x, N_y, N_{xy} loading separately
- Select layer orientations aligned with loading
- Initially account for thickness of aligned layers only

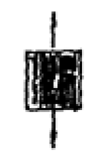




1. **For direct loading intensity N_x**

   Design for $\sigma_x = \frac{N_x}{t} \leq \sigma_1 * \frac{t_{0^\circ}}{t} \rightarrow t_{0^\circ} \geq \frac{N_x}{\sigma_1 *}$ for $t = t_{0^\circ}$ initially

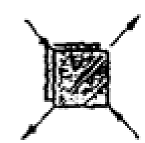
   i.e. required thickness of 0° layers



2. **For direct loading intensity N_y**

   Design for $\sigma_y = \frac{N_y}{t} \leq \sigma_1 * \frac{t_{90^\circ}}{t} \rightarrow t_{90^\circ} \geq \frac{N_y}{\sigma_1 *}$ for $t = t_{90^\circ}$ initially

   i.e. required thickness of 90° layers



3. For shear loading intensity N_{xy}

Design for $\sigma_{xy} = \frac{N_{xy}}{t} \le \sigma_1 * \frac{t_{45^\circ}}{t} \rightarrow t_{45^\circ} \ge \frac{N_{xy}}{\sigma_1 *}$ for $t = t_{45^\circ}$ initially

i.e. required thickness of 45° layers

*Note $t_{45^\circ} = t_{\pm 45^\circ} / 2$

4. Number of required layers at each angle θ of ply thickness t_p can then be calculated: i.e. $n_\theta = t_\theta / t_p$

Laminate 10% Rule of Mixtures

– Proposed by Hart-Smith

Application:

– To provide estimation of strengths and stiffness for QI laminates

Assumptions :

– Generally, off-axis layers contribute 10% of their strength and stiffness in the direction of loading.

Data :

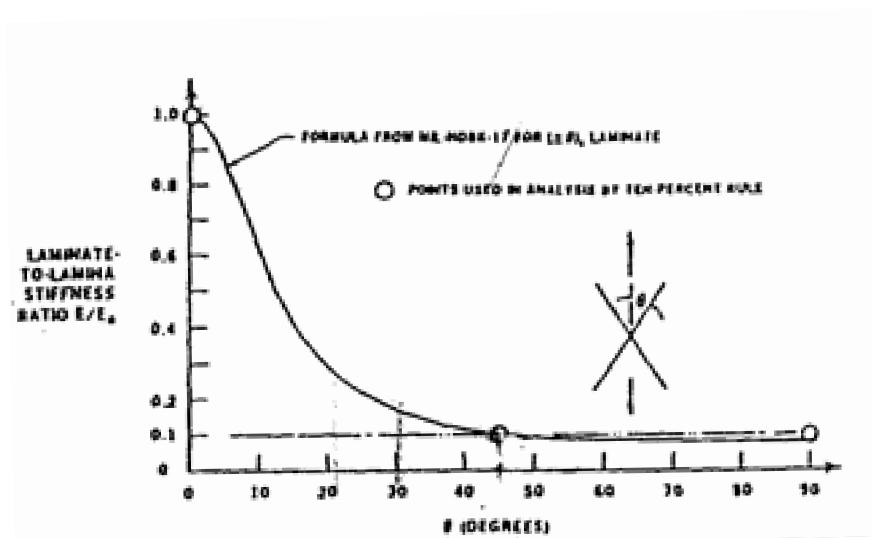
– Longitudinal tension and compression strength and stiffness

Limitations :

- 0,90,±45° fibre dominated layups only
- Extensional (membrane) loading only
- final ply failure (fibre failure) significant errors for matrix dominated properties.

Method:

- The fibre direction properties are factored by their thickness ratio and a scaling factor which provides an estimate of the layer contribution according to the orientation and loading system



10% Rule for laminate stiffness

Loading	Stiffness	% ply contribution factor 0°	±45°	90°	Apply to
	E_x	1.0	0.1	0.1	E_1
	E_Y	0.1	0.1	1.0	E_1
	G_{xy}	0.1	0.55	0.1	$\frac{E_1}{2(1+v^*)}$

x RoM ply thickness fraction

$\frac{t_{o^\circ}}{t}$ $\qquad \frac{t_{\pm 45^\circ}}{t}$ $\qquad \frac{t_{90^\circ}}{t}$

....

$$v_{xy} \text{ for QI laminates } = \frac{1}{\left[1+4\left(\frac{\%90^\circ}{\%\pm 45^\circ}\right)\right]}$$

i.e. with plies in all 0°,±45°, 90° directions

where $v*$ is the poisson ratio of the "complimentary layup" for doubly symmetric laminates, e.g.:

for ±45°	$v*$	$= v_{\pm 45^\circ}$	=0.05
for 0°,90°	$v*$	$= v_{0^\circ,90^\circ}$	=0.8
for 0°, ±45°,90°	$v*$	$= v_{0^\circ,\pm 45^\circ,90^\circ}$	=0.33

10% Rule for laminates strengths[17]

*Note, here layer contribution factor also depends on loading system

Loading	Strength	% ply contribution factor 0°	±45°	90°	Apply to
Uniaxial	σ^*_x	1.0	0.1	0.1	$\sigma_1{}^*$
	σ^*_{y}	0.1	0.1	1.0	$\sigma_1{}^*$
Bi-axial **Same sign**	σ^*_x	1.0	0.55	0.1	$\sigma_1{}^*$
	σ^*_y	0.1	0.55	1.0	$\sigma_1{}^*$
Bi-axial **Opposite sign** **i.e. Shear**	τ^*_{xy}	0.1	0.55	0.1	$\sigma_1{}^*/2$

x RoM ply thickness fraction

$\frac{t_{0^\circ}}{t}$ $\frac{t_{\pm 45^\circ}}{t}$ $\frac{t_{90^\circ}}{t}$

Examples:

For QI laminate strength under uni-axial loading in x direction :

$$E_x = \left(1.0 \times \frac{t_{0^\circ}}{t} + 0.1 \times \frac{t_{\pm 45^\circ}}{t} + 0.1 \times \frac{t_{90^\circ}}{t}\right) \times E_1$$

$$\sigma^*{}_x = (1.0 \times \frac{t_{0^\circ}}{t} + 0.1 \times \frac{t_{\pm 45^\circ}}{t} + 0.1 \times \frac{t_{90^\circ}}{t}) \times \sigma_1{}^*$$

etc.

Laminate Carpet Plots

Application :

- To provide preliminary design allowables for QI laminate families
- To select suitable laminates to satisfy design property requirements

Assumptions :

- Based on specific material system and specimen result specific failure criteria and associated to the strength curves

Data :

- Static mechanical tension, compression and shear test results for plain or notched or impacted specimens under room temperature/dry, hot/wet, or cold/dry conditions

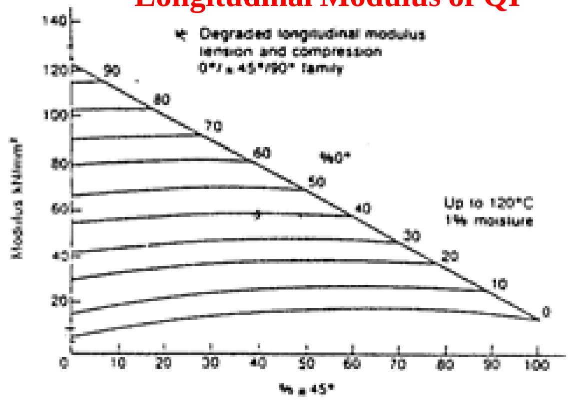
Limitations :

- Limited extrapolation of data for other material system

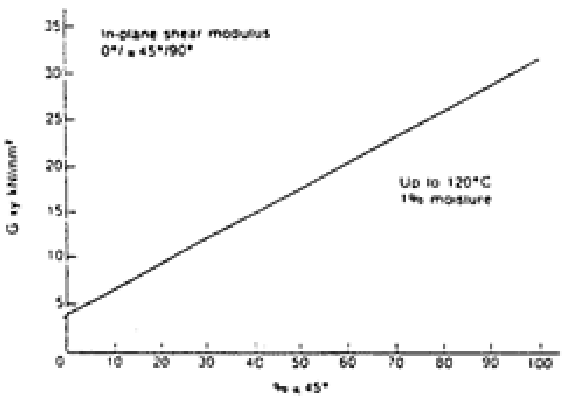
Method:

Read off plots to find required laminate layup or properties

Longitudinal Modulus of QI

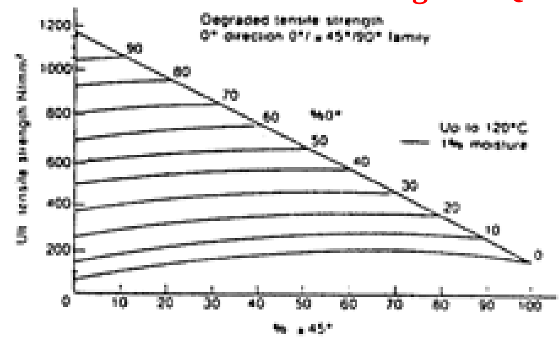


Shear Modulus of QI

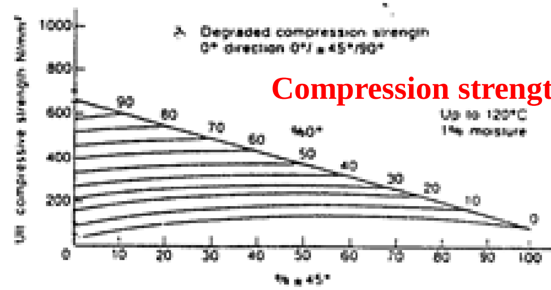


Degraded Mechanical Properties of typical Aerospace Approved HS Carbon Epoxy (175° cure)

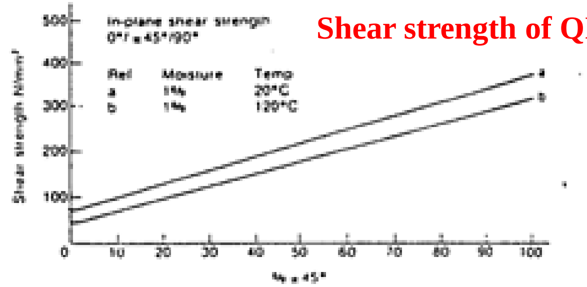
Tensile strength of QI



Compression strength of QI



Shear strength of QI



3. Strength Analysis of Laminate

- Laminate failure behavior is different from metal plate.
- **Even though a ply is failed, the laminate is not failure because other plies can endure the load by much higher stress.**
- The inter-laminar separation should be analyzed by 3-D stress analysis. Therefore this lecture does not treat this topic.

➢ Strength analysis of laminate will be discussed by the following calculation example.

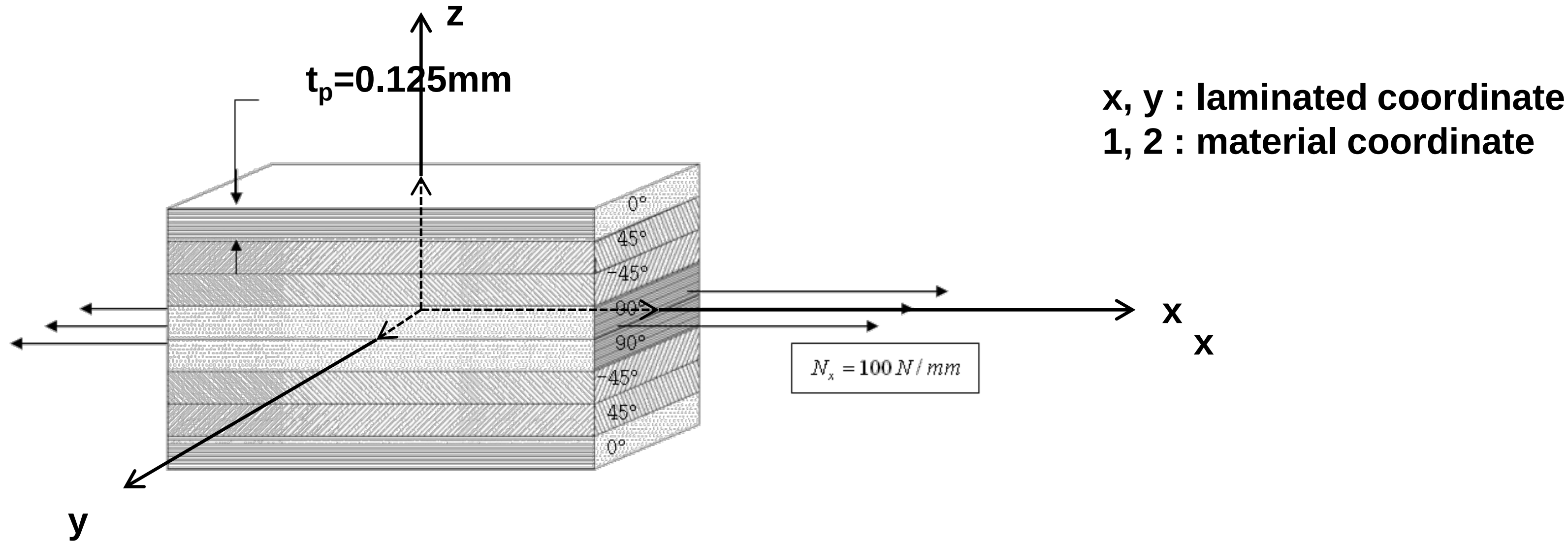
Appendix [Calculation Example of CLT]

Exercise 1 : Perform strength analysis of the following Quasi-isotropic laminate with lay-up sequence (0°/45°/-45°/90°)s. The laminate is considered as quasi-isotropic manufactured by laying-up with unidirectional carbon-epoxy prepreg ply.

Where axial load strength of N_X =100N/mm is applied.

, and use the following mechanical properties..

; E_1=140GPa, E_2=10GPa, G_{12}=5GPa, v_{12}=0.3, ply thickness; t_p=0.125mm, X_t=1500MPa, X_c=1200MPa, Y_t=50MPa, Y_c=250MPa, S=70MPa.



<Solution>

1st Step : Perform stress analysis by ply-by-ply of laminate.

- The Poisson ratio calculation uses the relationship of $\nu_{12}/E_1 = \nu_{21}/E_2$
  (∵ Reduced compliance must be symmetric)
- Reduced stiffness is calculated as follows;

$$Q_{11} = E_1/(1-\nu_{12}\nu_{21}) = 140/(1-0.3\times0.021) = 140.9KN/mm^2$$

$$Q_{22} = E_2/(1-\nu_{12}\nu_{21}) = 10/(1-0.3\times0.021) = 10.1KN/mm^2$$

$$Q_{33} = G_{12} = 5.0KN/mm^3$$

$$Q_{12} = \nu_{21}\cdot E_1/(1-\nu_{12}\nu_{21}) = (0.021\times140)/(1-0.3\times0.021) = 3.0KN/mm^2$$

$$Q = \begin{bmatrix} 140.9 & 3.0 & 0 \\ 3.0 & 10.1 & 0 \\ 0 & 0 & 5.0 \end{bmatrix}$$

∵ Reduced stiffness matrix has the following relationship.

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix}$$

$where\ f_1 = \sigma_1,\ f_2 = \sigma_2,\ f_3 = f_{12} = \tau_{12}$

$e_1 = \varepsilon_1,\ e_2 = \varepsilon_2,\ e_3 = e_{12} = \gamma_{12}$

$$
= \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix}
$$

To obtain $\overline{Q}$ it should be transformed by ply angle.

If $\cos\theta = m$ and $\sin\theta = n$,

$$
\begin{Bmatrix} \overline{Q}_{11} \\ \overline{Q}_{22} \\ \overline{Q}_{33} \\ \overline{Q}_{12} \\ \overline{Q}_{13} \\ \overline{Q}_{23} \end{Bmatrix} = \begin{bmatrix} m^4 & n^4 & 2m^2n^2 & 4m^2n^2 \\ n^4 & m^4 & 2m^2n^2 & 4m^2n^2 \\ m^2n^2 & m^2n^2 & -2m^2n^2 & \left(m^2-n^2\right)^2 \\ m^2n^2 & m^2n^2 & m^4+n^4 & -4m^2n^2 \\ m^3n & -mn^3 & mn^3-m^3n & 2\left(mn^3-m^3n\right) \\ mn^3 & -m^3n & m^3n-mn^3 & 2\left(m^3n-mn^3\right) \end{bmatrix} \begin{Bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{33} \end{Bmatrix}
$$

- The transformed reduced stiffness matrix with ply angle 0° can be obtained by using the above transformation matrix;

$$m = \cos 0^\circ = 1 \quad m^2 = 1 \quad m^4 = 1$$

$$n = \sin 0^\circ = 0 \quad n^2 = 0 \qquad n^4 = 1$$

$$m^2n^2 = 0, \quad 2m^2n^2 = 0, \quad 4m^2n^2 = 0$$

$$m^2 - n^2 = 1, \quad m^4 + n^4 = 1, \quad m^3n = 0, \quad mn^3 = 0$$

$$m^3n - mn^3 = 0, \quad 2(m^3n - mn^3) = 0$$

$$mn^3 - m^3n = 0, \quad 2(mn^3 - m^3n) = 0$$

$$\therefore \overline{Q}_{11} = m^4Q_{11} + n^4Q_{22} + 2m^2n^2Q_{12} + 4m^2n^2Q_{33} = (1\times140.9) + (0\times10.1) + (0\times3.0) + (0\times5.9)$$

$$= 140.9KN/mm^2 = Q_{11}$$

$$\overline{Q}_{22} = Q_{22}$$

$$\overline{Q}_{33} = Q_{33}$$

$$\overline{Q}_{12} = Q_{12}$$

$$\overline{Q}_{13} = Q_{13}$$

$$\overline{Q}_{23} = Q_{23}$$

$$\therefore (\overline{Q})_{0^\circ} = \begin{bmatrix} 140.9 & 3.0 & 0 \\ 3.0 & 10.1 & 0 \\ 0 & 0 & 5.0 \end{bmatrix} KN/mm^2$$

- Similarly, the transformed reduced stiffness matrix with ply angle $+45^\circ$
- can be obtained by using the above transformation matrix;

$$m = \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad m^2 = 0.5, \quad m^4 = 0.25$$

$$n = \sin 45^\circ = \frac{1}{\sqrt{2}}, \quad n^2 = 0.5, \quad n^4 = 0.25$$

$$m^2n^2 = 0.25, \quad 2m^2n^2 = 0.5, \quad 4m^2n^2 = 1$$

$$m^2 - n^2 = 0, \quad m^4 + n^4 = 0.5, \quad m^3n = 0.25, \quad mn^3 = 0.25, \quad m^3n - mn^3 = 0$$

$$2(m^3n - mn^3) = 0, \quad mn^3 - m^3n = 0, \quad 2(mn^3 - m^3n) = 0$$

$$\therefore \overline{Q}_{11} = (0.25 \times 140.9) + (0.25 \times 10.1) + (0.5 \times 3.0) + (1x5.0) = 44.3KN/mm^2$$

$$\overline{Q}_{22} = 44.3KN/mm^2$$

$$\overline{Q}_{33} = 36.3KN/mm^2$$

$$\overline{Q}_{12} = 34.3KN/mm^2$$

$$\overline{Q}_{13} = 32.7KN/mm^2$$

$$\overline{Q}_{23} = 32.7KN/mm^2$$

$$\therefore (\overline{Q})_{45^\circ} = \begin{bmatrix} 44.3 & 34.3 & 32.7 \\ 34.3 & 44.3 & 32.7 \\ 32.7 & 32.7 & 36.3 \end{bmatrix} KN/mm^2$$

- Similarly, the transformed reduced stiffness matrix with ply angle -45°and 90° can be obtained;

$$(\overline{Q})_{-45^\circ} = \begin{bmatrix} 44.3 & 34.3 & -32.7 \\ 34.3 & 44.3 & -32.7 \\ -32.7 & -32.7 & 36.3 \end{bmatrix} KN/mm^2$$

$$(\overline{Q})_{90^\circ} = \begin{bmatrix} 10.1 & 3.0 & 0 \\ 3.0 & 140.9 & 0 \\ 0 & 0 & 5.0 \end{bmatrix} KN/mm^2$$

- To obtain the extensional stiffness matrix $[A]$ of symmetric quasi-isotropic laminate, the transformed reduced stiffness matrix of each layer are arranged by the following table form;

$$A_{11}, A_{22}, A_{33}, A_{12}; A_{13} = A_{23} = 0$$

Ply	θ°	$\overline{Q}_{11}$	$\overline{Q}_{22}$	$\overline{Q}_{33}$	$\overline{Q}_{12}$	$\overline{Q}_{13}$	$\overline{Q}_{23}$	tp	$\overline{Z}_{p}$
1	0	140.9	10.1	5.0	3.0	0	0	0.125	-0.4375
2	45	44.3	44.3	36.3	34.3	32.7	32.7	0.125	-0.3125
3	-45	44.3	44.3	363.	34.3	-32.7	-32.7	0.125	-0.1875
4	90	10.1	140.9	5.0	3.0	0	0	0.125	-0.0625
5	90	10.1	140.9	5.0	3.0	0	0	0.125	+0.0625
6	-45	44.3	44.3	36.3	34.3	-32.7	-32.7	0.125	0.1875
7	45	44.3	44.3	36.3	34.3	32.7	32.7	0.125	0.3125
8	0	140.9	10.1	5.0	3.0	0	0	0.125	0.4375

$$A_{ij} = \sum_{p=1}^{N} t_p(\overline{Q}_{ij})_p$$

$$A_{11} = 2\{(0.125 \times 140.9)_{ply1} + (0.125 \times 44.3)_{ply2} + (0.125 \times 44.3)_{ply3} + (0.125 \times 10.1)_{ply4}\}$$

$$= 59.9KN/mm$$

$$A_{22} = 2 \times 0.125 \times \{(10.1)_{ply1} + (44.3)_{ply2} + (44.3)_{ply3} + (140.9)_{ply4}\} = 59.9KN/mm$$

$$A_{33} = 20.7KN/mm$$

$$A_{12} = 18.7KN/mm$$

$$A_{13} = 0$$

$$A_{23} = 0$$

$$\therefore A = \begin{bmatrix} 59.9 & 18.7 & 0 \\ 18.7 & 59.9 & 0 \\ 0 & 0 & 20.7 \end{bmatrix} KN/mm$$

$$\begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{Bmatrix} e_x^\circ \\ e_y^\circ \\ e_{xy}^\circ \end{Bmatrix}$$

where $f_x = \sigma_x$, $f_y = \sigma_y$, $f_{xy} = \tau_{xy}$
$e_x = \varepsilon_x$, $e_y = \varepsilon_y$, $e_{xy} = \gamma_{xy}$

or

$$\begin{Bmatrix} e_x^\circ \\ e_y^\circ \\ e_{xy}^\circ \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{Bmatrix} N_x \\ 0 \\ 0 \end{Bmatrix}$$

$$a_{11} = \frac{A_{22}}{(A_{11}A_{22} - A_{12}^2)} = \frac{59.9}{3238.3} = 0.0185$$

$$a_{22} = \frac{A_{11}}{(A_{11}A_{22} - A_{12}^2)} = 0.0185$$

$$a_{33} = \frac{1}{A_{33}} = \frac{1}{20.7} = 0.0483$$

$$a_{12} = \frac{-A_{12}}{(A_{11}A_{22} - A_{12}^2)} = -0.0058$$

$$a = \begin{bmatrix} 0.0185 & -0.0058 & 0 \\ -0.0058 & 0.0185 & 0 \\ 0 & 0 & 0.0483 \end{bmatrix} \frac{1}{KN/mm}$$

$$N_x = 100N/mm, N_y = N_{xy} = 0$$

If
$$\begin{Bmatrix} e_x^\circ \\ e_y^\circ \\ e_{xy}^\circ \end{Bmatrix} = \begin{bmatrix} 0.0185 & -0.0058 & 0 \\ -0.0058 & 0.0185 & 0 \\ 0 & 0 & 0.0483 \end{bmatrix} \times 10^{-3} \begin{Bmatrix} N_x = 100 \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore e_x^o = 1850 \times 10^{-6}$$

$$e_y^\circ = -580 \times 10^{-6}$$

$$e_{xy}^\circ = 0$$

-To get each ply stress as to material axis (1: fiber direction-2: fiber perpendicular direction), obtained strains as to laminate axis (x-y) are transformed to material axis.

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix}$$

->The material axis strains of ply 1 and 8 is calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = \begin{Bmatrix} 1850 \\ -580 \\ 0 \end{Bmatrix} \times 10^{-6}$$

->The material axis strains of ply 2 and 7 is calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 1850 \\ -580 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 635 \\ 635 \\ -2430 \end{Bmatrix} \times 10^{-6}$$

->The material axis strains of ply 3 and 6 is calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 1850 \\ -580 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 635 \\ 635 \\ 2430 \end{Bmatrix} \times 10^{-6}$$

->The material axis strains of ply 4 and 5 is calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{Bmatrix} -580 \\ 1850 \\ 0 \end{Bmatrix} \times 10^{-6}$$

- The relationship between ply strain and stress is;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix}$$

where

$$[Q]=\begin{bmatrix}140.9 & 3.0 & 0\\ 3.0 & 10.1 & 0\\ 0 & 0 & 5.0\end{bmatrix} KN/mm^2$$

-> Therefore stresses of ply 1 and 8 can be calculated as follows;

$$\begin{Bmatrix}f_1\\ f_2\\ f_{12}\end{Bmatrix}=\begin{bmatrix}140.9 & 3.0 & 0\\ 3.0 & 10.1 & 0\\ 0 & 0 & 5.0\end{bmatrix}\begin{Bmatrix}180\\ -580\\ 0\end{Bmatrix}\times 10^{-6}\times 10^{3}\,N/mm^2=\begin{Bmatrix}259\\ -0.3\\ 0\end{Bmatrix}N/mm^2$$

-> Stresses of ply 2 and 7 can be calculated as follows;

$$\begin{Bmatrix}f_1\\ f_2\\ f_{12}\end{Bmatrix}=\begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix}\begin{Bmatrix}635\\ 635\\ -2430\end{Bmatrix}\times 10^{-6}\times 10^{3}\,N/mm^2=\begin{Bmatrix}91\\ 8\\ -12\end{Bmatrix}N/mm^2$$

-> Stresses of ply 3 and 6 can be calculated as follows;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix} \begin{Bmatrix} 635 \\ 635 \\ 2430 \end{Bmatrix} \times 10^{-6} \times 10^{3}\, N/mm^2 = \begin{Bmatrix} 91 \\ 8 \\ 12 \end{Bmatrix} N/mm^2$$

-> Stresses of ply 1 and 8 can be calculated as follows;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix} \begin{Bmatrix} -580 \\ 1850 \\ 0 \end{Bmatrix} \times 10^{-6} \times 10^{3}\, N/mm^2 = \begin{Bmatrix} -79 \\ 17 \\ 0 \end{Bmatrix} N/mm^2$$

- Each ply stresses as to laminate axis (x-y) can be obtained by coordinate transforming the obtained each ply stress as to material axis (1-2).

$$\begin{Bmatrix} f_x \\ f_y \\ f_{xy} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix}$$

-> Stresses of ply 1 and 8 can be calculated as follows;

$$\begin{Bmatrix} f_x \\ f_y \\ f_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ e_{12} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{Bmatrix} 259 \\ -0.3 \\ 0 \end{Bmatrix} N/mm^2$$

-> Stresses of ply 2 and 7 can be calculated as follows;

$$\begin{Bmatrix} f_x \\ f_y \\ f_{xy} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & -1 \\ 0.5 & 0.5 & 1 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{Bmatrix} 91 \\ 8 \\ -12 \end{Bmatrix} = \begin{Bmatrix} 62 \\ 38 \\ 42 \end{Bmatrix} N/mm^2$$

-> Stresses of ply 3 and 6 can be calculated as follows;

$$\begin{Bmatrix} f_x \\ f_y \\ f_{xy} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -1 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{Bmatrix} 91 \\ 8 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 62 \\ 38 \\ -42 \end{Bmatrix} N/mm^2$$

-> Stresses of ply 4 and 5 can be calculated as follows;

$$\begin{Bmatrix} f_x \\ f_y \\ f_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} -76 \\ 17 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 17 \\ -76 \\ 0 \end{Bmatrix} N/mm^2$$

- **2nd Step: Strength Analysis of Laminate**

At the 1st Step, calculated each ply stresses as to 1-2 axis are as follows;

ply	θ°	f_1	f_2	f_3	F.I.1	F.I.2	F.I.12	MOF (mode of failure)
1	0	259	-0.3	0	0.17	0.01	0	LT
2	45	91	8	-12	0.06	0.16	0.17	S
3	-45	91	8	12	0.06	0.16	0.17	S
4	90	-76	17	0	0.06	0.34	0	TT
5	90	-76	17	0	0.06	0.34	0	TT
6	-45	91	8	12	0.06	0.16	0.17	S
7	45	91	8	-12	0.06	0.16	0.17	S
8	0	259	-0.3	0	0.17	0.01	0	LT

*** F.I: Failure Index = 1/RF**
MOF: Mode of Failure

- Ply plane failure modes are as follows;

· Longitudinal Tension Mode (LT) : Fiber direction
· Longitudinal Compression Mode(LC) : Fiber direction
· Transverse Tension Mode (TT) : Matrix direction
· Transverse Compression Mode (TC) : Matrix direction
· Plane Shear Mode (S) :

- **Failure Criteria and Failure Index (F.I.)**

(1) Maximum Stress Theory

- As to tensile stress :

$$f_1/X_t < 1$$
$$f_2/Y_t < 1$$

- As to compressive stress:

$$|f_1/X_c| < 1$$
$$|f_2/Y_c| < 1$$

- As to shear stress :

$$|f_{12}/S| < 1$$

Failure Index(F.I.):

- If F.I.1>1 ; Fiber failure
- If F.I.2>1 ; Matrix failure
- if F.I.12>1 ; Shear failure

(2) Maximum Strain Theory

- As to tensile strain:

$$e_1 < e_{x,t} \quad \text{or} \quad \frac{e_1}{e_{x,t}} < 1$$
$$e_2 < e_{y,t} \qquad\quad \frac{e_2}{e_{y,t}} < 1$$

- As to compressive strain : $|e_1| < |e_{x,c}|$ or $\left|\dfrac{e_1}{e_{x,c}}\right| < 1$

$|e_2| < |e_{y,c}|$ $\left|\dfrac{e_2}{e_{y,c}}\right| < 1$

- As to shear strain : $|e_{12}| < e_s$ or $\left|\dfrac{e_{12}}{e_s}\right| < 1$

Or Failure Index (F.I);,

- If F.I.1>1 : fiber directional failure
- If F.I.2>1 ; Matrix direction failure
- If F.I.12>1 ; Shear direction failure

(3) Tsai-Hill Theory

If Ply stresses satisfy the following criteria, the ply is not failed.

$$F.I. = \left(\frac{f_1}{X}\right)^2 + \left(\frac{f_2}{Y}\right)^2 + \left(\frac{f_{12}}{S}\right)^2 - \left(\frac{f_1}{X}\right)\left(\frac{f_2}{Y}\right) < 1$$

where $X = X_t$ (or X_c), $Y = Y_t$ (or Y_c) (this is determined by plus and minus sign of f_1 and f_2). Absolute values of X_c , Y_c are used.

(4) Tsai-Wu Theory

If Ply stresses satisfy the following criteria, the ply is not failed.

$$F.I. = F_1 f_1 + F_2 f_2 + F_{11} f_1^{\,2} + F_{22} f_2^{\,2} + F_{33} f_{12}^{\,2} + 2F_{12} f_1 f_2 < 1$$

where

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}$$

$$F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}$$

$$F_{11} = 1/(X_t + X_c)$$

$$F_{22} = 1/(Y_t + Y_c)$$

$$F_{33} = 1/S^2$$

$$F_{12} = F_{12}^{*}\sqrt{F_{11}F_{12}} = F_{12}^{*}\sqrt{X_t X_c Y_t Y_c} \approx -\frac{1}{2}/\sqrt{X_t X_c Y_t Y_c}$$

- Where $F_{12}^{\,*}$ is decided by 2-axis experimental test.
- If value of $F_{12}^{\,*}$ is not available, let it be -1/2

- Load Index (L.I.)

The load required to produce the first ply failure (FPF) can be obtained from F. I.,

· In case of Max Stress Theory,

Load Index = $\dfrac{1}{F.I.}$

· Incase of Max Strain Theory,

Load Index = $\dfrac{1}{\sqrt{F.I.}}$

- In this Exercise Max Stress theory is used, the axial load intensity of $N_x = 100 N/mm$ is applied.
- F. I.s of ply 1 and 8 (0°) are obtained,

  F.I.1=259/1500=0.17

  F.I.2=|-0.3/250|=0.01

  F.I.12=0

- F. I.s of ply 2 and 7(45°) are obtained,

  F.I.1=91/1500=0.06

  F.I.2=8/50=0.16

  F.I.12=|-12/70|=0.17

- F. I.s of ply 3 and 6 (- 45°) are obtained,

  F.I.1=91/1500=0.06

  F.I.2=8/50=0.16

  F.I.12=12/70=0.17

- F. I.s of ply 4 and 5 (90°) are obtained,

  F.I.1=|-76/1200|=0.06

  F.I.2=17/50=0.34

  F.I.12=0

- According to calculation results, max F. I. occurs at 90° ply.

  i. e. F.I.2=0.34 is TT failure mode .

- The load required to produce the first ply failure (FPF) using Max Stress Theory can be obtained, and the FPF occurs at 90° plies (4 and 5) in TT mode!

$$N_x = 100/0.34 = 294 N/mm$$ ← **FPF Load**

➢ 2nd Ply Failure

In the previous calculation, it was confirmed that The FPF occurs as TT mode at 90° plies (ply 4 and 5).

After FPF at plies 4 and 5, the elastic material properties of the failed 90 ° such as E_1 ,E_2 G_{12} are set as zero. Then the transformed reduced stiffness matrix of the degraded laminate are assumed using the 1st step calculation, and the extensional stiffness matrix can be obtained again.

ply	θ°	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{33}$	$\bar{Q}_{12}$	$\bar{Q}_{13}$	$\bar{Q}_{23}$
1	0	140.9	10.1	5	3	0	0
2	45	44.3	44.3	36.3	34.3	327	32.7
3	−45	44.3	44.3	36.3	34.3	−32.7	−32.7
4	90	0	0	0	0	0	0
5	90	0	0	0	0	0	0
6	−45	44.3	44.3	44.3	34.3	−32.7	−32.7
7	45	44.3	44.3	44.3	34.3	32.7	32.7
8	0	140.9	10.1	10.1	3	0	0

$$\therefore A_{11} = 2\{(0.125\times 140.9)_{ply1} + (0.125\times 44.3)_{ply2} + (0.125\times 44.3)_{ply3} + (0.125\times 0)_{ply4}\} = 57.4KN/mm$$

$$A_{22} = 24.7KN/mm$$

$$A_{33} = 19.4KN/mm$$

$$A_{12} = 17.9KN/mm$$

$$A_{13} = 0$$

$$A_{23} = 0$$

$$\therefore A = \begin{bmatrix} 57.4 & 17.9 & 0 \\ 17.9 & 24.7 & 0 \\ 0 & 0 & 19.4 \end{bmatrix} KN/mm$$

$$a = A^{-1} = \begin{bmatrix} 0.0225 & -0.0163 & 0 \\ -0.0163 & 0.0523 & 0 \\ 0 & 0 & 0.0515 \end{bmatrix} 1/(KN/mm)$$

$$\begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} = \begin{bmatrix} 0.0225 & -0.063 & 0 \\ -0.0163 & 0.0523 & 0 \\ 0 & 0 & 0.0515 \end{bmatrix} \times 10^{-3} \begin{Bmatrix} N_x = 294 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 6615 \\ -4792 \\ 0 \end{Bmatrix} \times 10^{-6}$$

-To get each ply stress as to material axis (1: fiber direction-2: fiber perpendicular direction), obtained strains as to laminate axis (x-y) are transformed to material axis.

->The material axis strains of ply 1 and 8 (0°) are calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{Bmatrix} transform \\ matrix \end{Bmatrix} \begin{Bmatrix} e_x^{\circ} \\ e_y^{\circ} \\ e_{xy}^{\circ} \end{Bmatrix} = \begin{Bmatrix} e_x^{\circ} \\ e_y^{\circ} \\ e_{xy}^{\circ} \end{Bmatrix} = \begin{Bmatrix} 6615 \\ -4792 \\ 0 \end{Bmatrix} \times 10^{-6}$$

->The material axis strains of ply 2 and 7 (45°) are calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 6615 \\ -4792 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 912 \\ 912 \\ -11407 \end{Bmatrix} \times 10^{-6}$$

->The material axis strains of ply 3 and 6 (- 45°) are calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 6615 \\ -4792 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} 912 \\ 912 \\ 11407 \end{Bmatrix} \times 10^{-6}$$

->The material axis strains of ply 4 and 5 (90°) are calculated as;

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 6615 \\ -4792 \\ 0 \end{Bmatrix} \times 10^{-6} = \begin{Bmatrix} -4892 \\ 6615 \\ 0 \end{Bmatrix} \times 10^{-6}$$

- Therefore stresses of ply 1 and 8 **(0°)** can be calculated as follows;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_{12} \end{Bmatrix} = \begin{bmatrix} 140.9 & 3.0 & 0 \\ 3.0 & 10.1 & 0 \\ 0 & 0 & 5.0 \end{bmatrix} \begin{Bmatrix} 6615 \\ -4792 \\ 0 \end{Bmatrix} \times 10^{-6} \times 10^{3} = \begin{Bmatrix} 912 \\ -29 \\ 0 \end{Bmatrix} N/mm^2$$

- **F. I.s of ply 1 and 8 (0°) are obtained using Max Stress Theory**

F.I.1=912/1500=0.61

F.I.2=29/250=0.12

F.I.12=0

- Stresses and F.I. of ply 2 and 7 (45**°)** can be calculated as follows;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{bmatrix} & & \\ & Q & \\ & & \end{bmatrix} \begin{Bmatrix} 912 \\ -912 \\ -11407 \end{Bmatrix} \times 10^{-6} \times 10^{-3} = \begin{Bmatrix} 131 \\ 12 \\ -57 \end{Bmatrix} N/mm^2$$

F.I.1=131/1500=0.09

F.I.2=12/50=0.24

F.I.12=|-57/70|=0.81

- Stresses and F.I. of ply 3 and 6 (- 45°) can be calculated as follows; -45°;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{Bmatrix} & Q & \end{Bmatrix} \begin{Bmatrix} 912 \\ 912 \\ 11407 \end{Bmatrix} \times 10^{-6} \times 10^{-3} = \begin{Bmatrix} 131 \\ 12 \\ 57 \end{Bmatrix} N/mm^2$$

F.I.1=131/1500=0.09
F.I.2=12/50=0.24
F.I.12=57/70=0.81

- Stresses and F.I. of ply 4 and 5 (90°) can be calculated as follows;

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = \begin{Bmatrix} & 0 & \end{Bmatrix} \begin{Bmatrix} -4792 \\ 6615 \\ 0 \end{Bmatrix} \times 10^{-6} \times 10^{-3} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} N/mm^2$$

- According to calculation results, max F. I. occurs at $\pm 45^\circ$ ply (2,3,6,7)

  i. e. F.I. 12=0.81 is S failure mode.

- The load required to produce the second ply failure using Max Stress Theory is obtained can be obtained, and the SPF occurs at $\pm 45^\circ$ plies (2,3,6,7) in S mode!
- **The failure load is;** $N_x = 294/0.81 = 363N/mm$

<Summary>

At $N_x = 294N/mm,$ ply 4 and 5(90°) are completely failed.

ply	θ°	f_1	f_2	f_{12}	F.I.1	F.I.2	F.I.12	MOF
1	0	918	−29	0	0.61	0.12	0	LT
2	45	131	12	−57	0.09	0.24	0.81	S
3	−45	131	12	57	0.09	0.24	0.81	S
4	90	0	0	0	0	0	0	
5	90	0	0	0	0	0	0	
6	−45	131	12	57	0.09	0.24	0.81	S
7	45	131	12	−57	0.09	0.24	0.81	S
8	0	918	−29	0	0.61	0.12	0	LT

➢Third-ply-failure

At $N_x = 294 / mm$, FPF occurs at 90° ply as TT mode ,
At $N_x = 363 / mm$, SPF occurs at ±45° ply as S mode.

However 2 plies (0°) still are not failed yet.
Again, similarly to the previous SPF calculation procedure, material properties such as E_1, E_2 G_{12} of ±45° are assumed as 'zero. Therefore after SPF, the transformed Reduced stiffness matrix at failed plies (90°and ±45°) are assumed as zero.

ply	θ°	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{33}$	$\bar{Q}_{12}$	$\bar{Q}_{13}$	$\bar{Q}_{23}$
1	0	140.9	10.1	5	3	0	0
2	45	0	0	0	0	0	0
3	−45	0	0	0	0	0	0
4	90	0	0	0	0	0	0
5	90	0	0	0	0	0	0
6	−45	0	0	0	0	0	0
7	45	0	0	0	0	0	0
8	0	140.9	10.1	5	3	0	0

$$A_{ij} = \sum_{p=1}^{N} t_p (\overline{Q}_{ij})_p$$

$$A_{11} = 2\{(0.125 \times 140.9)_{ply1} + (0.125 \times 0)_{ply2} + (0.125 \times 0)_{ply3} + (0.125 \times 0)_{ply}\}$$

$$= 35.2KN / mm$$

$$A_{22} = 2.5KN / mm$$

$$A_{33} = 1.3KN / mm$$

$$A_{12} = 0.8KN / mm$$

$$A_{13} = 0$$

$$A_{23} = 0$$

$$A = \begin{bmatrix} 35.2 & 0.8 & 0 \\ 0.8 & 2.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} KN / mm$$

$$\therefore a = \begin{bmatrix} 0.0286 & -0.0092 & 0 \\ -0.0092 & 0.4029 & 0 \\ 0 & 0 & 0.7692 \end{bmatrix} 1/(KN / mm)$$

$$\begin{Bmatrix} e^{\circ}{}_{x} \\ e^{\circ}{}_{y} \\ e^{\circ}{}_{xy} \end{Bmatrix} = \begin{bmatrix} 0.0286 & -0.0092 & 0 \\ -0.0092 & 0.4029 & 0 \\ 0 & 0 & 0.7692 \end{bmatrix} \times 10^{-3} \begin{Bmatrix} N_x = 363 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 10382 \\ -3340 \\ 0 \end{Bmatrix} \times 10^{-6}$$

► At Ply 1 and 8 (0° ply) the ply strains are;

$$\begin{Bmatrix} e_1^{\circ} \\ e_2^{\circ} \\ e_{12}^{\circ} \end{Bmatrix} = \begin{Bmatrix} e_x^{\circ} \\ e_y^{\circ} \\ e_{xy}^{\circ} \end{Bmatrix} = \begin{Bmatrix} 10382 \\ -13340 \\ 0 \end{Bmatrix} \times 10^{-6}$$

► At Ply 1 and 8 (0° ply) the ply 0° stress are,

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} e_1^{\circ} \\ e_2^{\circ} \\ e_{12}^{\circ} \end{Bmatrix} = \begin{bmatrix} 140.9 & 3.0 & 0 \\ 3.0 & 10.1 & 0 \\ 0 & 0 & 5.0 \end{bmatrix} \begin{Bmatrix} 10382 \\ -3340 \\ 0 \end{Bmatrix} \times 10^{-6} \times 10^{3}\, N/mm^2 = \begin{Bmatrix} 1453 \\ -3 \\ 0 \end{Bmatrix} N/mm^2$$

✓ F. I. can be obtained using Max Stress Failure Criteria.

F.I.1=1453/1500=0.97

F.I.2=0.12/50=0.01

F.I.12=0

<Summary>

At $N_x = 363N/mm$, 90° , ±45° plies are completely failed, the ply stress and F. I. are as follows;

ply	θ°	f_1	f_2	f_{12}	F.I.1	F.I.2	F.I.12	MOF
1	0	1453	−3	0	0.97	0.01	0	LT
2	45	0	0	0	0	0	0	
3	−45	0	0	0	0	0	0	
4	90	0	0	0	0	0	0	
5	90	0	0	0	0	0	0	
6	−45	0	0	0	0	0	0	
7	45	0	0	0	0	0	0	
8	0	1453	−3	0	0.97	0.01	0	LT

From the above table, max F. I.=0.97 occurs at 1 and 8 ply (0°) as LT mod.

Therefore the **3rd ply failure (TPF) load** can be calculated as;

$$N_x = 363/0.97 = 374N/mm$$

At this load the laminate is finally failed. → **Last Ply Failure (LPF)!**

Therefore the laminate strength F_x is obtained using Max Stress Failure Criteria when the ply is completely failed;

$$F_x = N_x / t = 374 / 1.0 = 374 N / mm^2$$

FPF: $N_x = 294N/mm, e_x = 0.0185 \times 294 \times 10^{-3} = 0.54\% (\because e_x = a_{11} N_x)$

SPF: $N_x = 363n/mm, e_x = 0.0225 \times 363 \times 10^{-3} = 0.82\%$

LPF: $N_x = 374N/mm, e_x = 0.0286 \times 374 \times 10^{-3} = 1.07\%$

$(*\ where\ E_x = \dfrac{1}{a_{11}t},\ t = 1mm\ )$

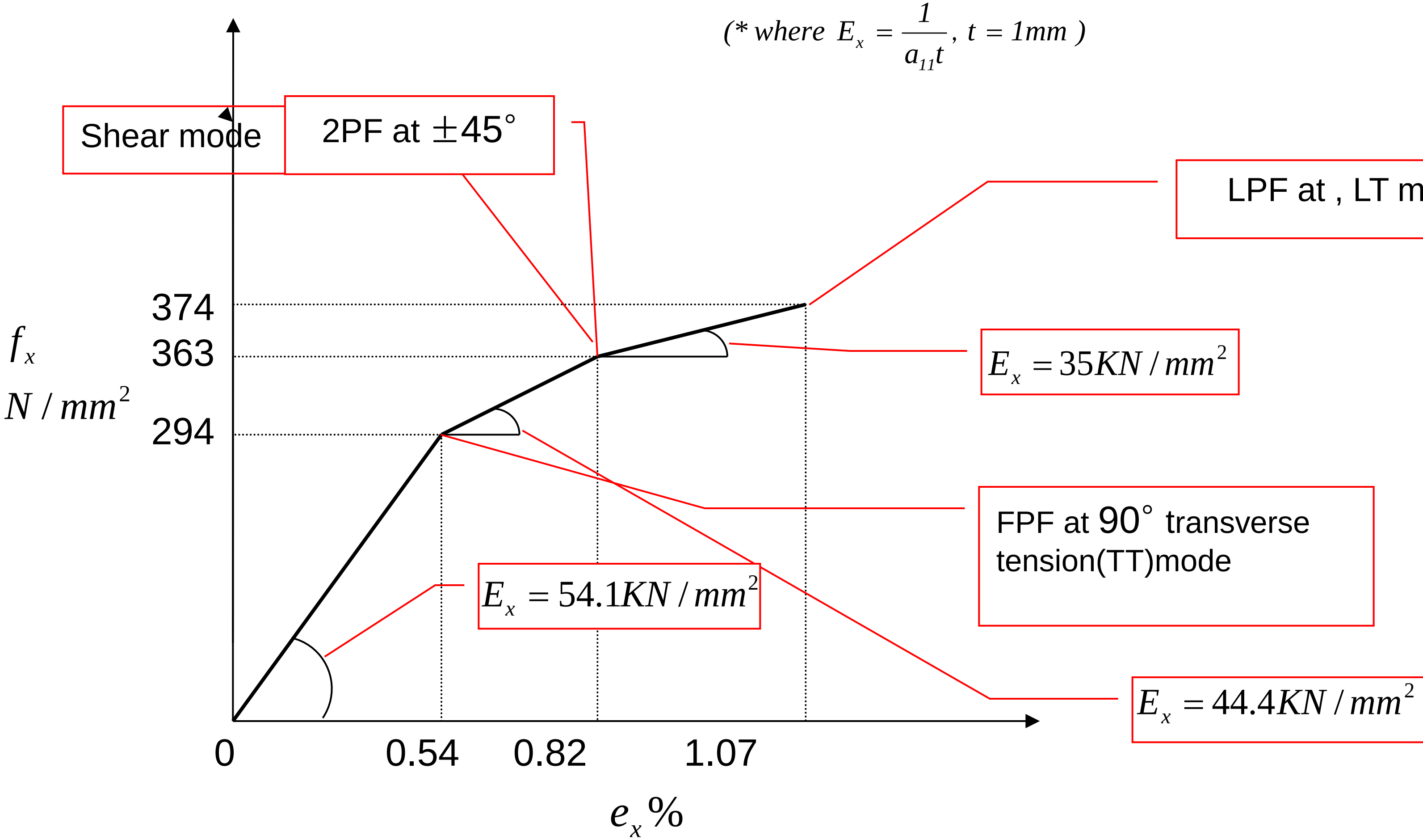


Figure :Longitudinal tensile strength by complete ply failure mode and maximum stress theory at (0/45/-45/90)s

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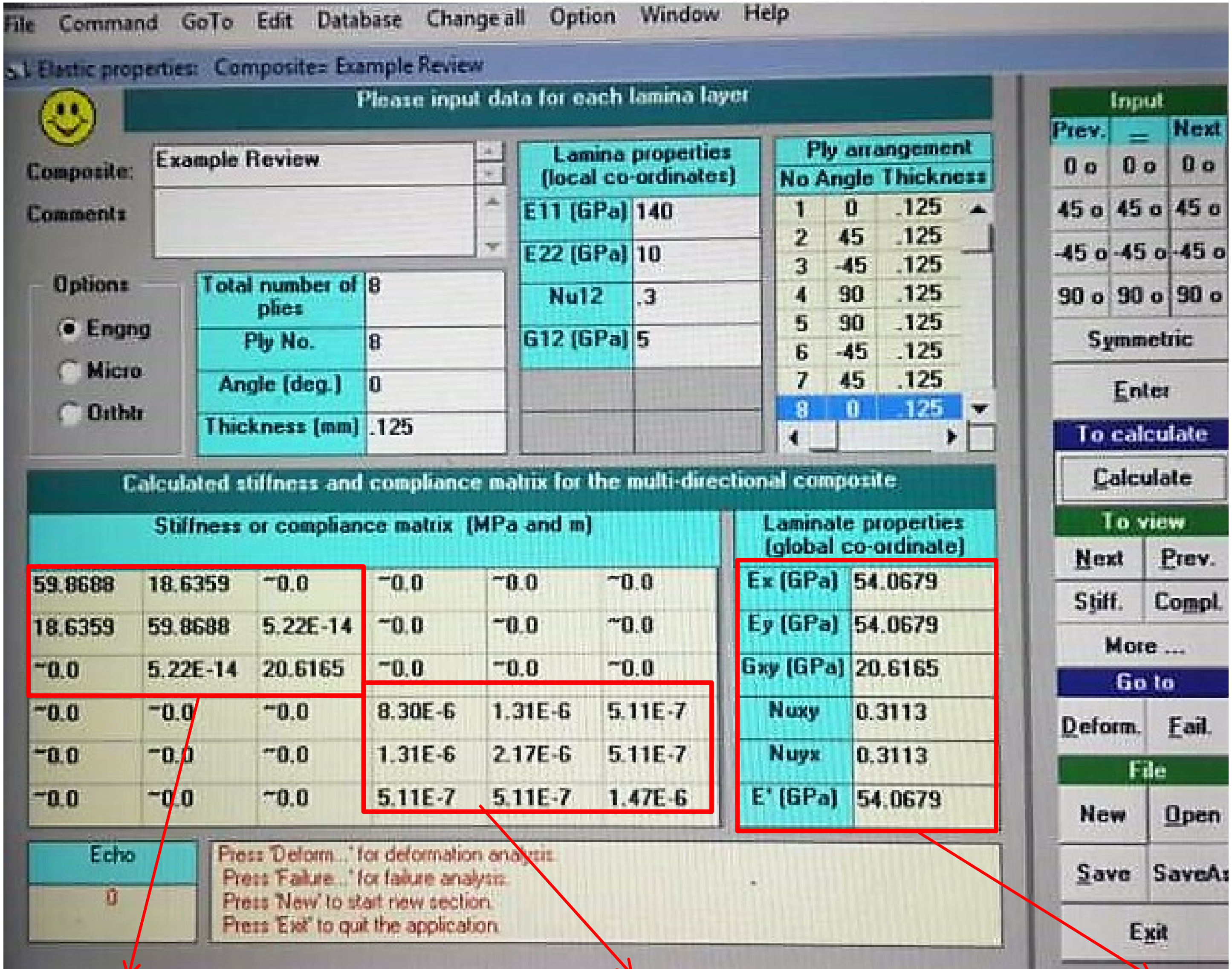
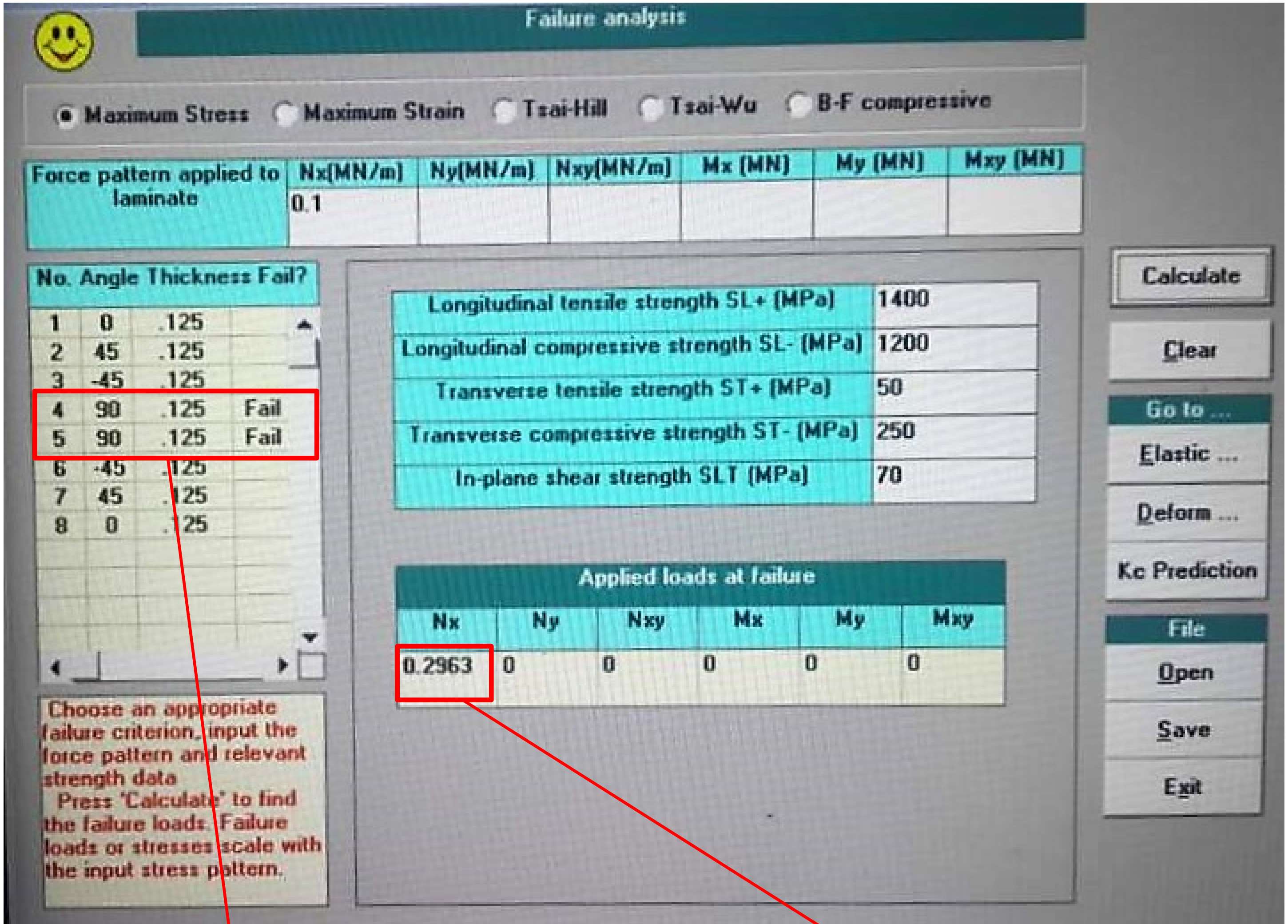


A_{ij}: to match unit with solution – multiplying by 10^3

D_{ij}: to match unit with solution – multiplying by 10^6

Engineering Properties of Laminate (extensional)



At 4th and 5th layer: TT mode-FPF

This failure load : 296 N/mm (FPF)

★ ★

Homework #1

Consider a $(0/45/-45/90)_s$ laminate configuration. The plies are unidirectional high strength carbon/epoxy of 0·125 mm thickness and have the same elastic properties as those in the previous exercise.

$$E_1 = 140, \qquad E_2 = 10, \qquad G_{12} = 5\ \text{kN/mm}^2; \qquad \nu_{12} = 0{\cdot}3$$

The symmetric laminate configuration is shown in Fig. 5.20, and the ordinate section properties are shown in Fig. 5.21.

Find the extensional stiffness A_{ij} and flexural stiffness D_{ij}, and membrane equivalent elastic constants and bending equivalent elastic constants :

$E_x, E_y, G_{xy}, \nu_{xy}, \nu_{yx}, m_x, m_y$

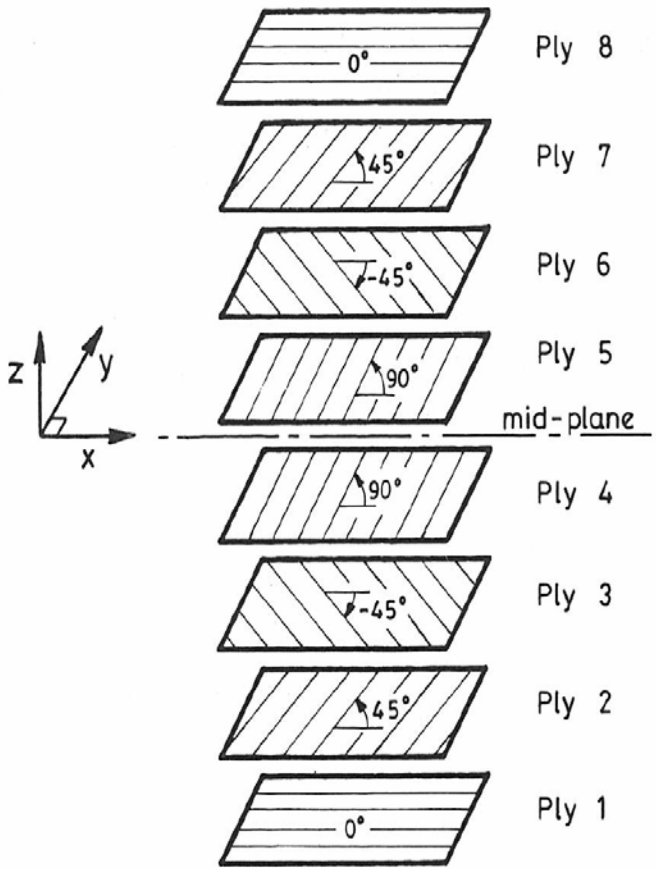


Fig. 5.20. Symmetric MOPL: quasi-isotropic $(0/45/-45/90)_s$.

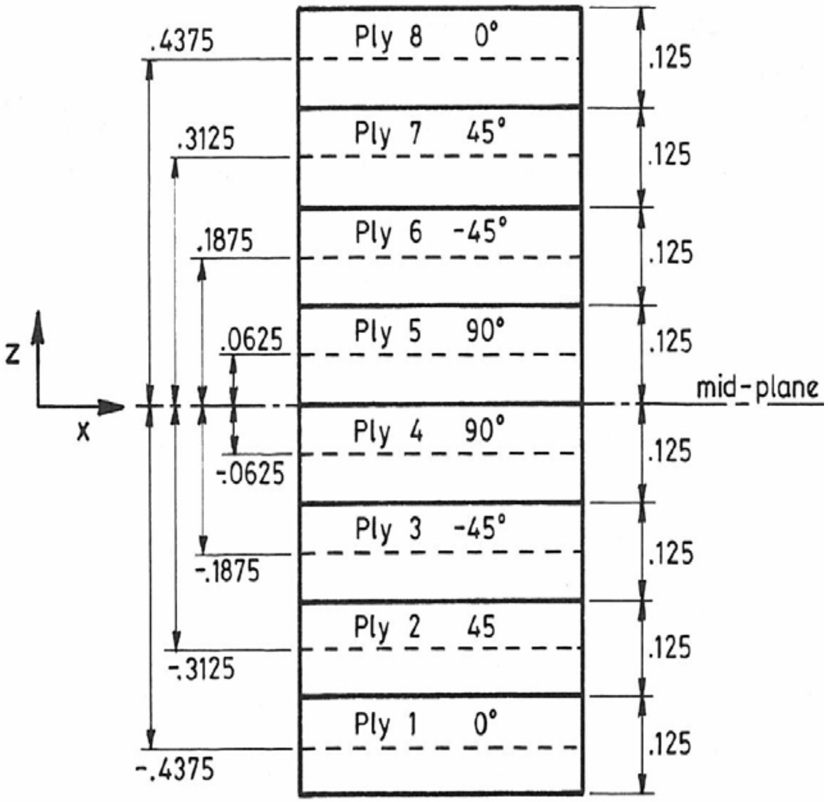


Fig. 5.21. Ordinate values (mm): quasi-isotropic $(0/45/-45/90)_s$.

(Solution)

The ply reduced stiffness matrix for this material as obtained from Example 5.2 are

$$Q = \begin{bmatrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{bmatrix} \text{kN/mm}^2 3$$

The transformed reduced stiffnesses have been determined previously for ply angles of 0° and 90° in Example 5.6, and for ply angles of 45° and −45° in Example 5.8; the results are given below:

$$(\bar{Q})_{0^\circ} = \begin{bmatrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{bmatrix} \text{kN/mm}^2$$

$$(\bar{Q})_{90^\circ} = \begin{bmatrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5{\cdot}0 \end{bmatrix} \text{kN/mm}^2$$

$$(\bar{Q})_{45^\circ} = \boxed{\begin{matrix} 44{\cdot}3 & 34{\cdot}3 & 32{\cdot}7 \\ 34{\cdot}3 & 44{\cdot}3 & 32{\cdot}7 \\ 32{\cdot}7 & 32{\cdot}7 & 36{\cdot}3 \end{matrix}} \ \mathrm{kN/mm^2}$$

$$(\bar{Q})_{-45^\circ} = \boxed{\begin{matrix} 44{\cdot}3 & 34{\cdot}3 & -32{\cdot}7 \\ 34{\cdot}3 & 44{\cdot}3 & -32{\cdot}7 \\ -32{\cdot}7 & -32{\cdot}7 & 36{\cdot}3 \end{matrix}} \ \mathrm{kN/mm^2}$$

The transformed reduced stiffness terms for all the plies in the laminate are given in Table 5.18.

Since the laminate is symmetric, then all the coupling terms $B_{ij} = 0$, so we only need to calculate the extensional and bending, A_{ij} and D_{ij}, stiffness terms.

Table 5.18

$\bar{Q}_{ij}$ Values (kN/mm^2) for a Symmetric MOPL: Quasi-isotropic $(0/45/-45/90)_s$

Ply	$\theta°$	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{33}$	$\bar{Q}_{12}$	$\bar{Q}_{13}$	$\bar{Q}_{23}$
1	0	140·9	10·1	5·0	3·0	0	0
2	45	44·3	44·3	36·3	34·3	32·7	32·7
3	−45	44·3	44·3	36·3	34·3	−32·7	−32·7
4	90	10·1	140·9	5·0	3·0	0	0
5	90	10·1	140·9	5·0	3·0	0	0
6	−45	44·3	4·3	36·3	34·3	−32·7	−32·7
7	45	44·3	44·3	36·3	34·3	32·7	32·7
8	0	140·9	10·1	5·0	3·0	0	0

* MOPL: Multioriented Ply Laminate

Table 5.19
Ordinate Values for a Symmetric MOPL: $(0/45/-45/90)_s$ (Basic dimensions: mm.)

Ply	t_p	$\bar{z}_p$	$(t_p\bar{z}_p^2 + t_p^3/12)$
1	0·125	−0·437 5	0·024 1
2	0·125	−0·312 5	0·012 4
3	0·125	−0·187 5	0·004 6
4	0·125	−0·062 5	0·000 7
5	0·125	0·062 5	0·000 7
6	0·125	0·187 5	0·004 6
7	0·125	0·312 5	0·012 4
8	0·125	0·437 5	0·024 1

The ply thickness and ply centroidal values are obtained from Fig. 5.21, and the required ordinate values, t_p and $(t_p\bar{z}_p^2 + t_p^3/12)$, for each ply are given in Table 5.19. We can now determine the extensional and bending stiffnesses, A_{ij} and D_{ij}, as defined in eqn (5.3).

A_{ij} *terms*

$$A_{ij} = \sum_{p=1}^{N} t_p(\bar{Q}_{ij})_p$$

From Table 5.19 we get the t_p term, the ply thickness value for each ply, and from Table 5.18 we get the $\bar{Q}_{ij}$ term, the transformed reduced stiffness value for each ply. We use these values for the plies in the lower half of the laminate, in this case Plies 1 to 4, in the above expression for A_{ij} in turn and then double the result to obtain the total laminate extensional stiffness value. Thus,

$$A_{11} = 2\{(0{\cdot}125 \times 140{\cdot}9)_{\text{Ply }1} + (0{\cdot}125 \times 44{\cdot}3)_{\text{Ply }2} + (0{\cdot}125 \times 44{\cdot}3)_{\text{Ply }3} + (0{\cdot}125 \times 10{\cdot}1)_{\text{Ply }4}\} = 59{\cdot}9 \text{ kN/mm}$$

$$A_{22} = 2\{(0{\cdot}125 \times 10{\cdot}1)_{\text{Ply }1} + (0{\cdot}125 \times 44{\cdot}3)_{\text{Ply }2} + (0{\cdot}125 \times 44{\cdot}3)_{\text{Ply }3} + (0{\cdot}125 \times 140{\cdot}9)_{\text{Ply }4}\} = 59{\cdot}9 \text{ kN/mm}$$

$$A_{33} = 2\{(0{\cdot}125 \times 5{\cdot}0)_{\text{Ply }1} + (0{\cdot}125 \times 36{\cdot}3)_{\text{Ply }2} + (0{\cdot}125 \times 36{\cdot}3)_{\text{Ply }3} + (0{\cdot}125 \times 5{\cdot}0)_{\text{Ply }4}\} = 20{\cdot}7 \text{ kN/mm}$$

$$A_{12} = 2\{(0{\cdot}125 \times 3{\cdot}0)_{\text{Ply }1} + (0{\cdot}125 \times 34{\cdot}3)_{\text{Ply }2} + (0{\cdot}125 \times 34{\cdot}3)_{\text{Ply }3} + (0{\cdot}125 \times 3{\cdot}0)_{\text{Ply }4}\} = 18{\cdot}7 \text{ kN/mm}$$

$$A_{13} = 2\{(0{\cdot}125 \times 0)_{\text{Ply }1} + (0{\cdot}125 \times 32{\cdot}7)_{\text{Ply }2} + (0{\cdot}125 \times -32{\cdot}7)_{\text{Ply }3} + (0{\cdot}125 \times 0)_{\text{Ply }4}\} = 0$$

$$A_{23} = 2\{(0{\cdot}125 \times 0)_{\text{Ply }1} + (0{\cdot}125 \times 32{\cdot}7)_{\text{Ply }2} + (0{\cdot}125 \times -32{\cdot}7)_{\text{Ply }3} + (0{\cdot}125 \times 0)_{\text{Ply }4}\} = 0$$

The extensional stiffness terms A_{ij} may be written in a boxed matrix notation form:

$$A = \boxed{\begin{matrix} 59{\cdot}9 & 18{\cdot}7 & 0 \\ 18{\cdot}7 & 59{\cdot}9 & 0 \\ 0 & 0 & 20{\cdot}7 \end{matrix}} \text{ kN/mm}$$

We again note here that the shear coupling terms sum to zero as the nonzero positive contribution of the $\bar{Q}_{13}$ (or $\bar{Q}_{23}$) in the product of $t_p\bar{Q}_{13}$ (or $t_p\bar{Q}_{23}$) from the +45° plies cancel out with the nonzero negative terms from the corresponding −45° ply contributions; the 0° and 90° plies have zero $\bar{Q}_{13}$ and $\bar{Q}_{23}$ values.

B_{ij} *terms*

All these coupling terms will be zero because the laminate is symmetric about its midplane.

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D_{ij} *terms*

$$D_{ij} = \sum_{p=1}^{N} (t_p \bar{z}_p^2 + t_p^3/12)(\bar{Q}_{ij})_p$$

From Table 5.19 we get the $(t_p \bar{z}_p^2 + t_p^3/12)$ term for each ply, and from Table 5.18 we get the $\bar{Q}_{ij}$ term, the transformeed reduced stiffness value for each ply. We use these values for the plies in the lower half of the laminate, in this case Plies 1 to 4, in the above expression for D_{ij} in turn and then double the result to obtain the total laminate bending stiffness value. Thus,

$$D_{11} = 2\{(0{\cdot}0241 \times 140{\cdot}9)_{\text{Ply 1}} + (0{\cdot}0124 \times 44{\cdot}3)_{\text{Ply 2}} + (0{\cdot}0046 \times 44{\cdot}3)_{\text{Ply 3}} + (0{\cdot}0007 \times 10{\cdot}1)_{\text{Ply 4}}\} = 8{\cdot}31 \text{ kN mm}$$

$$D_{22} = 2\{(0{\cdot}0241 \times 10{\cdot}1)_{\text{Ply 1}} + (0{\cdot}0124 \times 44{\cdot}3)_{\text{Ply 2}} + (0{\cdot}0046 \times 44{\cdot}3)_{\text{Ply 3}} + (0{\cdot}0007 \times 140{\cdot}9)_{\text{Ply 4}}\} = 2{\cdot}19 \text{ kN mm}$$

$$D_{33} = 2\{(0{\cdot}0241 \times 5{\cdot}0)_{\text{Ply 1}} + (0{\cdot}0124 \times 36{\cdot}3)_{\text{Ply 2}} + (0{\cdot}0046 \times 36{\cdot}3)_{\text{Ply 3}} + (0{\cdot}0007 \times 5{\cdot}0)_{\text{Ply 4}}\} = 1{\cdot}48 \text{ kN mm}$$

$$D_{12} = 2\{(0{\cdot}0241 \times 3{\cdot}0)_{\text{Ply 1}} + (0{\cdot}0124 \times 34{\cdot}3)_{\text{Ply 2}} + (0{\cdot}0046 \times 34{\cdot}3)_{\text{Ply 3}} + (0{\cdot}0007 \times 3{\cdot}0)_{\text{Ply 4}}\} = 1{\cdot}32 \text{ kN mm}$$

$$D_{13} = 2\{(0{\cdot}0241 \times 0)_{\text{Ply 1}} + (0{\cdot}0124 \times 32{\cdot}7)_{\text{Ply 2}} + (0{\cdot}0046 \times -32{\cdot}7)_{\text{Ply 3}} + (0{\cdot}0007 \times 0)_{\text{Ply 4}}\} = 0{\cdot}51 \text{ kN mm}$$

$$D_{23} = 2\{(0{\cdot}0241 \times 0)_{\text{Ply 1}} + (0{\cdot}0124 \times 32{\cdot}7)_{\text{Ply 2}} + (0{\cdot}0046 \times -32{\cdot}7)_{\text{Ply 3}} + (0{\cdot}0007 \times 0)_{\text{Ply 4}}\} = 0{\cdot}51 \text{ kN mm}$$

The bending stiffness terms written in a boxed matrix form are given as

$$D = \boxed{\begin{matrix} 8{\cdot}31 & 1{\cdot}32 & 0{\cdot}51 \\ 1{\cdot}32 & 2{\cdot}19 & 0{\cdot}51 \\ 0{\cdot}51 & 0{\cdot}51 & 1{\cdot}48 \end{matrix}} \text{ kN mm}$$

We use eqn (5.5) to obtain the extensional compliances as the extensional stiffness matrix is partially populated. Substituting A_{ij} for R_{ij} and a_{ij} for r_{ij} in eqn (5.5), we get

$$A_{11} = 59{\cdot}9$$
$$A_{22} = 59{\cdot}9$$
$$A_{33} = 20{\cdot}7$$
$$A_{12} = 18{\cdot}7$$

and

$$AA = A_{11}A_{22} - A_{12}^2 = (59{\cdot}9)(59{\cdot}9) - 18{\cdot}7^2 = 3238{\cdot}3$$

$$a_{11} = A_{22}/AA = 59{\cdot}9/3238{\cdot}3 = 0{\cdot}0185$$

$$a_{22} = A_{11}/AA = 59{\cdot}9/3238{\cdot}3 = 0{\cdot}0185$$

$$a_{33} = 1/A_{33} = 1/20{\cdot}7 = 0{\cdot}0483$$

$$a_{12} = -A_{12}/AA = -18{\cdot}7/3238{\cdot}3 = -0{\cdot}0058$$

Hence, the extensional compliance matrix is

$$a = \boxed{\begin{matrix} 0{\cdot}0185 & -0{\cdot}0058 & 0 \\ -0{\cdot}0058 & 0{\cdot}0185 & 0 \\ 0 & 0 & 0{\cdot}0483 \end{matrix}} \text{ } 1/(\text{kN/mm})$$

As the bending stiffness matrix is fully populated, we use eqn (5.4) to calculate the bending compliances. Substituting D_{ij} for R_{ij} and d_{ij} for r_{ij} in eqn (5.4), we get

$$D_{11} = 8{\cdot}31$$
$$D_{22} = 2{\cdot}19$$
$$D_{33} = 1{\cdot}48$$
$$D_{12} = 1{\cdot}32$$
$$D_{13} = 0{\cdot}51$$
$$D_{23} = 0{\cdot}51$$

and

$$\begin{aligned} DD &= D_{11}D_{22}D_{33} + 2D_{12}D_{23}D_{13} - D_{22}D_{13}^2 - D_{33}D_{12}^2 - D_{11}D_{23}^2 \\ &= (8{\cdot}31)(2{\cdot}19)(1{\cdot}48) + 2(1{\cdot}32)(0{\cdot}51)(0{\cdot}51) - (2{\cdot}19)(0{\cdot}51)^2 \\ &\quad - (1{\cdot}48)(1{\cdot}32)^2 - (8{\cdot}31)(0{\cdot}51)^2 = 22{\cdot}31 \end{aligned}$$

$$d_{11} = (D_{22}D_{33} - D_{23}^2)/DD = \{(2{\cdot}19)(1{\cdot}48) - (0{\cdot}51)^2\}/22{\cdot}31 = 0{\cdot}1336$$

$$d_{22} = (D_{11}D_{33} - D_{13}^2)/DD = \{(8{\cdot}31)(1{\cdot}48) - (0{\cdot}51)^2\}/22{\cdot}31 = 0{\cdot}5396$$

$$d_{33} = (D_{11}D_{22} - D_{12}^2)/DD = \{(8{\cdot}31)(2{\cdot}19) - (1{\cdot}32)^2\}/22{\cdot}31 = 0{\cdot}7376$$

$$d_{12} = (D_{13}D_{23} - D_{12}D_{33})/DD = \{(0{\cdot}51)(0{\cdot}51) - (1{\cdot}32)(1{\cdot}48)\}/22{\cdot}31 = -0{\cdot}0759$$

$$d_{13} = (D_{12}D_{23} - D_{22}D_{13})/DD = \{(1{\cdot}32)(0{\cdot}51) - (2{\cdot}19)(0{\cdot}51)\}/22{\cdot}31 = -0{\cdot}0199$$

$$d_{23} = (D_{12}D_{13} - D_{11}D_{23})/DD = \{(1{\cdot}32)(0{\cdot}51) - (8{\cdot}31)(0{\cdot}51)\}/22{\cdot}31 = -0{\cdot}1598$$

Hence, the bending compliance matrix is

$$d = \boxed{\begin{matrix} 0{\cdot}1336 & -0{\cdot}0759 & -0{\cdot}0199 \\ -0{\cdot}0759 & 0{\cdot}5396 & -0{\cdot}1598 \\ -0{\cdot}0199 & -0{\cdot}1598 & 0{\cdot}7376 \end{matrix}}\ 1/(\mathrm{kN\,mm})$$

The laminate equivalent elastic constants are obtained from eqns (5.6) and (5.7). The laminate thickness t for this example is $1{\cdot}0\ (= 8 \times 0{\cdot}125)$ mm.

For the membrane mode, substituting the appropriate values into Eqns (5.6), we get

$$E_x = 1/(ta_{11}) = 1/(1{\cdot}0 \times 0{\cdot}0185) = 54{\cdot}1\ \mathrm{kN/mm^2}$$

$$E_y = 1/(ta_{22}) = 1/(1{\cdot}0 \times 0{\cdot}0185) = 54{\cdot}1\ \mathrm{kN/mm^2}$$

$$G_{xy} = 1/(ta_{33}) = 1/(1{\cdot}0 \times 0{\cdot}0483) = 20{\cdot}7\ \mathrm{kN/mm^2}$$

$$\nu_{xy} = -a_{12}/a_{11} = -(-0{\cdot}0058)/0{\cdot}0185 = 0{\cdot}31$$

$$\nu_{yx} = -a_{12}/a_{22} = -(-0{\cdot}0058)/0{\cdot}0185 = 0{\cdot}31$$

$$m_x = -a_{13}/a_{11} = 0/0{\cdot}0185 = 0$$

$$m_y = -a_{23}/a_{22} = 0/0{\cdot}0185 = 0$$

For the bending mode, substituting the appropriate values into eqns (5.7), we get

$$E_x = 12/(t^3 d_{11}) = 12/(1{\cdot}0^3 \times 0{\cdot}1336) = 89{\cdot}8\ \mathrm{kN/mm^2}$$
$$E_y = 12/(t^3 d_{22}) = 12/(1{\cdot}0^3 \times 0{\cdot}5396) = 22{\cdot}2\ \mathrm{kN/mm^2}$$
$$G_{xy} = 12/(t^3 d_{33}) = 12/(1{\cdot}0^3 \times 0{\cdot}7376) = 16{\cdot}3\ \mathrm{kN/mm^2}$$
$$\nu_{xy} = -d_{12}/d_{11} = -(-0{\cdot}0759)/0{\cdot}1336 = 0{\cdot}57$$
$$\nu_{yx} = -d_{12}/d_{22} = -(-0{\cdot}0759)/0{\cdot}5396 = 0{\cdot}14$$
$$m_x = -d_{13}/d_{11} = -(-0{\cdot}0199)/0{\cdot}1336 = 0{\cdot}15$$
$$m_y = -d_{23}/d_{22} = -(-0{\cdot}1598)/0{\cdot}5396 = 0{\cdot}30$$

Thus, the stiffness characteristics of a symmetric quasi-isotropic laminate, observed from this example, are:

(1) There are no membrane-bending coupling effects due to the laminate symmetry about its midplane.

(2) There is only membrane isotropy; the conditions for which the laminate membrane equivalent elastic constants must satisfy simultaneously for membrane isotropy are:

$$E_x = E_y$$
$$G_{xy} = E_x/[2(1+\nu_{xy})]$$
$$m_x = m_y = 0$$

The above three conditions are satisfied in this case, as in the membrane mode:

$$E_x = E_y = 54{\cdot}1\ \text{kN/mm}^2$$

The value for $G_{xy} = 20{\cdot}7\ \text{kN/mm}^2$ and

$$E_x/[2(1+\nu_{xy})] = 54{\cdot}1/[2(1+0{\cdot}31)]$$
$$= 20{\cdot}6\ \text{kN/mm}^2 = G_{xy}\ \text{(allowing for round-off)}$$

Also, $m_x = m_y = 0$ in this case, implying that there are no shear couplings present. Hence, all the three isotropy conditions are satisfied simultaneously.

Alternatively, the laminate membrane isotropy conditions can be expressed in terms of the extensional stiffness terms:

$$A_{11} = A_{22}$$
$$A_{33} = (A_{11} - A_{12})/2$$
$$A_{13} = A_{23} = 0$$

The above three conditions are satisfied simultaneously in this case as $A_{11} = A_{22} = 18{\cdot}5\ \text{kN/mm}$; $A_{33} = 20{\cdot}7\ \text{kN/mm}$ and $(A_{11} - A_{12})/2 = (59{\cdot}9 - 18{\cdot}7)/2 = 20{\cdot}6\ \text{kN/mm}$ (calculated $A_{33} = 20{\cdot}7\ \text{kN/mm}$, allowing for round-off); and $A_{13} = A_{23} = 0$.

(3) There is no bending isotropy or orthotropy; the bending isotropy conditions are not satisfied in this case, and there are bend–twist coupling terms present, thereby destroying the bending orthotropy behaviour.

We thus see that a quasi-isotropic laminate will not behave completely in an isotropic fashion. It is only the membrane stiffness which exhibits isotropic behaviour, but in the bending mode there are no isotropy characteristics.

Homework #2

Consider the cross-ply laminate $(0/90)_s$ subjected to a positive moment intensity $M_x = 10$ Nmm/mm. Where the plies are unidirectional high strength carbon/epoxy, of 0.125mm thickness, and have the following elastic properties as ; $E_1 = 140,\ E_2 = 10,\ G_{12} = 5kN/mm^2,\ \nu_{12} = 0.3$

The ply reduced strengths for this material are;

$$X_t = 1500,\ X_c = 1200,\ Y_t = 50,\ Y_c = 250,\ S = 70N/mm^2$$

The symmetric laminate configuration is shown in Fig. 5.14.
Find the FPF mode and load.

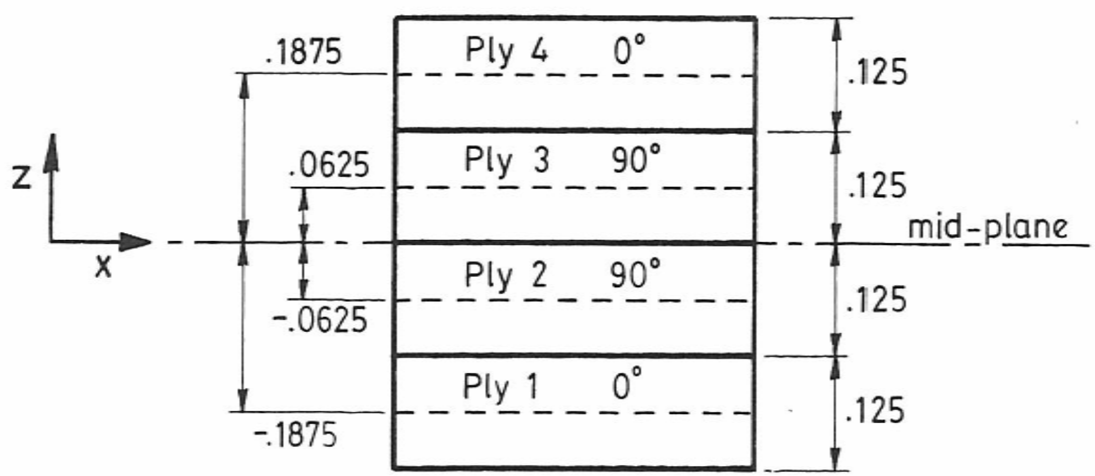

Fig. 5.14. Ordinate values (mm): $(0/90)_s$ laminate.

$$Q = \boxed{\begin{matrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \ \mathrm{kN/mm^2}$$

The transformed reduced stiffnesses for a ply angle of 0° have been previously determined in Example 5.2 where it was observed that the transformed reduced stiffnesses and the corresponding reduced stiffnesses for a 0° ply angle are identical. Thus,

$$(\bar{Q})_{0^\circ} = \boxed{\begin{matrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \ \mathrm{kN/mm^2}$$

The transformed reduced stiffness terms for a ply angle of 90° are obtained from eqn (5.2) with

$$\begin{array}{lll}
m = \cos 90^\circ = 0; & m^2 = 0; & m^4 = 0 \\
n = \sin 90^\circ = 1; & n^2 = 1; & n^4 = 1 \\
m^2n^2 = 0; & 2m^2n^2 = 0; & 4m^2n^2 = 0 \\
m^2 - n^2 = -1 & & \\
m^4 + n^4 = 1 & & \\
m^3n = 0; & mn^3 = 0; & \\
m^3n - mn^3 = 0; & 2(m^3n - mn^3) = 0 & \\
mn^3 - m^3n = 0; & 2(mn^3 - m^3n) = 0 &
\end{array}$$

Substituting the appropriate trigonometric values and the reduced stiffness values into eqn (5.2), we get

$$\bar{Q}_{11} = m^4Q_{11} + n^4Q_{22} + 2m^2n^2Q_{12} + 4m^2n^2Q_{33}$$
$$= (0 \times 140{\cdot}9) + (1 \times 10{\cdot}1) + (0 \times 3{\cdot}0) + (0 \times 5{\cdot}0) = 10{\cdot}1\ \text{kN/mm}^2 = Q_{22}$$

$$\bar{Q}_{22} = n^4Q_{11} + m^4Q_{22} + 2m^2n^2Q_{12} + 4m^2n^2Q_{33}$$
$$= (1 \times 140{\cdot}9) + (0 \times 10{\cdot}1) + (0 \times 3{\cdot}0) + (0 \times 5{\cdot}0) = 140{\cdot}9\ \text{kN/mm}^2 = Q_{11}$$

$$\bar{Q}_{33} = m^2n^2Q_{11} + m^2n^2Q_{22} - 2m^2n^2Q_{12} + (m^2 - n^2)^2Q_{33}$$
$$= (0 \times 140{\cdot}9) + (0 \times 10{\cdot}1) - (0 \times 3{\cdot}0) + ([-1]^2 \times 5{\cdot}0) = 5{\cdot}0\ \text{kN/mm}^2 = Q_{33}$$

$$\bar{Q}_{12} = m^2n^2Q_{11} + m^2n^2Q_{22} + (m^4 + n^4)Q_{12} - 4m^2n^2Q_{33}$$
$$= (0 \times 140{\cdot}9) + (0 \times 10{\cdot}1) + (1 \times 3{\cdot}0) - (0 \times 5{\cdot}0) = 3{\cdot}0\ \text{kN/mm}^2 = Q_{12}$$

$$\bar{Q}_{13} = m^3nQ_{11} - mn^3Q_{22} + (mn^3 - m^3n)Q_{12} + 2(mn^3 - m^3n)Q_{33}$$
$$= (0 \times 140{\cdot}9) - (0 \times 10{\cdot}1) + (0 \times 3{\cdot}0) + (0 \times 5{\cdot}0) = 0 = Q_{13}$$

$$\bar{Q}_{23} = mn^3Q_{11} - m^3nQ_{22} + (m^3n - mn^3)Q_{12} + 2(m^3n - mn^3)Q_{33}$$
$$= (0 \times 140{\cdot}9) - (0 \times 10{\cdot}1) + (0 \times 3{\cdot}0) + (0 \times 5{\cdot}0) = 0 = Q_{23}$$

Table 5.11

$\bar{Q}_{ij}$ Values (kN/mm^2) for a Symmetric MOPL: Cross-ply $(0/90)_s$

Ply	$\theta°$	$\bar{Q}_{11}$	$\bar{Q}_{22}$	$\bar{Q}_{33}$	$\bar{Q}_{12}$	$\bar{Q}_{13}$	$\bar{Q}_{23}$
1	0	140·9	10·1	5·0	3·0	0	0
2	90	10·1	140·9	5·0	3·0	0	0
3	90	10·1	140·9	5·0	3·0	0	0
4	0	140·9	10·1	5·0	3·0	0	0

Thus, the transformed reduced stiffness terms for the 90° ply are

$$(\bar{Q})_{90°} = \begin{bmatrix} 10\cdot1 & 3\cdot0 & 0 \\ 3\cdot0 & 140\cdot9 & 0 \\ 0 & 0 & 5\cdot0 \end{bmatrix} \text{kN/mm}^2$$

It is seen, therefore, that, for a 90° fibre orientation, the cross-stiffnesses Q_{11} and Q_{22} for a 0° ply angle change over in the transformed reduced stiffnesses of a 90° ply angle, and the remaining terms Q_{12} and Q_{33} remain the same.

The transformed reduced stiffness terms for the 0° and 90° plies in the four-ply laminate are given in Table 5.11.

Since the laminate is symmetric, then all the coupling terms $B_{ij} = 0$, so we only need to calculate the extensional and bending, A_{ij} and D_{ij}, stiffness terms. The ply thickness and ply centroidal values are obtained from Fig. 5.14, and the required ordinate values, t_p and $(t_p \bar{z}_p^2 + t_p^3/12)$, for each ply are given in Table 5.12.

We are now in a position to determine the extensional and bending stiffnesses, A_{ij} and D_{ij}, as defined in eqn (5.3).

A_{ij} *terms*

$$A_{ij} = \sum_{p=1}^{N} t_p (\bar{Q}_{ij})_p$$

From Table 5.12 we get the t_p term, the ply thickness value for each ply, and from Table 5.11 we get the $\bar{Q}_{ij}$ term, the transformed reduced stiffness value for each ply. We use these values for the plies in the lower half of the

Table 5.12
Ordinate Values for a Symmetric MOPL: Cross-ply $(0/90)_s$ (Basic dimensions: mm.)

Ply	$\theta°$	t_p	$\bar{z}_p$	$(t_p \bar{z}_p^2 + t_p^3/12)$
1	0	0·125	−0·187 5	0·004 56
2	90	0·125	−0·062 5	0·000 65
3	90	0·125	0·062 5	0·000 65
4	0	0·125	0·187 5	0·004 56

laminate, in this case Plies 1 and 2, in the above expression for A_{ij} in turn and then double the result to obtain the total laminate extensional stiffness. Thus,

$$A_{11} = 2\{(0{\cdot}125 \times 140{\cdot}9)_{\text{Ply 1}} + (0{\cdot}125 \times 10{\cdot}1)_{\text{Ply 2}}\} = 37{\cdot}8\ \text{kN/mm}$$

$$A_{22} = 2\{(0{\cdot}125 \times 10{\cdot}1)_{\text{Ply 1}} + (0{\cdot}125 \times 140{\cdot}9)_{\text{Ply 2}}\} = 37{\cdot}8\ \text{kN/mm}$$

$$A_{33} = 2\{(0{\cdot}125 \times 5{\cdot}0)_{\text{Ply 1}} + (0{\cdot}125 \times 5{\cdot}0)_{\text{Ply 2}}\} = 2{\cdot}5\ \text{kN/mm}$$

$$A_{12} = 2\{(0{\cdot}125 \times 3{\cdot}0)_{\text{Ply 1}} + (0{\cdot}125 \times 3{\cdot}0)_{\text{Ply 2}}\} = 1{\cdot}5\ \text{kN/mm}$$

$$A_{13} = A_{23} = 0 \quad \text{as} \quad \bar{Q}_{13} = \bar{Q}_{23} = 0 \text{ for the } 0^\circ \text{ and } 90^\circ \text{ plies}$$

The extensional stiffness terms A_{ij} may be written in a boxed matrix notation form:

$$A = \boxed{\begin{matrix} 37{\cdot}8 & 1{\cdot}5 & 0 \\ 1{\cdot}5 & 37{\cdot}8 & 0 \\ 0 & 0 & 2{\cdot}5 \end{matrix}}\ \text{kN/mm}$$

B_{ij} *terms*

All these coupling terms will be zero, as noted earlier in Example 5.1, because the laminate is symmetric about its midplane.

D$_{ij}$ *terms*

$$D_{ij} = \sum_{p=1}^{N} (t_p \bar{z}_p^2 + t_p^3/12)(\bar{Q}_{ij})_p$$

From Table 5.12 we get the $(t_p \bar{z}_p^2 + t_p^3/12)$ term for each ply, and from Table 5.11 we get the $\bar{Q}_{ij}$ term, the transformed reduced stiffness value for each ply. We use these values for the plies in the lower half of the laminate, in this case Plies 1 and 2, in the above expression for D_{ij} in turn and then double the result to obtain the total laminate bending stiffness.

$$D_{11} = 2\{(0{\cdot}004\,56 \times 140{\cdot}9)_{\text{Ply 1}} + 0{\cdot}004\,56 \times 10{\cdot}1)_{\text{Ply 2}}\} = 1{\cdot}2981 \text{ kN mm}$$

$$D_{22} = 2\{(0{\cdot}004\,56 \times 10{\cdot}1)_{\text{Ply 1}} + (0{\cdot}004\,56 \times 140{\cdot}9)_{\text{Ply 2}}\} = 0{\cdot}2753 \text{ kN mm}$$

$$D_{33} = 2\{(0{\cdot}004\,56 \times 5{\cdot}0)_{\text{Ply 1}} + (0{\cdot}004\,56 \times 5{\cdot}0)_{\text{Ply 2}}\} = 0{\cdot}0521 \text{ kN mm}$$

$$D_{12} = 2\{(0{\cdot}004\,56 \times 3{\cdot}0)_{\text{Ply 1}} + (0{\cdot}004\,56 \times 3{\cdot}0)_{\text{Ply 2}}\} = 0{\cdot}0313 \text{ kN mm}$$

$$D_{13} = D_{23} = 0 \text{ as } \bar{Q}_{13} = \bar{Q}_{23} = 0 \text{ for the } 0° \text{ and } 90° \text{ plies}$$

The bending stiffness terms written in a boxed matrix form are given as

$$D = \boxed{\begin{matrix} 1{\cdot}2981 & 0{\cdot}0313 & 0 \\ 0{\cdot}0313 & 0{\cdot}2753 & 0 \\ 0 & 0 & 0{\cdot}0521 \end{matrix}} \text{ kN mm}$$

The laminate compliances can now be obtained from the laminate stiffnesses. Since the laminate is uncoupled, and since the extensional and bending stiffness matrices are partially populated, then the membrane and bending compliances can be obtained from eqn (5.5).

Substituting A_{ij} for R_{ij} and a_{ij} for r_{ij} in eqn (5.5), we get

$$A_{11} = 37{\cdot}8$$
$$A_{22} = 37{\cdot}8$$
$$A_{33} = 2{\cdot}5$$
$$A_{12} = 1{\cdot}5$$

and

$$AA = A_{11}A_{22} - A_{12}^2 = (37{\cdot}8)(37{\cdot}8) - 1{\cdot}5^2 = 1426{\cdot}6$$

$$a_{11} = A_{22}/AA = 37{\cdot}8/1426{\cdot}6 = 0{\cdot}0265$$

$$a_{22} = A_{11}/AA = 37{\cdot}8/1426{\cdot}6 = 0{\cdot}0265$$

$$a_{33} = 1/A_{33} = 1/2{\cdot}5 = 0{\cdot}4000$$

$$a_{12} = -A_{12}/AA = -1{\cdot}5/1426{\cdot}6 = -0{\cdot}0011$$

Hence, the extensional compliance matrix is

$$a = \boxed{\begin{matrix} 0{\cdot}0265 & -0{\cdot}0011 & 0 \\ -0{\cdot}0011 & 0{\cdot}0265 & 0 \\ 0 & 0 & 0{\cdot}4000 \end{matrix}} \ 1/(\mathrm{kN/mm})$$

The bending compliance values are obtained from the bending stiffness values, using eqn (5.5), and substituting D_{ij} for R_{ij} and d_{ij} for r_{ij} in eqn (5.5), we get

$$D_{11} = 1{\cdot}2981$$
$$D_{22} = 0{\cdot}2753$$
$$D_{33} = 0{\cdot}0521$$
$$D_{12} = 0{\cdot}0313$$

and

$$DD = D_{11}D_{22} - D_{12}^2 = (1{\cdot}2981)(0{\cdot}2753) - (0{\cdot}0313)^2 = 0{\cdot}3564$$

$$d_{11} = D_{22}/DD = 0{\cdot}2753/0{\cdot}3564 = 0{\cdot}77$$

$$d_{22} = D_{11}/DD = 1{\cdot}2981/0{\cdot}3564 = 3{\cdot}64$$

$$d_{33} = 1/D_{33} = 1/0{\cdot}0521 = 19{\cdot}19$$

$$d_{12} = -D_{12}/DD = -0{\cdot}0313/0{\cdot}3564 = -0{\cdot}09$$

Hence, the bending compliance matrix is

$$d = \begin{vmatrix} 0{\cdot}77 & -0{\cdot}09 & 0 \\ -0{\cdot}09 & 3{\cdot}64 & 0 \\ 0 & 0 & 19{\cdot}19 \end{vmatrix} \ 1/(\text{kN mm})$$

The laminate equivalent elastic constants are obtained from eqns (5.6) and (5.7). The laminate thickness t for this example is 0·5 (=4 × 0·125) mm.

For the membrane mode, substituting the appropriate values into eqns (5.6), we get

$$
\begin{aligned}
E_x &= 1/(ta_{11}) = 1/(0{\cdot}5 \times 0{\cdot}0265) = 75{\cdot}5\ \mathrm{kN/mm^2} \\
E_y &= 1/(ta_{22}) = 1/(0{\cdot}5 \times 0{\cdot}0265) = 75{\cdot}5\ \mathrm{kN/mm^2} \\
G_{xy} &= 1/(ta_{33}) = 1/(0{\cdot}5 \times 0{\cdot}4000) = 5{\cdot}0\ \mathrm{kN/mm^2} \\
\nu_{xy} &= -a_{12}/a_{11} = -(-0{\cdot}0011)/0{\cdot}0265 = 0{\cdot}04 \\
\nu_{yx} &= -a_{12}/a_{22} = -(-0{\cdot}0011)/0{\cdot}0265 = 0{\cdot}04 \\
m_x &= -a_{13}/a_{11} = 0/0{\cdot}0265 = 0 \\
m_y &= -a_{23}/a_{22} = 0/0{\cdot}0265 = 0
\end{aligned}
$$

For the bending mode, substituting the appropriate values into Eqns (5.7), we get

$$
\begin{aligned}
E_x &= 12/(t^3 d_{11}) = 12/(0{\cdot}5^3 \times 0{\cdot}77) = 124{\cdot}7\ \mathrm{kN/mm^2} \\
E_y &= 12/(t^3 d_{22}) = 12/(0{\cdot}5^3 \times 3{\cdot}64) = 26{\cdot}4\ \mathrm{kN/mm^2} \\
G_{xy} &= 12/(t^3 d_{33}) = 12/(0{\cdot}5^3 \times 19{\cdot}19) = 5{\cdot}0\ \mathrm{kN/mm^2} \\
\nu_{xy} &= -d_{12}/d_{11} = -(-0{\cdot}09)/0{\cdot}77 = 0{\cdot}12 \\
\nu_{yx} &= -d_{12}/d_{22} = -(-0{\cdot}09)/3{\cdot}64 = 0{\cdot}02 \\
m_x &= -d_{13}/d_{11} = 0/0{\cdot}77 = 0 \\
m_y &= -d_{23}/d_{22} = 0/3{\cdot}64 = 0
\end{aligned}
$$

Using the bending compliances, converted into units of N and mm, in eqn (6.6), and with $M_x = 10$ N mm/mm, $M_y = M_{xy} = 0$, we get

	10	0	0	
k_x	0·77	−0·09	0	
k_y	−0·09	3·64	0	$\times 10^{-3}$
k_{xy}	0	0	19·19	

giving

$$
\begin{aligned}
k_x &= 7700 \times 10^{-6} \quad 1/\mathrm{mm} \\
k_y &= -900 \times 10^{-6} \quad 1/\mathrm{mm} \\
k_{xy} &= 0
\end{aligned}
$$

The strain induced by the curvatures is $-zk$, where z is the distance from the midplane. So, on the bottom surface where $z = -0{\cdot}25$ mm,

$$
\begin{aligned}
e_x &= -(-0{\cdot}25) \times 7700 \times 10^{-6} = 1925 \times 10^{-6} \\
e_y &= -(-0{\cdot}25) \times -900 \times 10^{-6} = -225 \times 10^{-6} \\
e_{xy} &= -(-0{\cdot}25) \times 0 = 0
\end{aligned}
$$

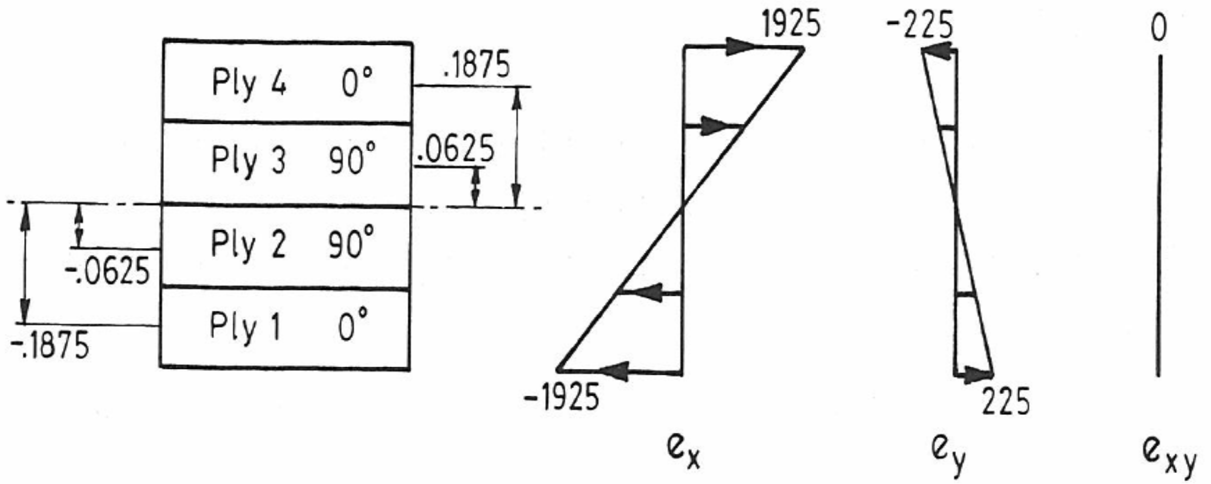

Fig. 6.5. Bending reference axes strains ($\times 10^{-6}$) through laminate thickness.

and on the top surface where $z = 0{\cdot}25$,

$$e_x = -(0{\cdot}25) \times 7700 \times 10^{-6} = -1925 \times 10^{-6}$$

$$e_y = -(0{\cdot}25) \times -900 \times 10^{-6} = 225 \times 10^{-6}$$

$$e_{xy} = -(0{\cdot}25) \times 0 = 0$$

This is a linear strain distribution through the laminate thickness in all the four plies, as shown in Fig. 6.5.

From Fig. 6.5 we get the ordinate distances for each ply centroid and hence the strains at the ply centroids:

Ply 1 0° $z = -0{\cdot}1875$:

$$e_x = -(-0{\cdot}1875) \times 7700 \times 10^{-6} = 1444 \times 10^{-6}$$
$$e_y = -(-0{\cdot}1875) \times -900 \times 10^{-6} = -169 \times 10^{-6}$$
$$e_{xy} = -(-0{\cdot}1875) \times 0 = 0$$

Ply 2 90° $z = -0{\cdot}0625$:

$$e_x = -(-0{\cdot}0625) \times 7700 \times 10^{-6} = 481 \times 10^{-6}$$
$$e_y = -(-0{\cdot}0625) \times -900 \times 10^{-6} = -56 \times 10^{-6}$$
$$e_{xy} = -(-0{\cdot}0625) \times 0 = 0$$

Ply 3 90° $z = 0{\cdot}0625$:

$$e_x = -(0{\cdot}0625) \times 7700 \times 10^{-6} = -481 \times 10^{-6}$$
$$e_y = -(0{\cdot}0625) \times -900 \times 10^{-6} = 56 \times 10^{-6}$$
$$e_{xy} = -(0{\cdot}0625) \times 0 = 0$$

Ply 4 0° $z = 0{\cdot}1875$:

$$e_x = -(0{\cdot}1875) \times 7700 \times 10^{-6} = -1444 \times 10^{-6}$$
$$e_y = -(0{\cdot}1875) \times -900 \times 10^{-6} = 169 \times 10^{-6}$$
$$e_{xy} = -(0{\cdot}1875) \times 0 = 0$$

We then go back to the individual plies and transform the strains into the material axes.

Ply 1 at 0°

We can transform strains from the reference axes to the material axes for a 0° ply angle by inspection, as we saw in Example 6.1; effectively, there is no change in the strain directions.

$$e_1 = 1444 \times 10^{-6}$$
$$e_2 = -169 \times 10^{-6}$$
$$e_{12} = 0$$

Ply 2 at 90°

Again, by inspection from the results in Example 6.1, for a ply angle of 90° the reference axes direct strain values get transposed and the shear value changes signs.

$$e_1 = -56 \times 10^{-6}$$
$$e_2 = 481 \times 10^{-6}$$
$$e_{12} = 0$$

The ply material strains in Plies 3 and 4 will have values of opposing signs to their corresponding ones in Plies 2 and 1, respectively, as evident by inspection of Fig. 6.5.

Next, we make use of eqn (6.10) to determine the ply stresses in the material axes, having got the ply strains in the material axes. Substituting the values of the ply reduced stiffnesses given in the beginning of this example (and converting the units to N and mm), and the ply strains in the material axes, for each ply, into eqn (6.10), we get:

Ply 1 at 0°

	1444	−169	0
f_1	140·9	3·0	0
f_2	3·0	10·1	0
f_{12}	0	0	5·0

$\times 10^{-6} \times 10^{3}\ \mathrm{N/mm^2}$

giving

$$f_1 = 203\ \mathrm{N/mm^2}$$
$$f_2 = 3\ \mathrm{N/mm^2}$$
$$f_{12} = 0$$

Using the maximum stress failure criterion:

$$\text{F.I. } 1 = 203/1500 = 0{\cdot}14$$
$$\text{F.I. } 2 = 3/50 = 0{\cdot}06$$
$$\text{F.I. } 12 = 0$$

Ply 2 at 90°

$$\begin{array}{c|ccc}
 & -56 & 481 & 0 \\
\hline
f_1 & 140{\cdot}9 & 3{\cdot}1 & 0 \\
f_2 & 3{\cdot}1 & 10{\cdot}1 & 0 \\
f_{12} & 0 & 0 & 5{\cdot}0
\end{array} \times 10^{-6} \times 10^{3}\ \text{N/mm}^2$$

giving

$$f_1 = -6\ \text{N/mm}^2$$

$$f_2 = 5\ \text{N/mm}^2$$

$$f_{12} = 0$$

Using the maximum stress failure criterion:

$$\text{F.I. } 1 = 6/1200 = 0{\cdot}01$$

$$\text{F.I. } 2 = 5/50 = 0{\cdot}10$$

$$\text{F.I. } 12 = 0$$

Ply 3 at 90°

The ply stresses will be of opposing signs to those in the corresponding ply,

$$f_1 = 6\ \mathrm{N/mm^2}$$
$$f_2 = -5\ \mathrm{N/mm^2}$$
$$f_{12} = 0$$

Using the maximum stress failure criterion:

$$\text{F.I. } 1 = 6/1500 = 0{\cdot}01$$
$$\text{F.I. } 2 = 5/250 = 0{\cdot}02$$
$$\text{F.I. } 12 = 0$$

Ply 4 at 0°
The ply stresses will be of opposing signs to those in the corresponding ply, Ply 1:

$$f_1 = -203\ \mathrm{N/mm^2}$$
$$f_2 = -3\ \mathrm{N/mm^2}$$
$$f_{12} = 0$$

Using the maximum stress failure criterion:

$$\text{F.I. } 1 = 203/1200 = 0{\cdot}17$$
$$\text{F.I. } 2 = 3/250 = 0{\cdot}01$$
$$\text{F.I. } 12 = 0$$

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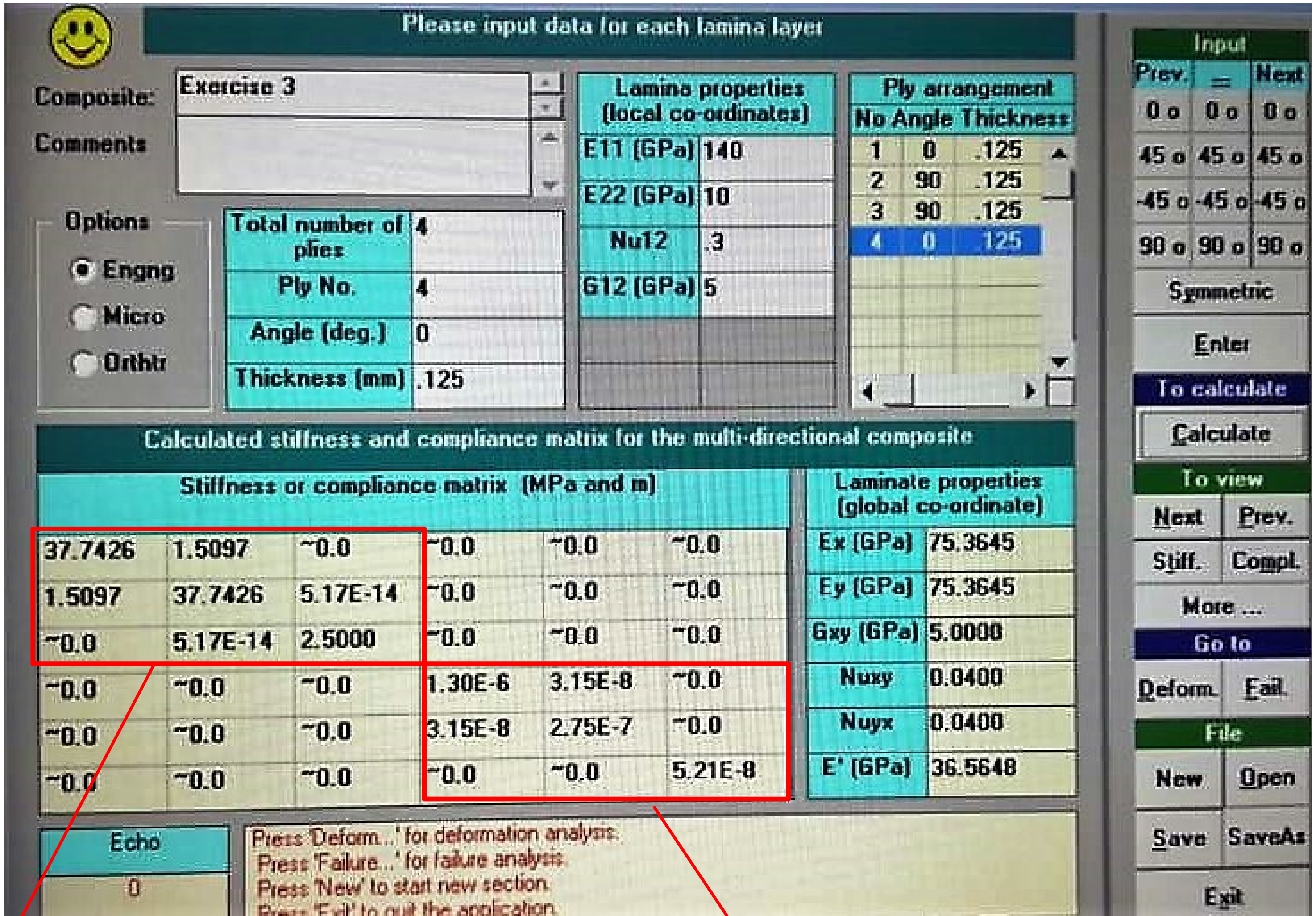
Table 6.3
Ply Stresses (N/mm^2) and Failure Indices: $(0/90)_s$ with $M_x = 10$ N mm/mm

Ply	$\theta°$	f_1	f_2	f_{12}	*F.I.*1	*F.I.*2	*F.I.*12	*MOF*
1	0	203	3	0	0·14	0·06	0	LT
2	90	−6	5	0	0·01	0·10	0	TT
3	90	6	−5	0	0·01	0·02	0	TC
4	0	−203	−3	0	**0·17**	0·01	0	LC

A summary of the stress values (in the material axes) and their corresponding F.I. for all the four plies in this cross-ply laminate configuration $(0/90)_s$ for a positive moment load $M_x = 10$ N mm/mm is given in Table 6.3.

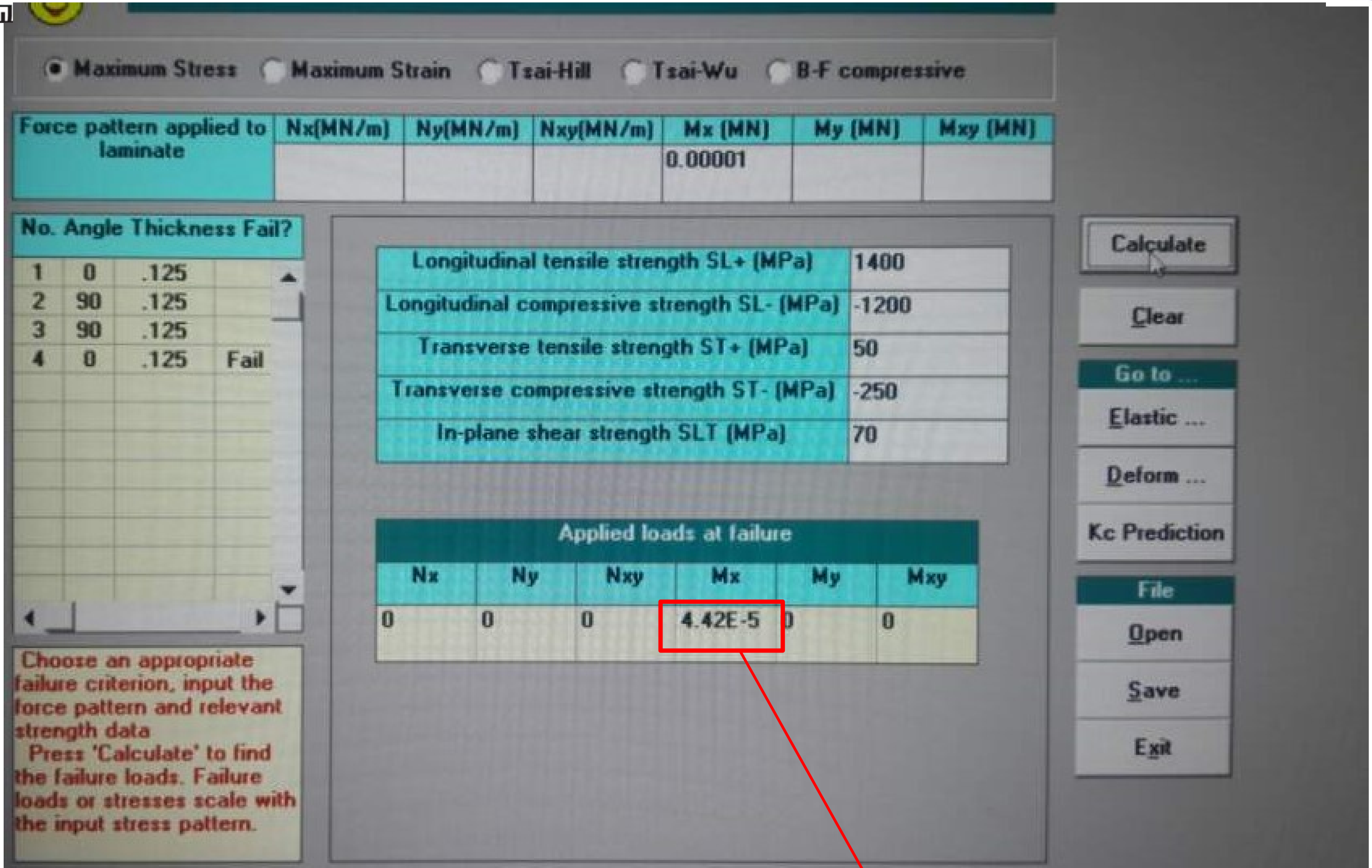
From the summary of results in Table 6.3 it is seen that the maximum failure index occurs in the top ply, the 0° ply, Ply 4, in the longitudinal direction, F.I. $1 = 0{\cdot}17$, in the compression mode as the stress in this direction f_1 is compressive. The correspondingly placed bottom ply, Ply 1, has stress values of opposing signs to Ply 4, and the ply strength analyses give different F.I. because of the different tensile and compressive strengths. Since the compressive strength in the fibre direction is lower than its tensile strength, Ply 4 with a compressive fibre stress is the critical one, rather than Ply 1 with a tensile fibre stress.

So, with the applied positive moment $M_x = 10$ N mm/mm, no ply failure has yet occurred. Therefore, the load can be increased by a factor of $1/0{\cdot}17 = 5{\cdot}88$ before FPF is predicted by the maximum stress theory. Hence, the FPF load is $10 \times 5{\cdot}88 = 58{\cdot}8$ N mm/mm and this will occur in the top ply, Ply 4: 0°, in the LC MOF.



A_{ij}: to match unit with solution – multiplying by 10^3

D_{ij}: to match unit with solution – multiplying by 10^9 (kN-mm)



This failure moment is at top skin of the 4th ply.
The hand calculation result (5.88×10^{-5} MN-mm/mm) is at mid of the 4th ply (if calculating at the top skin; $5.88\times10^{-5}\times(0.1875/0.25)=4.42\times10^{-5}$

Thermal Residual Stresses Calculation Example

Calculate laminate stiffnesses: A_{ij}, B_{ij}, D_{ij}

↓

Determine ply free thermal strains: $(e_1^T)_p$, $(e_2^T)_p$, 0

↓

Transform thermal strains to reference axes:
$(e_x^T)_p$, $(e_y^T)_p$, $(e_{xy}^T)_p$

↓

Evaluate ply equivalent free thermal loads:
$(N_x^T)_p$, $(N_y^T)_p$, $(N_{xy}^T)_p$
$(M_x^T)_p$, $(M_y^T)_p$, $(M_{xy}^T)_p$

↓

Obtain laminate equivalent free thermal loads:
N_x^T, N_y^T, N_{xy}^T
M_x^T, M_y^T, M_{xy}^T

↓

Solve for laminate common strains in reference axes x-y

↓

Calculate ply residual strains in reference axes x-y:
e_x^R, e_y^R, e_{xy}^R

↓

Transform ply residual strains to material axes 1-2:
e_1^R, e_2^R, e_{12}^R

↓

Convert ply residual strains to stresses: f_1^R, f_2^R, f_{12}^R

Fig. 7.15. Thermal residual stresses determination procedure.

조선대학교

Example 7.2 ★

Homework #3

Determine the thermal residual stresses in a symmetric cross-ply laminate $(0/90)_s$ made from high strength carbon/epoxy unidirectional plies. The ply is 0·125 mm thick and the elastic and strength properties are the same as in Example 6.1:

$$E_1 = 140, \qquad E_2 = 10, \qquad G_{12} = 5\ \text{kN/mm}^2; \qquad \nu_{12} = 0{\cdot}3$$

$$X_t = 1500; \qquad X_c = 1200; \qquad Y_t = 50; \qquad Y_c = 250; \qquad S = 70\ \text{N/mm}^2$$

The coefficients of thermal expansion are:

$$\alpha_1 = -0{\cdot}3 \times 10^{-6}\ \text{strain/°C}$$

$$\alpha_2 = 28 \times 10^{-6}\ \text{strain/°C}$$

(Note that the coefficient of thermal expansion in the fibre direction is a negative value.) Assume for the moment that the stress free temperature is the same as the curing temperature of 125°C and the laminate is then cooled to an ambient temperature of 25°C. Assume a constant temperature distribution through the laminate thickness.

The laminate therefore experiences a temperature drop of $\Delta T = 25° - 125° = -100°\text{C}$ and this is assumed to be constant in all the four plies as shown in Fig. 7.16.

Ref.: Constant temperature distribution and constant ply thickness

$$
\begin{array}{c|ccc:ccc}
 & e_x^o & e_y^o & e_{xy}^o & k_x & k_y & k_{xy} \\
\hline
N_x^T & A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\
N_y^T & A_{12} & A_{22} & A_{23} & B_{12} & B_{22} & B_{23} \\
N_{xy}^T & A_{13} & A_{23} & A_{33} & B_{13} & B_{23} & B_{33} \\
\hdashline
M_x^T & B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\
M_y^T & B_{12} & B_{22} & B_{23} & D_{12} & D_{22} & D_{23} \\
M_{xy}^T & B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33}
\end{array}
\qquad (7.24)
$$

where

$$A_{ij} = \sum_{p=1}^{N} t_p(\bar{Q}_{ij})_p$$

$$B_{ij} = \sum_{p=1}^{N} -t_p\bar{z}_p(\bar{Q}_{ij})_p$$

$$D_{ij} = \sum_{p=1}^{N} (t_p\bar{z}_p^2 + t_p^3/12)(\bar{Q}_{ij})_p$$

and N^T is given by eqn (7.20):

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = \sum_{p=1}^{N} (t_p) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_p \begin{bmatrix} e_x^T \\ e_y^T \\ e_{xy}^T \end{bmatrix}_p$$

and M^T is given by eqn (7.22):

$$\begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix} = \sum_{p=1}^{N} -(t_p\bar{z}_p) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_p \begin{bmatrix} e_x^T \\ e_y^T \\ e_{xy}^T \end{bmatrix}_p$$

Having obtained the laminate common deformation values given by the above equation, the ply residual strains are then evaluated from eqn (7.8):

$$e^{\mathrm{R}} = e^{\mathrm{o}} - zk - e^{\mathrm{T}}$$

from which the ply residual stress can be determined by the ply stress–strain relationship:

$$f^{\mathrm{R}} = \bar{Q}e^{\mathrm{R}}$$

(Solution)

The laminate configuration is symmetric about its midplane and so all the coupling terms B_{ij} will be zero in eqn (7.24). Also, the temperature distribution through the laminate thickness is constant and by careful examination of eqn (7.22), all the equivalent free thermal moments M^T will also be zero (although this will be shown to be the case). Now, since the coupling terms and the free thermal moments are both zero, then we need only consider the membrane deformations and free thermal forces contributions from eqn (7.24). Hence, we only need the laminate membrane stiffnesses and compliances.

The ply reduced stiffnesses Q_{ij}, the transformed reduced stiffnesses for each ply angle $\bar{Q}_{ij}$, the membrane stiffnesses A_{ij} and the compliances a_{ij}, for this cross-ply laminate have already been determined in Example 6.1 and are

$$Q = \begin{bmatrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{bmatrix} \text{ kN/mm}^2$$

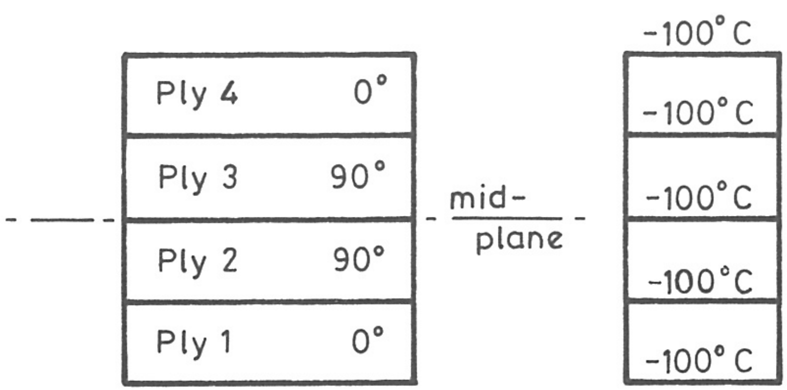


Fig. 7.16. Constant temperature distribution in $(0/90)_s$.

$$(\bar{Q})_{0^\circ} = \boxed{\begin{matrix} 140\cdot9 & 3\cdot0 & 0 \\ 3\cdot0 & 10\cdot1 & 0 \\ 0 & 0 & 5\cdot0 \end{matrix}} \ \mathrm{kN/mm^2}$$

$$(\bar{Q})_{90^\circ} = \boxed{\begin{matrix} 10\cdot1 & 3\cdot0 & 0 \\ 3\cdot0 & 140\cdot9 & 0 \\ 0 & 0 & 5\cdot0 \end{matrix}} \ \mathrm{kN/mm^2}$$

$$A = \boxed{\begin{matrix} 37\cdot8 & 1\cdot5 & 0 \\ 1\cdot5 & 37\cdot8 & 0 \\ 0 & 0 & 2\cdot5 \end{matrix}} \ \mathrm{kN/mm}$$

$$a = \boxed{\begin{matrix} 0\cdot0265 & -0\cdot0011 & 0 \\ -0\cdot0011 & 0\cdot0265 & 0 \\ 0 & 0 & 0\cdot4000 \end{matrix}} \ 1/(\mathrm{kN/mm})$$

Having obtained the laminate stiffnesses, the next step is to determine the free thermal strain in the material axes, using eqn (7.1):

$$e_1^{\mathrm{T}} = \alpha_1 \, \Delta T = -0{\cdot}3 \times 10^{-6} \times -100 = \quad 30 \quad \times 10^{-6}$$

$$e_2^{\mathrm{T}} = \alpha_2 \, \Delta T = \quad 28 \times 10^{-6} \times -100 = -2800 \times 10^{-6}$$

The free thermal strains in the material axes have now got to be transformed into the reference axes for each ply, using eqn (7.32):

	$(e_1^{\mathrm{T}})_{\mathrm{p}}$	$(e_2^{\mathrm{T}})_{\mathrm{p}}$	0
$(e_x^{\mathrm{T}})_{\mathrm{p}}$	m^2	n^2	$-mn$
$(e_y^{\mathrm{T}})_{\mathrm{p}}$	n^2	m^2	mn
$(e_{xy}^{\mathrm{T}})_{\mathrm{p}}$	$2mn$	$-2mn$	$m^2 - n^2$

Ply 1 at 0°

$$m = \cos 0° = 1 \quad \text{and} \quad n = \sin 0° = 0$$

$$m^2 = 1 \qquad n^2 = 0$$

$$mn = 0 \qquad 2mn = 0$$

$$m^2 - n^2 = 1$$

Substituting the appropriate trigonometric values and the material axes free thermal strain values into the above equation, we get

	30	−2800	0
$(e_x^T)_p$	1	0	0
$(e_y^T)_p$	0	1	0
$(e_{xy}^T)_p$	0	0	1

$\times 10^{-6}$.

giving

$$
\begin{aligned}
(e_x^T)_p &= 30 \times 10^{-6} \\
(e_y^T)_p &= -2800 \times 10^{-6} \\
(e_{xy}^T)_p &= 0
\end{aligned}
$$

Note that in the case when the ply angle is 0° the material axes strains will naturally be the same as the reference axes strains, as we have seen in previous examples. However, the detailed calculations have been repeated here, for the first occurence in this chapter, for completeness.

Ply 2 at 90°

The transformation of strains for a 90° ply angle can be obtained by inspection as we saw in previous examples, in which the orthogonal strains change directions when transposing from the material axes to the reference axes, and the shear strain changes sign (or can be alternatively computed by substituting appropriate trigonometric values with $m = \cos 90° = 0$ and $n = \sin 90° = 1$ in eqn (7.32)):

$$
\begin{aligned}
(e_x^T)_p &= -2800 \times 10^{-6} \\
(e_y^T)_p &= 30 \times 10^{-6} \\
(e_{xy}^T)_p &= 0
\end{aligned}
$$

There is no need to perform the calculations on Plies 3 and 4, because the laminate configuration and the temperature distribution about the laminate midplane is symmetric. Thus, the free thermal strains in the reference axes for Plies 1 and 4 with a ply angle of 0° will be the same, and the free thermal strains in the material axes for Plies 2 and 3 with a ply angle of 90° will be the same.

The equivalent free thermal loads can now be calculated for each ply. For the ply equivalent free thermal forces $(N^T)_p$ and with a constant temperature distribution, we make use of eqn (7.19):

$$
\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix}_p = (t_p) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_p \begin{bmatrix} e_x^T \\ e_y^T \\ e_{xy}^T \end{bmatrix}_p
$$

substituting the appropriate values into the above equation, we get:

Ply 1 at 0°

The free thermal strains in the reference axes for this ply, obtained earlier, are

$$(e_x^T)_p = 30 \times 10^{-6}$$
$$(e_y^T)_p = -2800 \times 10^{-6}$$
$$(e_{xy}^T)_p = 0$$

and the transformed reduced stiffnesses given at the beginning of this example are, converted to units of N and mm:

$$(\bar{Q})_{0°} = \boxed{\begin{matrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \times 10^3\ \mathrm{N/mm^2}$$

So the equivalent free thermal forces on this ply are

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix}_p = 0{\cdot}125 \begin{bmatrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 30 \\ -2800 \\ 0 \end{bmatrix} \times 10^3 \times 10^{-6}$$

giving

$$(N_x^T)_p = -0{\cdot}522\ \text{N/mm}$$
$$(N_y^T)_p = -3{\cdot}524\ \text{N/mm}$$
$$(N_{xy}^T)_p = 0$$

Ply 2 at 90°

The free thermal strain in the reference axes for this ply, obtained earlier, are

$$(e_x^T)_p = -2800 \times 10^{-6}$$
$$(e_y^T)_p = 30 \times 10^{-6}$$
$$(e_{xy}^T)_p = 0$$

and the transformed reduced stiffnesses given at the beginning of this example are, converted to units of N and mm:

$$(\bar{Q})_{90°} = \boxed{\begin{matrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \times 10^3\ \text{N/mm}^2$$

So the equivalent free thermal forces on this ply are

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix}_p = 0{\cdot}125 \begin{bmatrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -2800 \\ 30 \\ 0 \end{bmatrix} \times 10^3 \times 10^{-6}$$

giving

$$(N_x^T)_p = -3{\cdot}524 \text{ N/mm}$$
$$(N_y^T)_p = -0{\cdot}522 \text{ N/mm}$$
$$(N_{xy}^T)_p = 0$$

Due to the laminate configuration and temperature distribution symmetry about the midplane, we only need perform computations for one half of the laminate configuration, as the corresponding plies in the other half of the laminate configuration will have identical values. In this case then, the equivalent free thermal forces for Plies 1 and 4 with a ply angle of 0° will be the same, and the equivalent free thermal forces for Plies 2 and 3 with a ply angle of 90° will be the same.

The laminate equivalent free thermal forces is, then, the summation of all the corresponding ply values, given by eqn (7.20) in this case of constant temperature distribution. Because of the symmetry, we can sum all the corresponding values in one half of the laminate and then double the result to obtain the total values. Thus,

$$N_x^T = 2\{(-0{\cdot}522)_{\text{Ply 1}} + (-3{\cdot}524)_{\text{Ply 2}}\} = -8{\cdot}092 \text{ N/mm}$$
$$N_y^T = 2\{(-3{\cdot}524)_{\text{Ply 1}} + (-0{\cdot}522)_{\text{Ply 2}}\} = -8{\cdot}092 \text{ N/mm}$$
$$N_{xy}^T = 2\{(0)_{\text{Ply 1}} + (0)_{\text{Ply 2}}\} = 0$$

Before proceeding to determine the laminate common strains, let us first show that the equivalent free thermal moments are indeed zero as presumed, due to the symmetry of the laminate configuration and temperature distribution. From eqn (7.21) or (7.22), giving the free thermal moments, we see that we need the ordinate properties $(t_p \bar{z}_p)$ for each ply. The ordinate properties are shown in Fig. 7.17 and presented in Table 7.1.

The ply equivalent free thermal moments with a constant temperature distribution are obtained, using eqn (7.21):

$$\begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix}_p = -(t_p \bar{z}_p) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_p \begin{bmatrix} e_x^T \\ e_y^T \\ e_{xy}^T \end{bmatrix}_p$$

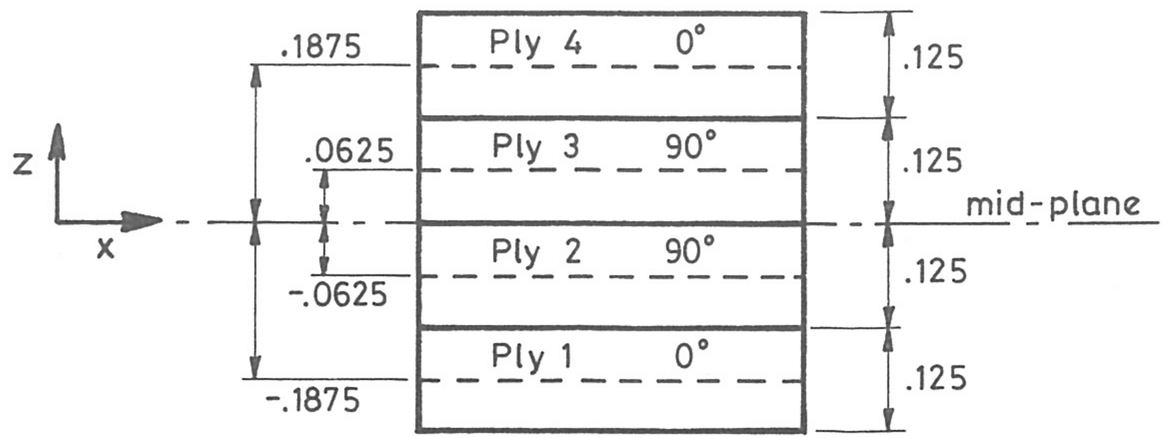


Fig. 7.17. Ordinate properties (mm) in $(0/90)_s$.

Table 7.1
Ordinate Values for a Symmetric Cross-ply $(0/90)_s$
(Basic dimensions: mm.)

Ply	$\theta°$	t_p	$\bar{z}_p$	$t_p\bar{z}_p$
1	0	0·125	−0·187 5	−0·023 44
2	90	0·125	−0·062 5	−0·007 81
3	90	0·125	0·062 5	0·007 81
4	0	0·125	0·187 5	0·023 44

Substituting the appropriate values into the above equation, we get:

Ply 1 at 0°
The ordinate $(t_p\bar{z}_p)$ value from Table 7.1 is −0·02344. The free thermal strain in the reference axes for this ply, obtained earlier, are

$$(e_x^T)_p = 30 \times 10^{-6}$$
$$(e_y^T)_p = -2800 \times 10^{-6}$$
$$(e_{xy}^T)_p = 0$$

and the transformed reduced stiffnesses given at the beginning of this example are, converted to units of N and mm:

$$(\bar{Q})_{0^\circ} = \boxed{\begin{matrix} 140\cdot9 & 3\cdot0 & 0 \\ 3\cdot0 & 10\cdot1 & 0 \\ 0 & 0 & 5\cdot0 \end{matrix}} \times 10^3\ \text{N/mm}^2$$

So the equivalent free thermal moments on this ply are

$$\begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix}_p = -(-0\cdot02344)\begin{bmatrix} 140\cdot9 & 3\cdot0 & 0 \\ 3\cdot0 & 10\cdot1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 30 \\ -2800 \\ 0 \end{bmatrix} \times 10^3 \times 10^{-6}$$

giving

$$(M_x^T)_p = -0\cdot098\ \text{N mm/mm}$$
$$(M_y^T)_p = -0\cdot661\ \text{N mm/mm}$$
$$(M_{xy}^T)_p = 0$$

Ply 2 at 90°

The ordinate $(t_p\bar{z}_p)$ value from Table 7.1 is $-0\cdot00781$. The free thermal strain in the reference axes for this ply, obtained earlier, are

$$(e_x^T)_p = -2800 \times 10^{-6}$$
$$(e_y^T)_p = 30 \times 10^{-6}$$
$$(e_{xy}^T)_p = 0$$

and the transformed reduced stiffnesses given at the beginning of this example are, converted to units of N and mm:

$$(\bar{Q})_{90^\circ} = \boxed{\begin{matrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \times 10^3\ \mathrm{N/mm^2}$$

So the equivalent free thermal moments on this ply are

$$\begin{bmatrix} M_x^T \\ M_y^T \\ M_{xy}^T \end{bmatrix}_p = -(-0{\cdot}00781)\begin{bmatrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -2800 \\ 30 \\ 0 \end{bmatrix} \times 10^3 \times 10^{-6}$$

giving

$$(M_x^T)_p = -0{\cdot}220\ \mathrm{N\ mm/mm}$$
$$(M_y^T)_p = -0{\cdot}033\ \mathrm{N\ mm/mm}$$
$$(M_{xy}^T)_p = 0$$

Ply 3 at 90°

By inspection, the ply equivalent free thermal moments will be of the same magnitude, but of different sign to those of the corresponding Ply 2 at 90°. This is so because the only difference in the computations between the corresponding plies is the sign of the ordinate value $(t_p \bar{z}_p)$, which changes sign for the correspondingly placed plies about the midplane. Hence,

$$(M_x^T)_p = 0{\cdot}220 \text{ N mm/mm}$$
$$(M_y^T)_p = 0{\cdot}033 \text{ N mm/mm}$$
$$(M_{xy}^T)_p = 0$$

Ply 4 at 0°

Again, by inspection, the equivalent free thermal moments for this ply will have opposing signs to those of the corresponding ply below the midplane, Ply 1 at 0°. Hence,

$$(M_x^T)_p = 0{\cdot}098 \text{ N mm/mm}$$
$$(M_y^T)_p = 0{\cdot}661 \text{ N mm/mm}$$
$$(M_{xy}^T)_p = 0$$

The laminate equivalent free thermal moments are then the summation of all the corresponding ply values, given by eqn (7.22) in this case of constant temperature distribution. So,

$$M_x^T = (-0{\cdot}098)_{\text{Ply 1}} + (-0{\cdot}220)_{\text{Ply 2}} + (0{\cdot}220)_{\text{Ply 3}} + (0{\cdot}098)_{\text{Ply 4}} = 0$$
$$M_y^T = (-0{\cdot}661)_{\text{Ply 1}} + (-0{\cdot}033)_{\text{Ply 2}} + (0{\cdot}033)_{\text{Ply 3}} + (0{\cdot}661)_{\text{Ply 4}} = 0$$
$$M_{xy}^T = 0$$

We have, therefore, shown numerically that, in symmetric laminate configurations with a symmetric temperature distribution through the laminate thickness, there are no net laminate equivalent free thermal moments. Hence, since the coupling terms B_{ij} and the free thermal moments are both zero, then eqn (7.24) simplifies to

	e_x^o	e_y^o	e_{xy}^o
N_x^T	A_{11}	A_{12}	A_{13}
N_y^T	A_{12}	A_{22}	A_{23}
N_{xy}^T	A_{13}	A_{23}	A_{33}

The laminate membrane common strains are, therefore, obtained by inverting the above matrix:

	N_x^T	N_y^T	N_{xy}^T
e_x^o	a_{11}	a_{12}	a_{13}
e_y^o	a_{12}	a_{22}	a_{23}
e_{xy}^o	a_{13}	a_{23}	a_{33}

The membrane compliances are given at the beginning of this example, and substituting the appropriate values, and working in units of N and mm, we get

	$-8{\cdot}092$	$-8{\cdot}092$	0	
e_x^o	$0{\cdot}0265$	$-0{\cdot}0011$	0	
e_y^o	$-0{\cdot}0011$	$0{\cdot}0265$	0	$\times 10^{-3}$
e_{xy}^o	0	0	$0{\cdot}4000$	

giving

$$e_x^o = -206 \times 10^{-6}$$
$$e_y^o = -206 \times 10^{-6}$$
$$e_{xy}^o = 0$$

These membrane common strains are constant through the laminate thickness. Since there are no coupling terms and no thermal moments present, then all the curvatures (k) are zero and so all the strains due to the curvature ($-zk$) are also zero, i.e.

$$-zk_x = -zk_y = -zk_{xy} = 0$$

We then go back to the individual plies and obtain the residual strains in the reference axes by using eqn (7.8):

$$e^R = e^o - zk - e^T$$

Plies 1 and 4 at 0°

Recalling the free thermal strain values in the reference axes obtained earlier: $(e_x^T)_p = 30 \times 10^{-6}$, $(e_y^T)_p = -2800 \times 10^{-6}$, $(e_{xy}^T)_p = 0$, and substituting the appropriate values, we get

$$e_x^R = e_x^o - zk_x - e_x^T = (-206 - 0 - 30) \times 10^{-6} = -236 \times 10^{-6}$$
$$e_y^R = e_y^o - zk_y - e_y^T = (-206 - 0 - -2800) \times 10^{-6} = 2594 \times 10^{-6}$$
$$e_{xy}^R = 0$$

Plies 2 and 3 at 90°

Recalling the free thermal strain values in the reference axes obtained earlier: $(e_x^T)_p = -2800 \times 10^{-6}$, $(e_y^T)_p = 30 \times 10^{-6}$, $(e_{xy}^T)_p = 0$, and substituting the appropriate values, we get

$$e_x^R = e_x^o - zk_x - e_x^T = (-206 - 0 - -2800) \times 10^{-6} = 2594 \times 10^{-6}$$
$$e_y^R = e_y^o - zk_y - e_y^T = (-206 - 0 - 30) \times 10^{-6} = -236 \times 10^{-6}$$
$$e_{xy}^R = 0$$

The next step is to transform the residual strains from the reference axes x–y to the material axes 1–2; this is generally done using eqn (7.33), but in the case of 0° and 90° ply angles, these transformations can be done by inspection:

Plies 1 and 4 at 0°

$$e_1^R = e_x^R = -236 \times 10^{-6}$$
$$e_2^R = e_y^R = 2594 \times 10^{-6}$$
$$e_{12}^R = e_{xy}^R = 0$$

Plies 2 and 3 at 90°

$$e_1^R = e_y^R = -236 \times 10^{-6}$$
$$e_2^R = e_x^R = 2594 \times 10^{-6}$$
$$e_{12}^R = e_{xy}^R = 0$$

The ply residual stresses are finally obtained by the specially orthotropic ply stress–strain relationship of eqn (7.34):

	e_1^R	e_2^R	e_{12}^R
f_1^R	Q_{11}	Q_{12}	0
f_2^R	Q_{12}	Q_{22}	0
f_{12}^R	0	0	Q_{33}

The ply reduced stiffnesses have been given at the beginning of this example. Note that the residual strains in the material axes for all the plies are the same, and, therefore, the residual stresses in the material axes will also be the same, as the plies are of the same material having the same reduced stiffnesses. Hence, substituting the residual strain values, and converting all units to N and mm, we get:

Plies 1 and 4 at 0° and Plies 2 and 3 at 90°

$$
\begin{array}{c|ccc|}
 & -236 & 2594 & 0 \\
\hline
f_1^R & 140{\cdot}9 & 3{\cdot}0 & 0 \\
f_2^R & 3{\cdot}0 & 10{\cdot}1 & 0 \\
f_{12}^R & 0 & 0 & 5
\end{array}
\times 10^3 \times 10^{-6}
$$

giving

$$
\begin{aligned}
f_1^R &= -26\ \mathrm{N/mm^2} \\
f_2^R &= 26\ \mathrm{N/mm^2} \\
f_{12}^R &= 0
\end{aligned}
$$

The laminate equivalent coefficients of thermal expansion for this cross-ply laminate configuration are obtained from eqn (7.30). In this case, there will be no bending equivalent values because there are no bending deformations induced. The common membrane strains obtained from the above are

$$e_x^o = -206 \times 10^{-6}$$
$$e_y^o = -206 \times 10^{-6}$$
$$e_{xy}^o = 0$$

and from eqn (7.30), with $\Delta T = -100$, we get

$$\alpha_x = e_x^o/\Delta T = (-206 \times 10^{-6})/-100 = 2{\cdot}06 \times 10^{-6}\ \text{strain/°C}$$
$$\alpha_y = e_y^o/\Delta T = (-206 \times 10^{-6})/-100 = 2{\cdot}06 \times 10^{-6}\ \text{strain/°C}$$
$$\alpha_{xy} = e_{xy}^o/\Delta T = 0$$

The membrane equivalent coefficient of thermal expansion values show that they are the same in the orthogonal x- and y-directions; this is as would be expected because of the equal reinforcements in the 0° and 90° directions in a cross-ply laminate. This is analogous to the equivalent membrane elastic constants being the same in the x- and y-directions in a cross-ply laminate. This implies that, as a result of a temperature change, the laminate as a whole will deform by equal amounts in the x- and y-directions. Furthermore, α_{xy} is zero, as would be expected in a symmetric cross-ply laminate because of the laminate orthotropy; this signifies that a temperature change will not induce any shear coupling effect on the laminate, in that a direct deformation will not cause an associated shear deformation, and vice versa. If the value of α_{xy} were nonzero, then a shear coupling affect would be present.

Recall that we stipulated that, in the absence of external loads, a system of residual stresses should form a net equilibrating system of forces and moments. It should be noted, however, that the summation of loads can only be performed in a common direction for all the plies in a laminate. So, for example, we cannot consider the residual stress in the material axes for the 0° and 90° plies, as the fibre directions in each ply are different. We will, therefore, have to obtain the ply residual stresses in the common reference axes x–y.

The transformation of the ply residual stresses from the material axes to the reference axes can be performed by using eqn (2.20) or by inspection for ply angles of 9° and 90°.

Plies 1 and 4 at 0°

In this case of a 0° ply angle, the reference axes and material axes stress values are the same:

$$f_x^R = f_1^R = -26\ \mathrm{N/mm^2}$$
$$f_y^R = f_2^R = \ \ 26\ \mathrm{N/mm^2}$$
$$f_{xy}^R = f_{12}^R = \ \ 0$$

Plies 2 and 3 at 90°

In the case of 90° ply angle, the reference axes and material axes direct stress values change in the orthogonal direction, and the shear stress values changes sign:

$$f_x^R = f_2^R = 26\ \mathrm{N/mm^2}$$
$$f_y^R = f_1^R = -26\ \mathrm{N/mm^2}$$
$$f_{xy}^R = f_{12}^R = 0$$

The distribution of the residual stresses in the x-, y- and x–y directions in all the four plies through the laminate thickness is shown in Fig. 7.18. Note that as there are no bending deformations, the stresses across each ply are constant.

The net forces due to the residual stresses are the product of the residual stress and the ply thickness. This gives the force per unit width of the section. Since the stress across the ply is constant (as shown in Fig. 7.18), then the

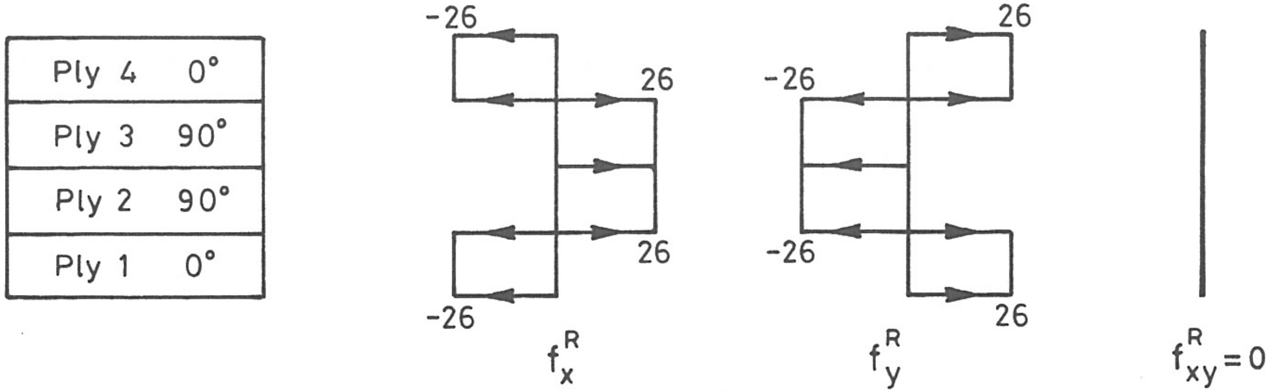


Fig. 7.18. Thermal residual stresses (N/mm²) through thickness of $(0/90)_s$.

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resultant force will act at the ply centroid. It can be seen by inspection of Fig. 7.18 that the forces contributed by the residual stresses in Plies 1 and 4 cancel out with the force contributions arising from the residual stresses of opposing signs in Plies 2 and 3. For example, the net residual force in the x-direction, N_x^R, for all the plies, is

$$\begin{aligned}
&\text{Ply 1:} \quad -26 \times 0{\cdot}125 = -3{\cdot}3\ \mathrm{N/mm}\\
&\text{Ply 2:} \quad 26 \times 0{\cdot}125 = 3{\cdot}3\ \mathrm{N/mm}\\
&\text{Ply 3:} \quad 26 \times 0{\cdot}125 = 3{\cdot}3\ \mathrm{N/mm}\\
&\text{Ply 4:} \quad -26 \times 0{\cdot}125 = -3{\cdot}3\ \mathrm{N/mm}
\end{aligned}$$

Each of the ply forces acts at the respective ply centroid, as shown in Fig. 7.19. So the summation of all the forces in the x-direction due to the residual stress is

$$N_x^R = (-3{\cdot}3)_{\text{Ply 1}} + (3{\cdot}3)_{\text{Ply 2}} + (3{\cdot}3)_{\text{Ply 3}} + (-3{\cdot}3)_{\text{Ply 4}} = 0$$

The moment of these residual forces about the midplane is, therefore, the product of the magnitude of the residual force and the lever arm to the point of line action of the force, which in this case is at the respective ply centroid, taking into account the direction of the moment. The ply centroidal values are taken from Fig. 7.17 and shown in Fig. 7.19. Now assuming that clockwise moment is positive, then residual momet about the midplane M_x^R, is

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$$M_x^R = (3{\cdot}3 \times 0{\cdot}1875)_{\text{Ply 1}} - (3{\cdot}3 \times 0{\cdot}0625)_{\text{Ply 2}} + (3{\cdot}3 \times 0{\cdot}0625)_{\text{Ply 3}} - (3{\cdot}3 \times 0{\cdot}1875)_{\text{Ply 4}} = 0$$

Therefore, as would have been expected, in the absence of external loads, a system of residual stresses forms an equilibrating system of forces and moments about the laminate midplane.

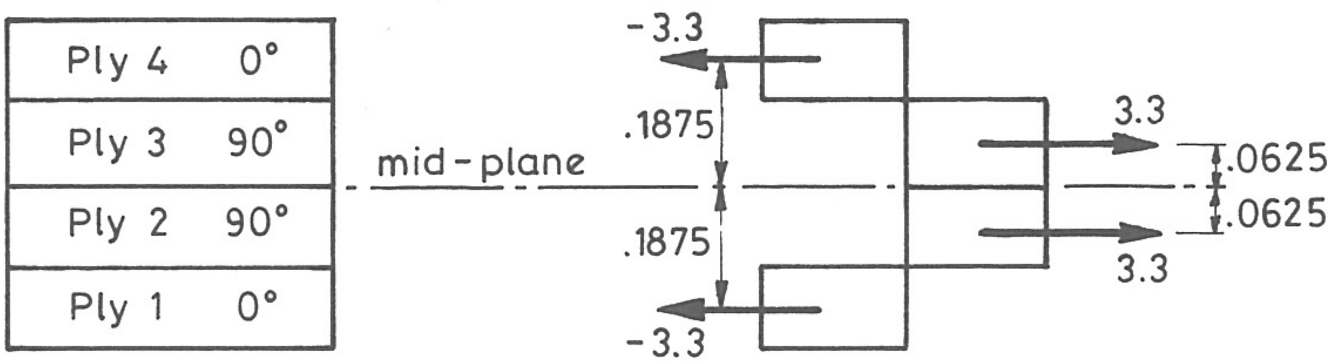

Fig. 7.19. Thermal residual forces (N/mm) in x-direction through thickness of $(0/90)_s$.

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- **Hygroscopic Residual Stresses Calculation Example**

Example 7.5

Determine the hygroscopic residual stresses in a symmetric cross-ply laminate $(0/90)_s$ made from high strength carbon/epoxy unidirectional plies. The ply is 0·125 mm thick and the elastic and strength properties are the same as in Example 7.2:

$$E_1 = 140, \quad E_2 = 10, \quad G_{12} = 5\ \mathrm{kN/mm^2}; \quad \nu_{12} = 0{\cdot}3$$
$$X_t = 1500; \quad X_c = 1200; \quad Y_t = 50; \quad Y_c = 250; \quad S = 70\ \mathrm{N/mm^2}$$

The coefficients of hygroscopic expansions are:

$$\beta_1 = 0{\cdot}01$$
$$\beta_2 = 0{\cdot}30$$

A constant moisture intake content distribution of $m = 0{\cdot}5\%$ is assumed to occur in all the four plies through the laminate thickness, as shown in Fig. 7.24.

The laminate configuration is symmetric about its midplane and so all the coupling terms B_{ij} will be zero in eqn (7.47). Also, since the moisture distribution through the laminate thickness is constant and therefore symmetric about the midplane, there will, therefore, be no equivalent free hygroscopic moments induced (note that this was proved in Example 7.2 when considering a symmetric laminate configuration with a symmetric temperature distribution

Constant moisture intake content distribution and constant thickness

The common state of strain in a laminate as a result of the hygroscopic environment is given by adapting eqn (7.24), which will now give the relationship between the free hygroscopic loads and the resulting common deformations related by the laminate stiffnesses:

$$\begin{array}{c|ccc:ccc}
 & e_x^o & e_y^o & e_{xy}^o & k_x & k_y & k_{xy} \\
\hline
N_x^H & A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\
N_y^H & A_{12} & A_{22} & A_{23} & B_{12} & B_{22} & B_{23} \\
N_{xy}^H & A_{13} & A_{23} & A_{33} & B_{13} & B_{23} & B_{33} \\
\hdashline
M_x^H & B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\
M_y^H & B_{12} & B_{22} & B_{23} & D_{12} & D_{22} & D_{23} \\
M_{xy}^H & B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33}
\end{array} \qquad (7.47)$$

where

$$A_{ij} = \sum_{p=1}^{N} t_p(\bar{Q}_{ij})_p$$

$$B_{ij} = \sum_{p=1}^{N} -t_p\bar{z}_p(\bar{Q}_{ij})_p$$

$$D_{ij} = \sum_{p=1}^{N} (t_p\bar{z}_p^2 + t_p^3/12)(\bar{Q}_{ij})_p$$

and N^{H} is given by eqn (7.44):

$$\begin{bmatrix} N_{\mathrm{x}}^{\mathrm{H}} \\ N_{\mathrm{y}}^{\mathrm{H}} \\ N_{\mathrm{xy}}^{\mathrm{H}} \end{bmatrix} = \sum_{p=1}^{N} (t_{\mathrm{p}}) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_{\mathrm{p}} \begin{bmatrix} e_{\mathrm{x}}^{\mathrm{H}} \\ e_{\mathrm{y}}^{\mathrm{H}} \\ e_{\mathrm{xy}}^{\mathrm{H}} \end{bmatrix}_{\mathrm{p}}$$

and M^{H} is given by eqn (7.46):

$$\begin{bmatrix} M_{\mathrm{x}}^{\mathrm{H}} \\ M_{\mathrm{y}}^{\mathrm{H}} \\ M_{\mathrm{xy}}^{\mathrm{H}} \end{bmatrix} = \sum_{p=1}^{N} -(t_{\mathrm{p}}\bar{z}_{\mathrm{p}}) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_{\mathrm{p}} \begin{bmatrix} e_{\mathrm{x}}^{\mathrm{H}} \\ e_{\mathrm{y}}^{\mathrm{H}} \\ e_{\mathrm{xy}}^{\mathrm{H}} \end{bmatrix}_{\mathrm{p}}$$

Having obtained the laminate common deformation values given by the above equation, the ply residual strains are then evaluated from eqn (7.41):

$$e^{\mathrm{R}} = e^{\mathrm{o}} - zk - e^{\mathrm{H}}$$

from which the ply residual stress can be determined by the expression:

$$f^{\mathrm{R}} = \bar{Q}e^{\mathrm{R}}$$

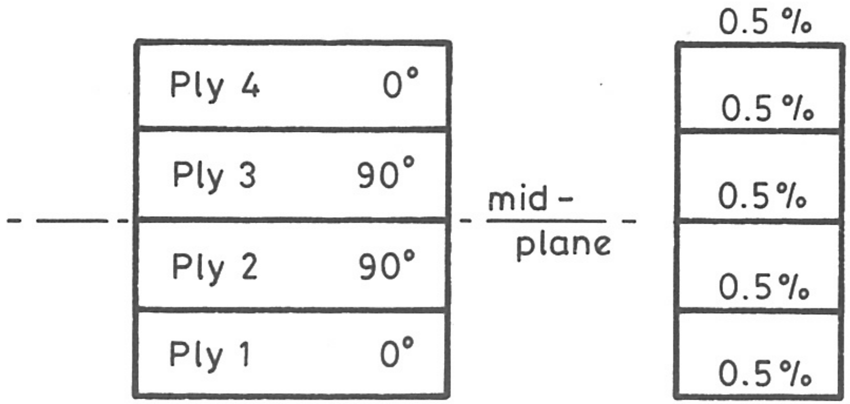

Fig. 7.24. Constant moisture distribution in $(0/90)_s$.

about the midplane, in which the laminate equivalent free thermal moments, similar to the equivalent free hygroscopic moment analyses, were shown to be zero).

As a result of this laminate and moisture content profile symmetry, the coupling terms and the free hygroscopic moments will be zero as a consequence, and therefore then, we need only be concerned with the membrane related properties with the associated free hygroscopic forces contributions from eqn (7.47).

The ply reduced stiffnesses Q_{ij}, the transformed reduced stiffnesses for each ply angle $\bar{Q}_{ij}$, the membrane stiffnesses A_{ij}, and the compliances a_{ij}, for this cross-ply laminate have been presented earlier in Example 7.2 and are

$$Q = \boxed{\begin{matrix} 140{\cdot}0 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \text{ kN/mm}^2$$

$$(\bar{Q})_{0^\circ} = \boxed{\begin{matrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \text{ kN/mm}^2$$

$$(\bar{Q})_{90^\circ} = \boxed{\begin{matrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \text{ kN/mm}^2$$

$$A = \boxed{\begin{matrix} 37{\cdot}8 & 1{\cdot}5 & 0 \\ 1{\cdot}5 & 37{\cdot}8 & 0 \\ 0 & 0 & 2{\cdot}5 \end{matrix}} \text{ kN/mm}$$

$$a = \begin{bmatrix} 0{\cdot}0265 & -0{\cdot}0011 & 0 \\ -0{\cdot}0011 & 0{\cdot}0265 & 0 \\ 0 & 0 & 0{\cdot}4000 \end{bmatrix} 1/(\mathrm{kN/mm})$$

The free hygroscopic strain in the material axes, using eqn (7.40), and observing that the moisture content m is in percentages, are

$$e_1^{\mathrm{H}} = \beta_1 m = 0{\cdot}01 \times 0{\cdot}005 = 0{\cdot}05 \times 10^{-3}$$

$$e_2^{\mathrm{H}} = \beta_2 m = 0{\cdot}30 \times 0{\cdot}005 = 1{\cdot}50 \times 10^{-3}$$

The free hygroscopic strains in the material axes have now got to be transformed into the reference axes for each ply, either by using the strain transformations of eqn (7.32), in which the free thermal strains are substituted for the free hygroscopic strains, or by inspection when the ply angle is 0° or 90°:

Ply 1 at 0°

For a ply angle of 0°, the reference axes strains will naturally be the same as the material axes strains, as shown in Example 7.2:

$$(e_x^H)_p = (e_1^H)_p = 0{\cdot}05 \times 10^{-3}$$
$$(e_y^H)_p = (e_2^H)_p = 1{\cdot}50 \times 10^{-3}$$
$$(e_{xy}^H)_p = (e_{12}^H)_p = 0$$

Ply 2 at 90°

For a ply angle of 90°, the orthogonal strains change directions when transposing from the material axes to the reference axes, and the shear strain changes sign as we saw in Example 7.2:

$$(e_x^H)_p = (e_2^H)_p = 1{\cdot}50 \times 10^{-3}$$
$$(e_y^H)_p = (e_1^H)_p = 0{\cdot}05 \times 10^{-3}$$
$$(e_{xy}^H)_p = (e_{12}^H)_p = 0$$

There is no need to perform the calculations on Plies 3 and 4 because the laminate and the moisture distribution are symmetric about the midplane. So the free hygroscopic strains in the reference axes for Plies 1 and 4 with a ply angle of 0° will be the same, and the free hygroscopic strains in the material axes for Plies 2 and 3 with a ply angle of 90° will be the same.

The equivalent free hygroscopic forces can now be calculated for each ply. With a constant moisture distribution, we make use of eqn (7.43) to determine the ply equivalent free hygroscopic forces N^H:

$$\begin{bmatrix} N_x^H \\ N_y^H \\ N_{xy}^H \end{bmatrix}_p = (t_p) \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix}_p \begin{bmatrix} e_x^H \\ e_y^H \\ e_{xy}^H \end{bmatrix}_p$$

Substituting the appropriate values into the above equation, we get:

Ply 1 at 0°

The free hygroscopic strain in the reference axes for this ply, obtained ealier, are

$$(e_x^H)_p = 0{\cdot}05 \times 10^{-3}$$
$$(e_y^H)_p = 1{\cdot}50 \times 10^{-3}$$
$$(e_{xy}^H)_p = 0$$

and the transformed reduced stiffnesses given at the beginning of this example are, converted to units of N and mm:

$$(\bar{Q})_{0^\circ} = \boxed{\begin{matrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \times 10^3\ \mathrm{N/mm^2}$$

and so the equivalent free hygroscopic forces on this ply are

$$\begin{bmatrix} N_x^H \\ N_y^H \\ N_{xy}^H \end{bmatrix}_p = 0{\cdot}125 \begin{bmatrix} 140{\cdot}9 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 10{\cdot}1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0{\cdot}05 \\ 1{\cdot}50 \\ 0 \end{bmatrix} \times 10^3 \times 10^{-3}$$

giving

$$(N_x^H)_p = 1{\cdot}443\ \text{N/mm}$$
$$(N_y^H)_p = 1{\cdot}913\ \text{N/mm}$$
$$(N_{xy}^H)_p = 0$$

Ply 2 at 90°

The free hygroscopic strain in the reference axes for this ply, obtained earlier, are

$$(e_x^H)_p = 1{\cdot}50 \times 10^3$$
$$(e_y^H)_p = 0{\cdot}05 \times 10^{-3}$$
$$(e_{xy}^H)_p = 0$$

and the transformed reduced stiffnesses given at the beginning of this example are, converted to units of N and mm:

$$(\bar{Q})_{90^\circ} = \boxed{\begin{matrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5{\cdot}0 \end{matrix}} \times 10^3\ \text{N/mm}^2$$

and so the equivalent free hygroscopic forces on this ply are

$$\begin{bmatrix} N_x^H \\ N_y^H \\ N_{xy}^H \end{bmatrix}_p = 0{\cdot}125 \begin{bmatrix} 10{\cdot}1 & 3{\cdot}0 & 0 \\ 3{\cdot}0 & 140{\cdot}9 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1{\cdot}50 \\ 0{\cdot}05 \\ 0 \end{bmatrix} \times 10^3 \times 10^{-3}$$

giving

$$(N_x^H)_p = 1{\cdot}913\ \mathrm{N/mm}$$

$$(N_y^H)_p = 1{\cdot}443\ \mathrm{N/mm}$$

$$(N_{xy}^H)_p = 0$$

Due to the laminate configuration and moisture distribution symmetry about the midplane, we only need consider one half of the laminate configuration, as the corresponding plies in the top half of the laminate configuration will have identical values. In this case then, the free hygroscopic forces for Plies 1 and 4 with a ply angle of 0° will be the same, and the free hygroscopic forces for Plies 2 and 3 with a ply angle of 90° will be the same.

The laminate equivalent free hygroscopic forces are then the summation of all the corresponding ply values, given by eqn (7.44) in this case of constant moisture distribution. Because of the symmetry, we can sum all the corresponding values in one half of the laminate and then double the result to obtain the total values. Thus,

$$N_x^H = 2\{(1{\cdot}443)_{\text{Ply 1}} + (1{\cdot}913)_{\text{Ply 2}}\} = 6{\cdot}712\ \mathrm{N/mm}$$

$$N_y^H = 2\{(1{\cdot}913)_{\text{Ply 1}} + (1{\cdot}443)_{\text{Ply 2}}\} = 6{\cdot}712\ \mathrm{N/mm}$$

$$N_{xy}^H = 2\{(0)_{\text{Ply 1}} + (0)_{\text{Ply 2}}\} = 0$$

As mentioned earlier in this example, there are no net laminate equivalent free hygroscopic moments and the coupling terms B_{ij} are zero, therefore then, eqn (7.47) simplifies to

	e_x^o	e_y^o	e_{xy}^o
N_x^H	A_{11}	A_{12}	A_{13}
N_y^N	A_{12}	A_{22}	A_{23}
N_{xy}^H	A_{13}	A_{23}	A_{33}

The laminate membrane common strains are, therefore, obtained by inverting the above matrix:

	N_x^H	N_y^H	N_{xy}^H
e_x^o	a_{11}	a_{12}	a_{13}
e_y^o	a_{12}	a_{22}	a_{23}
e_{xy}^o	a_{13}	a_{23}	a_{33}

We then go back to the individual plies and obtain the residual strains in the reference axes by using eqn (7.41):

$$e^{R} = e^{o} - zk - e^{H}$$

Plies 1 and 4 at 0°

Recalling the free hygroscopic strain values in the reference axes obained earlier: $(e_x^H)_p = 0{\cdot}05 \times 10^{-3}$, $(e_y^H)_p = 1{\cdot}50 \times 10^{-3}$ and $(e_{xy}^H)_p = 0$, and substituting the appropriate values, we get

$$e_x^R = e_x^o - zk_x - e_x^H = (0{\cdot}170 - 0 - 0{\cdot}05) \times 10^{-3} = 0{\cdot}120 \times 10^{-3}$$

$$e_y^R = e_y^o - zk_y - e_y^H = (0{\cdot}170 - 0 - 1{\cdot}50) \times 10^{-3} = -1{\cdot}330 \times 10^{-3}$$

$$e_{xy}^R = 0$$

Plies 2 and 3 at 90°

Recalling the free hygroscopic strain values in the reference axes obtained earlier: $(e_x^H)_p = 1{\cdot}50 \times 10^{-3}$, $(e_y^H)_p = 0{\cdot}05 \times 10^{-3}$ and $(e_{xy}^H)_p = 0$, and substituting the appropriate values, we get

$$e_x^R = e_x^o - zk_x - e_x^H = (0{\cdot}170 - 0 - 1{\cdot}50) \times 10^{-3} = -1{\cdot}330 \times 10^{-3}$$

$$e_y^R = e_y^o - zk_y - e_y^H = (0{\cdot}170 - 0 - 0{\cdot}05) \times 10^{-3} = 0{\cdot}120 \times 10^{-3}$$

$$e_{xy}^R = 0$$

The transformation of the hygroscopically induced residual strains from the reference axes x–y to the material axes 1–2 can be done by inspection for the cases of 0° and 90° ply angles:

Plies 1 and 4 at 0°

$$
\begin{aligned}
e_1^R &= e_x^R = 0{\cdot}120 \times 10^{-3} \\
e_2^R &= e_y^R = -1{\cdot}330 \times 10^{-3} \\
e_{12}^R &= e_{xy}^R = 0
\end{aligned}
$$

Plies 2 and 3 at 90°

$$
\begin{aligned}
e_1^R &= e_y^R = 0{\cdot}120 \times 10^{-3} \\
e_2^R &= e_x^R = -1{\cdot}330 \times 10^{-3} \\
e_{12}^R &= e_{xy}^R = 0
\end{aligned}
$$

The ply residual stresses are finally obtained by the specially orthotropic ply stress–strain relationship of eqn (7.34):

	e_1^R	e_2^R	e_{12}^R
f_1^R	Q_{11}	Q_{12}	0
f_2^R	Q_{12}	Q_{22}	0
f_{12}^R	0	0	Q_{33}

The ply reduced stiffnesses have been given at the beginning of this example. Note that since the residual strains in the material axes for all the plies are the same, therefore, the residual stresses in the material axes will also be the same as the plies are all of the same material having the same reduced stiffnesses. Hence, substituting the residual strain values, and converting all units to N and mm, we get:

Plies 1 and 4 at 0° and Plies 2 and 3 at 90°

	0·120	−1·330	0
f_1^R	140·9	3·0	0
f_2^R	3·0	10·1	0
f_{12}^R	0	0	5

$\times 10^3 \times 10^{-3}$

giving

$$f_1^R = 13\ \mathrm{N/mm^2}$$
$$f_2^R = -13\ \mathrm{N/mm^2}$$
$$f_{12}^R = 0$$

The ply reduced stiffnesses have been given at the beginning of this example. Note that since the residual strains in the material axes for all the plies are the same, therefore, the residual stresses in the material axes will also be the same as the plies are all of the same material having the same reduced stiffnesses. Hence, substituting the residual strain values, and converting all units to N and mm, we get:

Plies 1 and 4 at 0° and Plies 2 and 3 at 90°

	0·120	−1·330	0	
f_1^R	140·9	3·0	0	
f_2^R	3·0	10·1	0	$\times 10^3 \times 10^{-3}$
f_{12}^R	0	0	5	

giving

$$f_1^R = 13\ \mathrm{N/mm^2}$$
$$f_2^R = -13\ \mathrm{N/mm^2}$$
$$f_{12}^R = 0$$

The laminate membrane equivalent coefficients of hygroscopic expansion for this cross-ply laminate configuration are obtained from eqn (7.50). In this case, there will be no bending equivalent values because there are no bending deformations induced. The common membrane strains obtained from above are

$$e_x^o = 0{\cdot}170 \times 10^{-3}$$
$$e_y^o = 0{\cdot}170 \times 10^{-3}$$
$$e_{xy}^o = 0$$

and from eqn (7.50), with $m = 0{\cdot}005$, we get

$$\beta_x = e_x^o/m = (0{\cdot}170 \times 10^{-3})/0{\cdot}005 = 0{\cdot}034$$
$$\beta_y = e_y^o/m = (0{\cdot}170 \times 10^{-3})/0{\cdot}005 = 0{\cdot}034$$
$$\beta_{xy} = e_{xy}^o/m = 0$$

Again, in the absence of external loads, a system of hygroscopically induced residual stresses should form a net equilibrating system of forces and moments. Note that we will have to obtain the ply residual stresses in the common reference axes x–y to resolve the ply forces in the common direction. The transformation of the ply residual stresses from the material axes to the reference axes can be performed by using eqn (2.20) or by inspection for ply angles of 0° and 90°.

Plies 1 and 4 at 0°

In this case of a 0° ply angle, the reference axes and material axes stress values are the same:

$$
\begin{aligned}
f_{x}^{R} &= f_{1}^{R} = 13\ \mathrm{N/mm^2} \\
f_{y}^{R} &= f_{2}^{R} = -13\ \mathrm{N/mm^2} \\
f_{xy}^{R} &= f_{12}^{R} = 0
\end{aligned}
$$

Plies 2 and 3 at 90°

In the case of a 90° ply angle, the reference axes and material axes direct stress values change in the orthogonal directions and the shear stress values change sign:

$$
\begin{aligned}
f_{x}^{R} &= f_{2}^{R} = -13\ \mathrm{N/mm^2} \\
f_{y}^{R} &= f_{1}^{R} = 13\ \mathrm{N/mm^2} \\
f_{xy}^{R} &= f_{12}^{R} = 0
\end{aligned}
$$

The distribution of the residual stresses in the x–, y– and x–y directions in all the four plies through the laminate thickness is shown in Fig. 7.25. Note that as there are no bending deformations, the stresses across each ply are constant.

It can be seen by inspection of Fig. 7.25 that the forces contributed by the residual stresses in Plies 1 and 4 cancel out with the force contributions arising from the residual stresses of opposing signs in Plies 2 and 3. Also, the

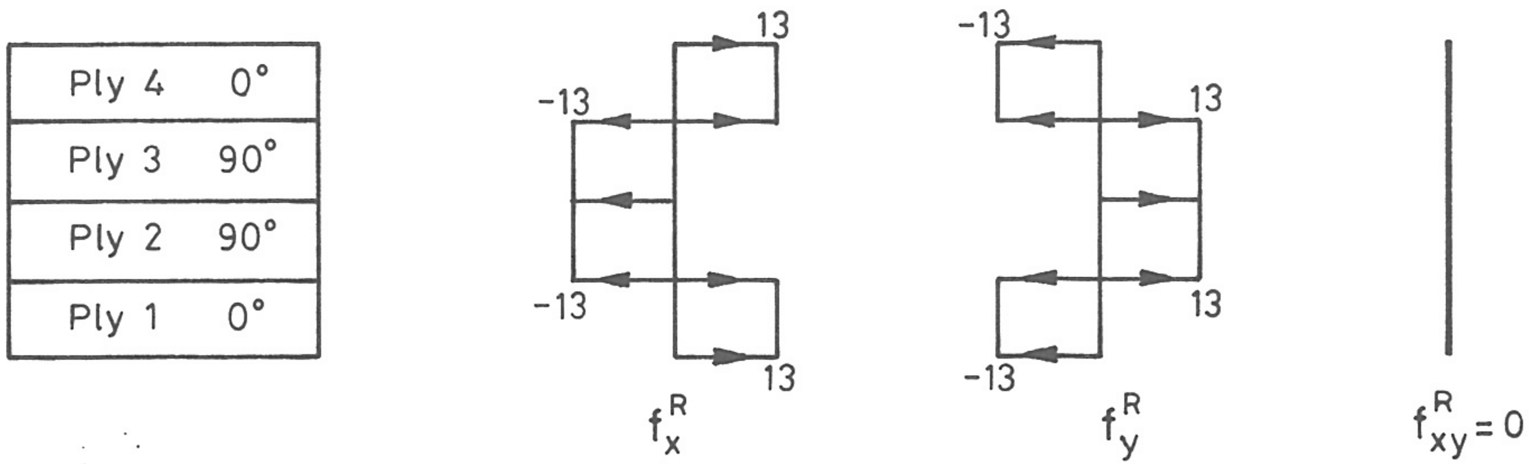

Fig. 7.25. Hygroscopic residual stresses (N/mm^2) through thickness of $(0/90)_s$.

summation of moments of the equivalent forces about the laminate midplane can also, by inspection, be seen to be zero.