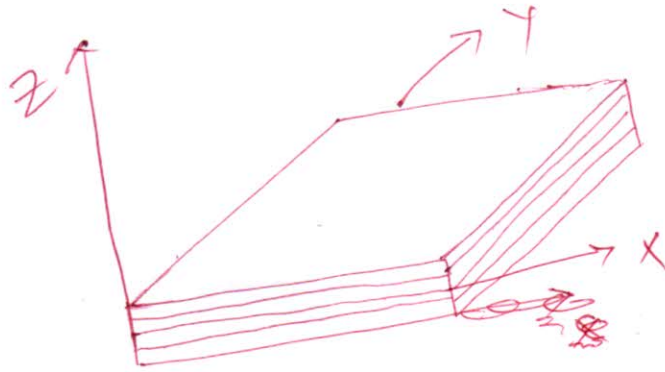


①



$u \rightarrow$ displacement along x
 $v \rightarrow$ " " " y
 $w \rightarrow$ " " " z

Assume no shear deformation

$$u = u_0 - z \frac{\partial w}{\partial x} \quad v = v_0 - z \frac{\partial w}{\partial y}$$

Strain displacement relationship

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_s \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$= \begin{Bmatrix} \epsilon_{x0} \\ \epsilon_{y0} \\ \epsilon_{s0} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{x0} \\ \epsilon_{y0} \\ \epsilon_{s0} \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{Bmatrix}$$

Stress-strain relationship for ~~each~~ ^{kl} laminate

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_s \end{Bmatrix}$$

Relations between stress resultants and strains

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} \\ A_{xy} & A_{yy} & A_{ys} \\ A_{xs} & A_{ys} & A_{ss} \end{bmatrix} \begin{Bmatrix} \epsilon_{x0} \\ \epsilon_{y0} \\ \epsilon_{s0} \end{Bmatrix} + \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{xy} & B_{yy} & B_{ys} \\ B_{xs} & B_{ys} & B_{ss} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} B_{xx} & B_{xy} & B_{xs} \\ B_{xy} & B_{yy} & B_{ys} \\ B_{xs} & B_{ys} & B_{ss} \end{bmatrix} \begin{Bmatrix} \epsilon_{x0} \\ \epsilon_{y0} \\ \epsilon_{s0} \end{Bmatrix} + \begin{bmatrix} D_{xx} & D_{xy} & D_{xs} \\ D_{xy} & D_{yy} & D_{ys} \\ D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{Bmatrix}$$

Energy Principles

Principle of minimum potential energy (static case)

$$\delta(\Pi) = 0 \Rightarrow \delta(U + V) = 0$$

U = strain energy

V = Potential of the applied load

$$\Rightarrow \delta \left[\int_V \frac{1}{2} \{\epsilon\}^T \{\sigma\} dV - \int_\Omega \{u_0 \ v_0 \ w\} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} d\Omega \right] = 0$$

A similar principle for a dynamic (time dependent) known as the Hamilton's principle is

$$\int_{t_1}^{t_2} \delta(T - \Pi) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta \left[\int_\Omega \frac{1}{2} m (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) d\Omega - \int_V \frac{1}{2} \{\epsilon\}^T \{\sigma\} dV + \int_\Omega \{u_0 \ v_0 \ w\} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} d\Omega \right] dt$$

= 0

t_1 and t_2 are arbitrary time instants

$m = \text{mass per area}$
 $= \int_{-h/2}^{h/2} \rho dz$

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Virtual Work PrincipleInternal ~~work~~ virtual work = External virtual work

$$\Rightarrow \int_V \delta \{ \epsilon \}^T \{ \sigma \} dv = \int_{\Omega} \delta \{ u_0 \ v_0 \ w \} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} d\Omega$$

time dependent -
For a ~~time dependent~~ (dynamic) problem same equation can be written as:

$$\underbrace{\int_V \rho \delta \{ u \ v \ w \}^T \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} dv}_{\downarrow \int_{\Omega} m \delta \{ u \ v \ w \}^T \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix}} + \int_V \delta \{ \epsilon \}^T \{ \sigma \} dv = \int_{\Omega} \delta \{ u_0 \ v_0 \ w \} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} d\Omega$$

Governing Differential Equations:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_s}{\partial y} + q_x = m \ddot{u}_0$$

$$\frac{\partial N_s}{\partial x} + \frac{\partial N_y}{\partial y} + q_y = m \ddot{v}_0$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z = m \ddot{w}_0$$

$$\frac{\partial M_s}{\partial y} + \frac{\partial M_x}{\partial x} = V_x$$

$$\frac{\partial M_s}{\partial x} + \frac{\partial M_y}{\partial y} = V_y$$

$$\Rightarrow \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_s}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q_z = m \ddot{w}_0$$

Boundary conditions

All the edges are simply supported

$$\text{at } x=0, x=a, y=0, y=b \quad w=0$$

$$x=0, x=a \quad v_0=0$$

$$y=0, y=b \quad u_0=0$$

Essential BCs

$$\text{at } x=0, x=a \quad M_x=0$$

$$y=0, y=b \quad M_y=0$$

Natural BCs

$$x=0, x=a \quad N_x=0$$

$$y=0, y=b \quad N_y=0$$

Weak Form

Test functions S_u, S_v, S_w

~~S_u satisfies the~~ Test functions ^{are} arbitrary functions
only restrictions are —

S_u should satisfy the homogeneous version of the essential BC for u_0

S_v — — — — —

v_0

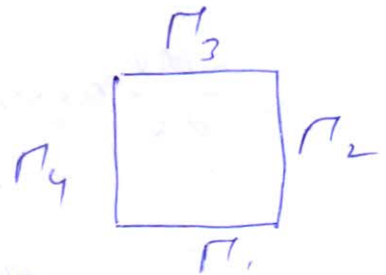
S_w — — — — —

w

(3)

$$\int_{\Omega} S_u \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_s}{\partial y} + q_x - m\ddot{u}_0 \right) d\Omega = 0$$

$$\Rightarrow \int_{\Gamma} S_u N_x n_x d\Gamma - \int_{\Omega} \frac{\partial S_u}{\partial x} N_x d\Omega + \int_{\Gamma} S_u N_s n_y d\Gamma - \int_{\Omega} \frac{\partial S_u}{\partial y} N_s d\Omega + \int_{\Omega} q_x S_u d\Omega - \int_{\Omega} m\ddot{u}_0 S_u d\Omega = 0$$



at Γ_1 and Γ_3 $n_x = 0$

at Γ_2 $n_x = 1$ but ~~$u_0 \neq 0 \Rightarrow S_u = 0$~~
 $N_x = 0$

Γ_4 $n_x = -1$ but ~~$u_0 \neq 0 \Rightarrow S_u = 0$~~
 $N_x = 0$

at Γ_2 and Γ_4 $n_y = 0$
at Γ_3 $n_y = 1$, but $S_u = 0$
 Γ_1 $n_y = -1$, but $S_u = 0$

Hence

$$\int_{\Omega} m\ddot{u}_0 S_u d\Omega + \int_{\Omega} N_x \frac{\partial S_u}{\partial x} d\Omega + \int_{\Omega} N_s \frac{\partial S_u}{\partial y} d\Omega - \int_{\Omega} q_x S_u d\Omega = 0$$

WEAK FORM

$$\int_{\Omega} S_v \left(\frac{\partial N_s}{\partial x} + \frac{\partial N_y}{\partial y} + q_y - m\ddot{v}_0 \right) d\Omega = 0$$

$$\Rightarrow \int_{\Gamma} S_v N_s n_x d\Gamma - \int_{\Omega} N_s \frac{\partial S_v}{\partial x} d\Omega + \int_{\Gamma} S_v N_y n_y d\Gamma - \int_{\Omega} \frac{\partial S_v}{\partial y} N_y d\Omega + \int_{\Omega} q_y S_v d\Omega - \int_{\Omega} m\ddot{v}_0 S_v d\Omega = 0$$

at Γ_2 and Γ_4 $n_y = 0$

at Γ_3 $n_y = 1$ but ~~$v_0 \neq 0 \Rightarrow S_v = 0$~~
 $N_y = 0$

Γ_1 $n_y = -1$ but ~~$v_0 \neq 0 \Rightarrow S_v = 0$~~
 $N_y = 0$

at Γ_1 and Γ_3 $n_x = 0$
at Γ_2 $n_x = 1$, but $S_v = 0$
 Γ_4 $n_x = -1$, but $S_v = 0$

Hence,
$$\int_{\Omega} m \ddot{u}_0 S_u d\Omega + \int_{\Omega} N_x \frac{\partial S_u}{\partial x} d\Omega + \int_{\Omega} N_y \frac{\partial S_u}{\partial y} d\Omega - \int_{\Omega} q_z S_u d\Omega = 0$$

WEAK FORM

$$\int_{\Omega} S_w \left(\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_s}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - m \ddot{w} + q_z \right) d\Omega = 0$$

$$\begin{aligned} \Rightarrow & \int_{\Gamma_0} S_w \frac{\partial M_x}{\partial x} n_x d\Gamma \quad \text{I} \quad - \int_{\Gamma_0} \frac{\partial S_w}{\partial x} M_x n_x d\Gamma \quad \text{II} \quad + \int_{\Omega} \frac{\partial^2 S_w}{\partial x^2} M_x d\Omega \\ & + \int_{\Gamma_0} S_w \frac{\partial M_s}{\partial y} n_x d\Gamma \quad \text{III} \quad - \int_{\Gamma_0} \frac{\partial S_w}{\partial x} M_s n_y d\Gamma \quad \text{IV} \quad + \int_{\Omega} \frac{\partial^2 S_w}{\partial x \partial y} M_s d\Omega \\ & + \int_{\Omega} S_w \frac{\partial M_s}{\partial x} n_y d\Gamma \quad \text{V} \quad - \int_{\Omega} \frac{\partial S_w}{\partial y} M_s n_x d\Gamma \quad \text{VI} \quad + \int_{\Omega} \frac{\partial^2 S_w}{\partial x \partial y} M_s d\Omega \\ & + \int_{\Omega} S_w \frac{\partial M_y}{\partial y} n_y d\Gamma \quad \text{VII} \quad - \int_{\Omega} \frac{\partial S_w}{\partial y} M_y n_y d\Gamma \quad \text{VIII} \quad + \int_{\Omega} \frac{\partial^2 S_w}{\partial y^2} M_y d\Omega \\ & - \int_{\Omega} m \ddot{w} S_w d\Omega + \int_{\Omega} q_z S_w d\Omega = 0 \end{aligned}$$

at $\Gamma_1 \cup \Gamma_3$ $n_x = 0$

~~$\Gamma_1 \cup \Gamma_3$~~

at Γ_2 $n_x = 1$, but $M_x = 0$

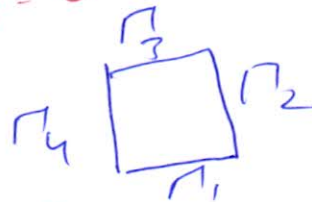
Γ_4 $n_x = -1$ but $M_x = 0$

Hence the term II is zero

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at Γ_2 and Γ_4 $n_y = 0$ at Γ_3 $n_y = 1$, but $M_y = 0$
 at Γ_1 $n_y = -1$, but $M_y = 0$

Hence the term \textcircled{VIII} is zero



at Γ_2 and Γ_4 $n_y = 0$ at Γ_3 $n_y = 1$ but $\frac{\partial S_w}{\partial x} = 0$
 at Γ_1 $n_y = -1$ but $\frac{\partial S_w}{\partial x} = 0$

Hence, the term \textcircled{IV} is zero

at Γ_1 and Γ_3 $n_x = 0$ at Γ_2 $n_x = 1$ but $\frac{\partial S_w}{\partial y} = 0$
 at Γ_4 $n_x = -1$ but $\frac{\partial S_w}{\partial y} = 0$

Hence, the term \textcircled{VI} is zero

Adding the terms \textcircled{I} and \textcircled{III} $\int_{\Gamma} S_w v_x n_x d\Gamma$

at $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ S_w is zero Hence, sum of

terms \textcircled{I} and \textcircled{III} is zero

Adding the terms \textcircled{V} and \textcircled{VII} $\int_{\Gamma} S_w v_y n_y d\Gamma$

at $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ S_w is zero Hence, sum of

terms \textcircled{V} and \textcircled{VII} is zero

Hence,

$$\int_{\Omega} M_x \frac{\partial^2 S_w}{\partial x^2} d\Omega + \int_{\Omega} 2M_s \frac{\partial^2 S_w}{\partial x \partial y} d\Omega + \int_{\Omega} M_y \frac{\partial^2 S_w}{\partial y^2} d\Omega - \int_{\Omega} m \ddot{w} S_w d\Omega + \int_{\Omega} q S_w d\Omega = 0$$

WEAK FORM

Let us assume $u_0 = \sum_{m=1}^M \sum_{n=1}^N U_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \sum_{j=1}^P b_{uj} \phi_j(x, y)$

$v_0 = \sum_{m=1}^M \sum_{n=1}^N V_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} = \sum_{j=1}^P b_{vj} \phi_j(x, y)$ ~~$P=MM$~~ , $P=MM$

$w = \sum_{m=1}^M \sum_{n=1}^N W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \sum_{j=1}^P b_{wj} \phi_j(x, y)$

The assumptions for u_0, v_0, w should be such that they satisfy the essential boundary conditions at least

Let us assume $S_u = \sum_{m=1}^M \sum_{n=1}^N U_{smn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \sum_{j=1}^P g_{uj} \phi_j(x, y)$

$S_v = \sum_{m=1}^M \sum_{n=1}^N U_{smn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} = \sum_{j=1}^P g_{vj} \phi_j(x, y)$

$S_w = \sum_{m=1}^M \sum_{n=1}^N W_{smn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \sum_{j=1}^P g_{wj} \phi_j(x, y)$

S_u, S_v, S_w should satisfy the homogenous version of the essential boundary conditions satisfied by u_0, v_0, w respectively.

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these approximations $\epsilon_{x0} = \frac{\partial u_0}{\partial x} = \sum_{j=1}^P b_{uj} \phi_{uj,x}$
 $\epsilon_{y0} = \frac{\partial v_0}{\partial y} = \sum_{j=1}^P b_{vj} \phi_{vj,y}$

$$= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = \sum_{j=1}^P (b_{uj} \phi_{uj,y} + b_{vj} \phi_{vj,x})$$

$$= -\frac{\partial^2 w}{\partial x^2} = -\sum_{j=1}^P b_{wj} \phi_{wj,xx} \quad k_y = -\frac{\partial^2 w}{\partial y^2} = -\sum_{j=1}^P b_{vj} \phi_{vj,yy}$$

$$= -2 \frac{\partial^2 w}{\partial x \partial y} = -2 \sum_{j=1}^P b_{wj} \phi_{wj,xy}$$

Using these approximations in the first weak form

$$\int_{\Omega} m \sum_{j=1}^P \bar{g}_{uj} \psi_{uj} \sum_{j=1}^P \ddot{b}_{uj} \phi_{uj} d\Omega + \int_{\Omega} \left\{ A_{xx} \sum_{j=1}^P b_{uj} \phi_{uj,x} + A_{xy} \sum_{j=1}^P b_{uj} \phi_{uj,y} \right.$$

$$+ A_{xs} \sum_{j=1}^P (b_{uj} \phi_{uj,y} + b_{vj} \phi_{vj,x}) - B_{xx} \sum_{j=1}^P b_{wj} \phi_{wj,xx} - B_{xy} \sum_{j=1}^P b_{wj} \phi_{wj,xy}$$

$$\left. - 2B_{xs} \sum_{j=1}^P b_{wj} \phi_{wj,xy} \right\} d\Omega + \int_{\Omega} \left\{ A_{xs} \sum_{j=1}^P b_{uj} \phi_{uj,x} + A_{ys} \sum_{j=1}^P b_{vj} \phi_{vj,y} + A_{ss} \sum_{j=1}^P (b_{uj} \phi_{uj,y} + b_{vj} \phi_{vj,x}) \right.$$

$$+ \int_{\Omega} \left\{ A_{xs} \sum_{j=1}^P b_{uj} \phi_{uj,x} + A_{ys} \sum_{j=1}^P b_{vj} \phi_{vj,y} + A_{ss} \sum_{j=1}^P (b_{uj} \phi_{uj,y} + b_{vj} \phi_{vj,x}) \right.$$

$$\left. - B_{xs} \sum_{j=1}^P b_{wj} \phi_{wj,xx} - B_{ys} \sum_{j=1}^P b_{wj} \phi_{wj,yy} - 2B_{ss} \sum_{j=1}^P b_{wj} \phi_{wj,xy} \right\} \sum_{j=1}^P \bar{g}_{uj} \psi_{uj} d\Omega$$

$$- \int_{\Omega} q_x \sum_{j=1}^P \bar{g}_{uj} \psi_{uj} d\Omega = 0$$

$$\begin{aligned}
\Rightarrow \sum_{i=1}^P \delta u_i \left[\int_{\Omega} m \psi_{ui} \sum_{j=1}^P \phi_{uj} \ddot{b}_{uj} d\Omega + \int_{\Omega} \rho \psi_{ui,x} \left\{ A_{xx} \sum_{j=1}^P \phi_{uj,x} b_{uj} \right. \right. \\
+ A_{xy} \sum_{j=1}^P \phi_{uj,y} b_{uj} + A_{xs} \sum_{j=1}^P (b_{uj} \phi_{uj,y} + b_{uj} \phi_{uj,x}) - B_{xx} \sum_{j=1}^P b_{uj} \phi_{uj,xx} \\
- B_{xy} \sum_{j=1}^P b_{uj} \phi_{uj,yy} - 2B_{xs} \sum_{j=1}^P b_{uj} \phi_{uj,xy} \left. \right\} d\Omega + \int_{\Omega} \psi_{ui,y} \left\{ A_{xs} \sum_{j=1}^P \phi_{uj,x} b_{uj} \right. \\
+ A_{ys} \sum_{j=1}^P \phi_{uj,y} b_{uj} + A_{ss} \sum_{j=1}^P (b_{uj} \phi_{uj,y} + b_{uj} \phi_{uj,x}) - B_{xs} \sum_{j=1}^P b_{uj} \phi_{uj,xx} \\
- B_{ys} \sum_{j=1}^P b_{uj} \phi_{uj,yy} - 2B_{ss} \sum_{j=1}^P b_{uj} \phi_{uj,xy} \left. \right\} d\Omega - \int_{\Omega} \rho_x \psi_{ui} d\Omega \Big] = 0 \\
\Rightarrow \sum_{i=1}^P \delta u_i L_i = 0
\end{aligned}$$

δu_i is arbitrary, hence, $\sum_{i=1}^P \delta u_i L_i$ ~~can~~ always be zero only when $L_i = 0 \quad i=1, \dots, P$

This gives us P equations

Similarly putting these approximations in the second weak form can be written as $\sum_{i=1}^P \delta v_i L_i = 0$

the third weak form can be written as $\sum_{i=1}^P \delta w_i L_i = 0$

Hence, we get total $3P$ equations of the form

$$[M] \begin{Bmatrix} \ddot{b}_{u1} \\ \ddot{b}_{up} \\ \ddot{b}_{v1} \\ \ddot{b}_{vp} \\ \ddot{b}_{w1} \\ \ddot{b}_{wp} \end{Bmatrix} + [K] \begin{Bmatrix} b_{u1} \\ b_{up} \\ b_{v1} \\ b_{vp} \\ b_{w1} \\ b_{wp} \end{Bmatrix} = \{F\}$$

(6)

where

$$[M]_{3p \times 3p} = \begin{bmatrix} [M_{uu}]_{p \times p} & [0]_{p \times p} & [0]_{p \times p} \\ [0]_{p \times p} & [M_{vv}]_{p \times p} & [0]_{p \times p} \\ [0]_{p \times p} & [0]_{p \times p} & [M_{ww}]_{p \times p} \end{bmatrix}$$

$$M_{uij} = \int_{\Omega} m \psi_{ui} \phi_{uj} d\Omega \quad M_{vij} = \int_{\Omega} m \psi_{vi} \phi_{vj} d\Omega$$

$$M_{wij} = \int_{\Omega} m \psi_{wi} \phi_{wj} d\Omega$$

$$[K]_{3p \times 3p} = \begin{bmatrix} [K_{uu}]_{p \times p} & [K_{uv}]_{p \times p} & [K_{uw}]_{p \times p} \\ [K_{vu}]_{p \times p} & [K_{vv}]_{p \times p} & [K_{vw}]_{p \times p} \\ [K_{wu}]_{p \times p} & [K_{wv}]_{p \times p} & [K_{ww}]_{p \times p} \end{bmatrix}$$

$$K_{uij} = \int_{\Omega} \psi_{ui,x} \{ A_{xx} \phi_{uj,x} + A_{xs} \phi_{uj,y} \} d\Omega$$

$$K_{vij} = \int_{\Omega} \psi_{vi,y} \{ A_{xs} \phi_{uj,x} + A_{ss} \phi_{uj,y} \} d\Omega$$

$$K_{wij} = \int_{\Omega} \psi_{wi,x} (A_{xy} \phi_{uj,y} + A_{xs} \phi_{uj,x}) d\Omega + \int_{\Omega} \psi_{wi,y} (A_{ys} \phi_{uj,y} + A_{ss} \phi_{uj,x}) d\Omega$$

$$K_{uij} = \int_{\Omega} -\psi_{ui,x} (B_{xx} \phi_{wj,xx} + B_{xy} \phi_{wj,xy} + 2B_{xs} \phi_{wj,xy}) d\Omega + \int_{\Omega} -\psi_{ui,y} (B_{xs} \phi_{wj,xx} + B_{ys} \phi_{wj,xy} + 2B_{ss} \phi_{wj,xy}) d\Omega$$

$$K_{vuij} = K_{vji}$$

$$K_{vuij} = \int_{\Omega} \psi_{vi,y} \{ A_{yy} \phi_{vj,y} + A_{ys} \phi_{vj,x} \} d\Omega \\ + \int_{\Omega} \psi_{vi,x} \{ A_{ys} \phi_{vj,y} + A_{ss} \phi_{vj,x} \} d\Omega$$

$$K_{wuij} = \int_{\Omega} -\psi_{vi,y} (B_{yy} \phi_{wj,yy} + B_{yx} \phi_{wj,xx} + 2B_{ys} \phi_{wj,xy}) d\Omega \\ + \int_{\Omega} -\psi_{vi,x} (B_{ys} \phi_{wj,yy} + B_{xs} \phi_{wj,xx} + 2B_{ss} \phi_{wj,xy}) d\Omega$$

$$K_{wuij} = \int_{\Omega} \psi_{wi,xx} (D_{xx} \phi_{wj,xx} + D_{xy} \phi_{wj,yy} + 2D_{xs} \phi_{wj,xy}) d\Omega \\ + \int_{\Omega} \psi_{wi,yy} (D_{xy} \phi_{wj,xx} + D_{yy} \phi_{wj,yy} + 2D_{ys} \phi_{wj,xy}) d\Omega$$

$$+ \int_{\Omega} 2\psi_{wi,xy} (D_{xs} \phi_{wj,xx} + D_{ys} \phi_{wj,yy} + 2D_{ss} \phi_{wj,xy}) d\Omega$$

$$K_{wuij} = K_{wji}$$

$$K_{wuij} = K_{wji}$$

$$\{F\}_{3 \times p \times 1} = \begin{Bmatrix} \{F_u\}_{p \times 1} \\ \{F_v\}_{p \times 1} \\ \{F_w\}_{p \times 1} \end{Bmatrix}$$

$$F_{ui} = \int_{\Omega} q_x \psi_{ui} d\Omega$$

$$F_{vi} = \int_{\Omega} q_y \psi_{vi} d\Omega$$

$$F_{wi} = \int_{\Omega} q_z \psi_{wi} d\Omega$$