Constitutive Modeling Summary (Fluid Part)

1. Cell complexes

When examing fluids, it is often be wise to split the space up in cells. If we then know the relation between cells, we can perform calculations with them. This chapter examines the tricks we can apply with such cells. We also examine the relation with the actual integral equations.

1.1 Introduction

1.1.1 What is constitutive modelling?

Let's examine some mechanical system. The state of the system is described by **configuration variables**. These are variables like position, velocity and acceleration. This state is influenced by the so-called **source terms**. These are variables like force, stress and pressure.

Each of these two types of variables lives in its own realm. They can be equated with each other. By using integral relations and such. That's what we'll look at first in this summary.

However, equating configuration variables to source terms is more difficult. To equate variables from different realms, we need material parameters and other physical constants. The resulting relations form the **constitutive model**.

1.1.2 Mathematical operators

Many equations apply on different kinds of object. Some equations apply on volumes, others on planes and others on lines. Luckily, there are mathematical equations with which these equations can be related. For example, the **divergence theorem** relates a vector field $\mathbf{A}$ on a surface to a scalar field div $\mathbf{A}$ on a volume. It does this according to

$$\int_{\partial\Omega} \mathbf{A} \cdot d\mathbf{S} = \int_{\Omega} \text{div } \mathbf{A}\, dV. \tag{1.1.1}$$

So the **divergence operator** div relates the **space of surfaces** H_S to the **space of volumes** H_V. Similarly, according to **Stokes' theorem**, the **curl operator** curl relates the **space of lines** H_L to the space of surfaces H_L. Finally, the **gradient operator** grad relates the **space of points** H_P to the space of lines. This gives us the following mappings.

$$H_P \xrightarrow{\text{grad}} H_L \xrightarrow{\text{curl}} H_S \xrightarrow{\text{div}} H_V. \tag{1.1.2}$$

The above mapping is kind of special. When we map two operators in a row, we always get zero. (Mathematically speaking, we say that the **null space** of one operator coincides with the **range** of the other.) So we have

$$\text{curl grad } = 0 \qquad \text{and} \qquad \text{div curl } = 0. \tag{1.1.3}$$

For this reason, the above sequence is called an **exact sequence**.

1.2 Definitions

1.2.1 Cell complexes and dual complexes

Let's consider the n-dimensional space $\mathbb{R}^n$. We can divide this space in several p**-cells**, where p indicates the dimension of the cell. For example, a point is a 0-cell, a line is a 1-cell, a surface is a 2-cell, a volume is a 3-cell, and so on.

The collection of all the p-cells is called a **(primal) cell complex**. A cell complex is usually denoted by K. We also define α_p as t he number of p-cells in K.

There can also be a **dual cell complex** $\tilde{K}$ corresponding to a primal cell complex K. In this case, the p-cells of $\tilde{K}$ lie in the $(n-p)$-cells of K, and vice verse. For example, in a 2-dimensional space (a plane), the points (0-cells) of $\tilde{K}$ lie in the surfaces (2-cells) of K and vice verse. Also, the lines (1-cells) of $\tilde{K}$ cross the lines (1-cells) of K.

The p-cells in a cell complex K are usually numbered. This numbering can be done arbitrarily. However, the numbering of the p-cells in a dual cell complex $\tilde{K}$ is not arbitrary. We just saw that p-cells in $\tilde{K}$ correspond to $(n-p)$-cells in K. These correspondings cells are (by convention) given the same number.

1.2.2 Faces and cofaces

Let's examine a p-cell Q. (For example, a 2-cell, or a surface.) The **faces** of Q are the $(p-1)$-cells that form the boundary of Q. (So the faces of Q are the boundaries of Q.) The **cofaces** of Q are the $(p+1)$-cells that have Q as a face. (So the cofaces of Q are the objects which Q bounds.)

1.2.3 Chains and cochains

Let's examine a cell complex K. The collection of all α_p p-cells in K is called a p**-chain**. A collection of only some p-cells is called a p**-sub-chain**. The **boundary** δC of a p-sub-chain C consists of all $p-1$-cells which bound C. (More strictly speaking, it consists of all $p-1$-cells which have exactly one item of C as a coface.)

With each p-chain, we can associate a set of α_p numbers/vectors $(b_1, b_2, \ldots, b_{\alpha_p})$. The function that assigns the numbers/vectors to the p-cells is called a p**-cochain**. Two p-cochains $a^{(p)}$ and $b^{(p)}$ can be added up. To do this, you simply have to add up the individual elements. So

$$a^{(p)} + b^{(p)} = (a_1 + b_1, a_2 + b_2, \ldots, a_{\alpha_p} + b_{\alpha_p}). \tag{1.2.1}$$

We can also take the integral over a certain p-chain, or p-sub-chain. To do this, we have to add up all the elements in the corresponding co-chain. So we have

$$\int_p a^{(p)} = \sum a_i. \tag{1.2.2}$$

1.2.4 The incidence matrix and the coboundary operator

p-cells are usually also given an orientation. For 0-cells (points), this can be either inward or outward. For 1-cells (lines), this can be in one direction along the line, or in the opposite direction. For 2-cells (surfaces), this can be clockwise or counterclockwise. And so on.

These orientations are necessary to find the **incidence coefficients** $e_{ij}^{(p+1),(p)}$. To find them, we need to

examine the i-th $(p+1)$-cell and the j-th p-cell. The coefficients are then defined as

$$e_{ij}^{(p+1),(p)} = \begin{cases} 0 & \text{if the } p-\text{cell is not a face of the } (p+1)-\text{cell,} \\ 1 & \text{if the cells have the same orientation,} \\ -1 & \text{if the cells have opposite orientation.} \end{cases} \tag{1.2.3}$$

We can find the incidence coefficients for all combinations of $(p+1)$-cells i and p-cells j. If we put all these coefficients in an $\alpha_{p+1} \times \alpha_p$ matrix, we have found the **incidence matrix** $E^{(p+1),(p)}$.

The incidence matrix allows us to generate $(p+1)$-cochains from p-cochains. For example, we can say that

$$b^{(p+1)} = E^{(p+1),(p)} a^{(p)} = \delta a^{(p)}, \tag{1.2.4}$$

where the δ, called the **coboundary operator**, is just another way of writing the above equation. We find that the incidence matrix and the coboundary operator have some interesting properties. We can, for example, multiply the matrices $E^{(p+2),(p+1)}$ and $E^{(p+1),p}$. It can then be shown that

$$E^{(p+2),(p+1)} E^{(p+1),(p)} a^{(p)} = E^{(p+2),(p)} a^{(p)} = 0^{(p+2)}, \qquad \text{or} \qquad \delta\delta a^{(p)} = 0^{(p+2)}. \tag{1.2.5}$$

So applying the coboundary operator twice always leads to the **null cochain** (the cochain filled with only zeroes). This actually makes sense. Because we're in fact finding the boundary of a boundary. Imagine some arbitrary surface. Now take its boundary. (It's a line.) This line doesn't have any end points. So the boundary of the boundary of the surface simply doesn't exist. It works the same for objects in other dimensions.

1.2.5 Coboundaries and cocycles

Let's take a look at the coboundary operator more closely. Suppose we apply it on a $(p-1)$-cochain $a^{(p-1)}$. The resulting p-cochain $b^p = \delta a^{(p-1)}$ is the so-called **coboundary** of $a^{(p-1)}$. In fact, we call any p-cochain b^p that is the coboundary of some $(p-1)$-cochain $a^{(p-1)}$ a **p-coboundary**.

Some p-cochains $b(p)$ have a zero coboundary. So $\delta b^{(p)} = 0^{(p+1)}$. Such cochains are called **cocycles**. It is interesting to note that any p-coboundaries b^p is a cocycle. (This is because $\delta b^p = \delta\delta a^{(p-1)} = 0^{(p+1)}$. We'll see in the next paragraph why this holds.) However, not any cocycle is a coboundary.

Finally, we say that two p-cochains are **cohomologous** if their difference is a coboundary. So $a^{(p)}$ and $b^{(p)}$ are cohomologous if there is a $(p-1)$-cochain $c^{(p-1)}$ for which $a^{(p)} - b^{(p)} = \delta c^{(p-1)}$.

1.3 Connections to the real world

1.3.1 Again an exact sequence

Previously, we have seen that we can use incidence matrices (or equivalently, the coboundary operator) to transform p-cochains to $(p+1)$-cochains. We can, for example, transform 1-cochains, connected to lines, to 2-cochains, which correspond to surface. So we can map $H_L \to H_S$. Let's visualize what other mappings we can do. We then find that

$$H_P \xrightarrow{\delta} H_L \xrightarrow{\delta} H_S \xrightarrow{\delta} H_V. \tag{1.3.1}$$

These mappings are exactly the same as the mappings we could do with grad, curl and div. So the coboundary operator seems to be a good replacement for all the integral theorems. By the way, since $\delta\delta = 0$, the above sequence again is an exact sequence.

1.3.2 The connection with integrals

Let's examine some cell complex. We can choose a p-sub-chain C in this path. (For example, a set of connected lines.) We associate with this p-sub-chain, the p-cochain $\delta a^{(p-1)}$, where $a^{(p-1)}$ is some $(p-1)-cochain$. We can then integrate over the p-sub-chain C. It can then be shown that

$$\int_C \delta a^{(p-1)} = \delta_{ij} = \int_{\delta C} a^{(p-1)}. \tag{1.3.2}$$

δ_{ij} is the so-called **Kronecker delta**. This important relation is called the **(generalized) Stokes' theorem**. But what does this mean? Well, we can integrate over the sub-chain of $\delta a^{(p-1)}$. But we can also integrate over the boundary of the sub-chain of $a^{(p-1)}$. And, according to the above theorem, both ways give exactly the same result.

The generalized Stokes' theorem is, as its name implies, a generalized version of many integral relations. We can, for example, recall the **gradient operator** grad. It satisfies

$$\int_C (\text{grad } p) \cdot d\mathbf{s} = p(C_{end}) - p(C_{begin}). \tag{1.3.3}$$

This integral relation says the same as the generalized Stokes' theorem for $p = 1$. We can examine δp along a one-dimensional curve. We can also simply examine p at the endpoints of the curve. It gives the same result.

Transforming other integral relations to integrals of sub-chains goes similar. For example, consider the conservation law

$$\frac{d}{dt}\int_\Omega \phi \, d\Omega + \int_{\delta\Omega} \mathbf{F}\, d\mathbf{S} = \int_\Omega q \, d\Omega. \tag{1.3.4}$$

If we use cochains, then the above equation turns into

$$\frac{d}{dt}\int_\Omega \phi^{(3)} + \int_{\delta\Omega} F^{(2)} = \int_\Omega q^{(3)} \qquad \Rightarrow \qquad \frac{d}{dt}\phi^{(3)} + \delta F^{(2)} = q^{(3)}. \tag{1.3.5}$$

Note that we have used the generalized Stokes' theorem on the middle part. After this, we were allowed to remove the integral. The final equation has no integrals anymore. Nor is there any connection to the volume Ω. So, we see that using cell complexes and cochains is a great way to get rid of integrals and geometry. Since there is no

1.3.3 The Laplace equation

Now let's consider the Laplace equation $\Delta\phi$, or equivalently, div grad ϕ. How can we transform this equation? We can't use $\delta\delta\phi$, since that would simply give zero. (You may wonder why the Laplacian does not give zero. This is because grad maps from points to lines, and div from surfaces to volumes.) So how do we tackle this equation?

The trick lies in the dual complex. We saw that (in 3D space), lines correspond to surfaces, and points correspond to volumes. So we can start with points. Then we use the grad operator to get lines. We then move to the dual complex to get surfaces. After this, we use the div operator to get volumes. Finally, we can move back to our original cell complex. We can display these mappings as

$$\begin{array}{ccccccc} H_P & \xrightarrow{\text{grad}} & H_L & \xrightarrow{\text{curl}} & H_S & \xrightarrow{\text{div}} & H_V \\ \updownarrow * & & \updownarrow * & & \updownarrow * & & \updownarrow * \\ H_{\tilde{V}} & \xleftarrow{\text{div}} & H_{\tilde{S}} & \xleftarrow{\text{curl}} & H_{\tilde{L}} & \xleftarrow{\text{grad}} & H_{\tilde{P}}. \end{array} \tag{1.3.6}$$

In this mapping, every square corresponds to the Laplace operator. So in fact, $\Delta\phi = *\delta * \delta\ \phi$. The so-called **Hodge operator** $*$ is used to transfer between the primal complex K and the dual complex

$\tilde{K}$. A **constitutive model** is a model which maps between the primal complex and the dual complex. So we find that the Hodge operator $*$ is, in fact, a constitutive model.

2. Coordinates, vectors and tensors

To express properties like location in our world, we need coordinates. How do coordinates work? And what fun things can we do with them? That's what this chapter is about.

2.1 Coordinates and 1-vectors

2.1.1 Coordinate systems

Let's consider an n-dimensional space. A **coordinate system** is a function $X(x^1, x^2, \ldots, x^n)$, which assigns to every point in space n numbers $x^1, x^2, \ldots, x^n$. These numbers are called the **coordinates**. A point can have different representations in different coordinate systems.

Suppose we have a coordinate system. We can then draw coordinate lines. **Coordinate lines** are lines for which $n-1$ coordinates are fixed. As the non-fixed coordinate varies, a line is drawn. (Note that these aren't always straight lines.)

2.1.2 Base vectors

A coordinate system also has **base vectors**. Thse vectors are vectors tangent to the coordinate lines. Together, they form the **(covariant) basis** of the system. The base vectors are defined as

$$\mathbf{e_1} = \frac{\partial \mathbf{X}}{\partial x^1}, \qquad \mathbf{e_2} = \frac{\partial \mathbf{X}}{\partial x^2}, \qquad \ldots, \qquad \mathbf{e_n} = \frac{\partial \mathbf{X}}{\partial x^n}. \tag{2.1.1}$$

The base vectors can be different at different points in the coordinate system. Also, they do not necessarily have length 1.

A covariant basis also has a corresponding **contravariant basis** (also known as the **dual basis**). Th contravariant base vectors $\mathbf{e^j}$ are defined such that

$$\mathbf{e^j} \cdot \mathbf{e_i} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{2.1.2}$$

Note that the contravariant basis is denoted by superscripts, while the covariant basis uses subscripts. To find the contravariant basis, you could take the matrix of covariant base vectors $[\mathbf{e_1} \ \ldots \ \mathbf{e_n}]$. If we invert it, we get the matrix of contravariant base vectors.

2.1.3 Normal vectors

Suppose that we have two points A and B. We can indicate their relative position by a vector. We can write down a vector $\mathbf{a}$ in the covariant basis as

$$\mathbf{a} = a^1\mathbf{e_1} + a^2\mathbf{e_2} + \ldots + a^n\mathbf{e_n} = \sum_{i=1}^{n} a^i\mathbf{e_i}. \tag{2.1.3}$$

The coefficients a^i are called the **contravariant coefficients**, since they have a superscript. We could also express the vector in the contravariant basis. We would then write it as

$$\mathbf{a} = a_1\mathbf{e^1} + a_2\mathbf{e^2} + \ldots + a_n\mathbf{e^n} = \sum_{i=1}^{n} a_i\mathbf{e^i}. \tag{2.1.4}$$

The coefficient a_i are the **covariant coefficients**, since they have a subscript. Of course, there is a relation between these coefficients. By using the definition of the dual basis, we can find that

$$a_j = \sum_{i=1}^{n} a_i \mathbf{e^i} \cdot \mathbf{e_j} = \mathbf{a} \cdot \mathbf{e_j} = \sum_{i=1}^{n} a^i \mathbf{e_i} \cdot \mathbf{e_j}. \tag{2.1.5}$$

2.1.4 The Einstein summation convention

We just saw that, to express a vector, we needed to add up n values. For that, we could use dots ... or the summation sign $\sum$. However, doing this every time could be a bit tiring. Therefore, from now on, we will use the **Einstein summation convention**. When, in a single term, there is both a subscript and an equal superscript, we make a summation. This means that

$$\sum_{i=1}^{n} a^i \mathbf{e_i} \quad \text{means the same as} \quad a^i \mathbf{e_i} \quad \text{and} \quad \sum_{i=1}^{n} a_i \mathbf{e^i} \quad \text{means the same as} \quad a_i \mathbf{e^i} \quad \text{as well.} \tag{2.1.6}$$

That should save us some ink.

2.1.5 Change of coordinates

Let's suppose we have a coordinate system $\mathbf{X}(x^1, \ldots, x^n)$. However, we move to a new set of coordinates $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n$. The functions $\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n)$ are given. In this case, the new base vectors become

$$\tilde{\mathbf{e}}_\mathbf{k} = \frac{\partial \mathbf{X}}{\partial \tilde{x}^k} = \frac{\partial \mathbf{X}}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^k} = \mathbf{e_i} \frac{\partial x^i}{\partial \tilde{x}^k}, \qquad \text{and similarly,} \qquad \mathbf{e_k} = \tilde{\mathbf{e}}_\mathbf{i} \frac{\partial \tilde{x}^i}{\partial x^k}. \tag{2.1.7}$$

It is important to note that we have used the Einstein summation convention in the above equation. So keep in mind that the above equation actually is a sum.

We can also express any vector $\mathbf{a}$ in our new coordinates. We simply need to find the new coefficients $\tilde{a}^i$. In this case, we have

$$\mathbf{a} = a^k \mathbf{e_k} = a^k \left(\tilde{\mathbf{e}}_\mathbf{i} \frac{\partial \tilde{x}^i}{\partial x^k} \right) = \left(a^k \frac{\partial \tilde{x}^i}{\partial x^k} \right) \tilde{\mathbf{e}}_\mathbf{i} = \tilde{a}^i \tilde{\mathbf{e}}_\mathbf{i}, \qquad \text{which implies that} \qquad \tilde{a}^i = a^k \frac{\partial \tilde{x}^i}{\partial x^k}. \tag{2.1.8}$$

And the transformation is complete.

2.2 Multi-vectors

2.2.1 2-vectors

We now know how to describe points (with three coordinates) and lines (with a vector). But how would we describe a surface? For this, we use **2-vectors**. We do this using the **wedge operator** $\wedge$. Let's suppose we have two vectors $\mathbf{a}$ and $\mathbf{b}$. Together, they can form the 2-vector $(\mathbf{a} \wedge \mathbf{b})$.

The two-vector is subject to several rules. The most important rules are

$$\begin{aligned}
c_1(\mathbf{a} \wedge \mathbf{b}) + c_2(\mathbf{a} \wedge \mathbf{b}) &= (c_1 + c_2)(\mathbf{a} \wedge \mathbf{b}), && (2.2.1)\\
(\mathbf{a} \wedge \mathbf{d}) + (\mathbf{b} \wedge \mathbf{d}) &= ((\mathbf{a} + \mathbf{b}) \wedge \mathbf{d}), && (2.2.2)\\
(c\mathbf{a} \wedge \mathbf{b}) = (\mathbf{a} \wedge c\mathbf{b}) &= c(\mathbf{a} \wedge \mathbf{b}), && (2.2.3)\\
(\mathbf{a} \wedge \mathbf{b}) &= -(\mathbf{b} \wedge \mathbf{a}), && (2.2.4)\\
(\mathbf{a} \wedge \mathbf{a}) &= 0. && (2.2.5)
\end{aligned}$$

One way to think of the 2-vector $(\mathbf{a} \wedge \mathbf{b})$ is as the surface spanned by the two vectors $\mathbf{a}$ and $\mathbf{b}$. It then also makes sense why $(\mathbf{a} \wedge \mathbf{a}) = 0$. A single vector can't span a surface by itself.

Let's suppose that $\mathbf{u} = a\mathbf{e_1} + b\mathbf{e_2}$ and $\mathbf{v} = c\mathbf{e_1} + d\mathbf{e_2}$. We can then simplify $(\mathbf{u} \wedge \mathbf{v})$ to

$$((a\mathbf{e_1}+b\mathbf{e_2})\wedge(c\mathbf{e_1}+d\mathbf{e_2})) = ac(\mathbf{e_1}\wedge\mathbf{e_1})+ad(\mathbf{e_1}\wedge\mathbf{e_2})+bc(\mathbf{e_2}\wedge\mathbf{e_1})+bd(\mathbf{e_2}\wedge\mathbf{e_2}) = (ad-bc)(\mathbf{e_1}\wedge\mathbf{e_2}). \quad (2.2.6)$$

Another way to represent a surface, is by using the normal vector. Let's examine the surface $(\mathbf{e_1}, \mathbf{e_2})$. The normal vector of this surface is $\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}$. (Similarly, $\mathbf{e_2} \times \mathbf{e_3} = \mathbf{e_1}$ and $\mathbf{e_3} \times \mathbf{e_1} = \mathbf{e_2}$. So, instead of taking the wedge operator, we could use the cross product to respresent surfaces. In this case, we would also find that

$$\mathbf{u} \times \mathbf{v} = (a\mathbf{e_1} + b\mathbf{e_2}) \times (c\mathbf{e_1} + d\mathbf{e_2}) = (ad - bc)\mathbf{e_3}. \quad (2.2.7)$$

We see that this matches with what we found earlier.

2.2.2 3-vectors

Just like a 2-vector represents a surface, so does a 3-vector represent a volume. We denote such a 3-vector by $(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})$. There are rules for 3-vectors as well. The most important ones are

$$\begin{aligned} a(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) + b(\mathbf{u'} \wedge \mathbf{v} \wedge \mathbf{w}) &= ((a\mathbf{u} + b\mathbf{u'}) \wedge \mathbf{v} \wedge \mathbf{w}), & (2.2.8)\\ (\mathbf{u} \wedge \mathbf{u} \wedge \mathbf{v}) = (\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{u}) = (\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{u}) &= 0, & (2.2.9)\\ (\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) = (\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v}) = (\mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u}) &= -(\mathbf{w} \wedge \mathbf{v} \wedge \mathbf{u}). & (2.2.10) \end{aligned}$$

Now let's suppose $\mathbf{u} = a\mathbf{e_1} + b\mathbf{e_2} + c\mathbf{e_3}$, $\mathbf{v} = k\mathbf{e_1} + l\mathbf{e_2} + m\mathbf{e_3}$ and $\mathbf{v} = p\mathbf{e_1} + q\mathbf{e_2} + e\mathbf{e_3}$. We can then simplify their wedge product to

$$(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) = \det \begin{vmatrix} a & b & c \\ k & l & m \\ p & q & r \end{vmatrix} (\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}). \quad (2.2.11)$$

2.3 Tensors

2.3.1 Tensor definitions

Tensors can be used to transform one vector to another. For example, we can say that the tensor $\mathbf{A}$ transforms vector $\mathbf{a}$ to vector $\mathbf{b}$. We write this as $\mathbf{b} = A\mathbf{a}$. We assume the tensor transforms vectors linearly. So,

$$A(c\mathbf{a}) = cA\mathbf{a} \qquad \text{and} \qquad A(\mathbf{a} + \mathbf{b}) = A\mathbf{a} + A\mathbf{b}. \quad (2.3.1)$$

Let's suppose that we have $\mathbf{b} = A\mathbf{a}$. The **inverse** of a tensor A (denoted by A^{-1}) is defined such that $\mathbf{a} = A^{-1}\mathbf{b}$, for every $\mathbf{a}$ and $\mathbf{b}$. The **transpose** of a vector A (denoted by A^T) is the vector which satisfies $\mathbf{b} \cdot A\mathbf{a} = \mathbf{a} \cdot A^T\mathbf{b}$, for every $\mathbf{a}$ and $\mathbf{b}$. If a tensor satisfies $A^T = A$, then it is called **symmetric**. If $A^T = -A$, then it is **anti-symmetric** (also known as **skew-symmetric**). If we have $A^T = A^{-1}$, then A is an **orthogonal** tensor. Finally, we define the identity tensor I as the tensor satisfying $\mathbf{a} = I\mathbf{a}$, for every $\mathbf{a}$.

2.3.2 Adding coordinate systems

The rules of the previous paragraph don't require any coordinate system. If we, however, do add a coordinate system, then we can represent a tensor A as a matrix. You should be careful with this though, as the matrix differs per coordinate system.

Let's suppose we know how a tensor A transforms vectors. How can we find the appropriate matrix? To find that out, we examine $\mathbf{b} = A\mathbf{a}$. Rewriting this, using the Einstein summation convention, gives

$$b^j \mathbf{e_j} = A\left(a^i \mathbf{e_i}\right) = a^i \left(A\mathbf{e_i}\right). \tag{2.3.2}$$

Left-multiplying by the dual basis vector $\mathbf{e^k}$ gives

$$b^k = b^j \mathbf{e^k} \cdot \mathbf{e_j} = a^i \left(\mathbf{e^k} \cdot A\mathbf{e_i}\right). \tag{2.3.3}$$

We thus find that A_i^k (being the component of A in the k-th row and the i-th column) is

$$A_i^k = \mathbf{e^k} \cdot A\mathbf{e_i}. \tag{2.3.4}$$

Note that we can now also write $b^k = A_i^k a^i$.

2.3.3 Change of variables

Let's suppose we know all the coefficients A_i^k. But now we move to a new coordinate system, having coordinates $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n$. Again, the functions $\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n)$ are given. How can we find the new components of the transformation matrix $\tilde{A}_i^k$?

To do this, we write $\tilde{A}_i^k$ as $\tilde{\mathbf{e}}^\mathbf{k} A \tilde{\mathbf{e}}_\mathbf{i}$. We can then apply the change of base vector equation (2.1.7) for base vectors. If we also work things out, we will find that

$$\tilde{A}_i^k = \tilde{\mathbf{e}}^\mathbf{k} A \tilde{\mathbf{e}}_\mathbf{i} = \left(\mathbf{e^l} \frac{\partial x^l}{\partial \tilde{x}^k}\right) A \left(\mathbf{e_j} \frac{\partial x_j}{\partial \tilde{x}_i}\right) = \frac{\partial x^l}{\partial \tilde{x}^k} \left(\mathbf{e^l} A \mathbf{e_j}\right) \frac{\partial x_j}{\partial \tilde{x}_i} = \frac{\partial x^l}{\partial \tilde{x}^k} \frac{\partial x_j}{\partial \tilde{x}_i} A_j^l. \tag{2.3.5}$$

The final relation above may look simple. But do remember that you need to sum up 9 individual parts to find the single component $\tilde{A}_i^k$, due to the summation convention. We could, of course, also reverse the above relation. We then would have

$$A_j^l = \frac{\partial \tilde{x}^k}{\partial x^l} \frac{\partial \tilde{x}_i}{\partial x_j} \tilde{A}_i^k. \tag{2.3.6}$$

3. Velocity gradients and stress tensors

It's time to look at an actual constitutive model: the stress tensor. How can we relate it to a physical phenomenon like velocity?

3.1 Indifferent vectors and tensors

3.1.1 Configuration and source variable notation

It's time to turn our attention to some less abstract stuff. First let's examine configuration variables (like position and velocity). Suppose we have some point P. We can indicate the position of P by a vector $\mathbf{x}$. Similarly, the velocity of P is denoted by $\mathbf{v} = d\mathbf{x}/dt = \dot{\mathbf{x}}$. These vectors (and all other vectors belonging to configuration variables) live in the **primal space**. We can also add a coordinate system to our primal space. We can then write a position as $\mathbf{x} = x^i \mathbf{e_i}$ and a velocity as $\mathbf{v} = v^i \mathbf{e_i}$.

Source variables (like forces $\mathbf{f}$ and stresses τ) are, however, something different. They live in the **dual space**. For that reason, we can't represent them by vectors. But instead, we use **covectors**. Luckily, these covectors are quite similar to vectors. Only the indices have changed position. We should, for example, write the force covector as $\mathbf{f} = f_i \mathbf{e^i}$.

3.1.2 Links between primal and dual space

The question remains, how do we go from the primal space to the dual space? To do that, we use tensors. An example is the **stress tensor** σ. This tensor relates a normal vector $\mathbf{n}$ (in the primal space) to a stress covector $\mathbf{f}$ (in the dual space). It does this according to

$$\mathbf{f} = \sigma \mathbf{n}. \tag{3.1.1}$$

When we use coordinate systems, we can represent σ by a matrix. In this case we can also write that

$$f_i = \sigma_{ij} n^j. \tag{3.1.2}$$

3.1.3 Indifferent vectors

The relation that $\mathbf{f} = \sigma \mathbf{n}$ does not depend on the coordinate system we use. In other words, if we change coordinate systems, it should still be satisfied. That is, as long as the distances remain the same. (In stretched space strange things occur.) So let's examine such a (non-stretched) transformation. In fact, let's examine the worst transformation possible. Let's consider

$$\mathbf{x}^* = Q(t)(\mathbf{x} - \mathbf{z}) + \mathbf{y}(t), \tag{3.1.3}$$

where $\mathbf{x}$ denotes the initial vector, and $\mathbf{x}^*$ denotes the transformed vector. What does this transformation do? Well, first it moves the origin of the coordinate system to $\mathbf{z}$. Then the orthogonal **transformation matrix** $Q(t)$, which might even vary with time, causes things to rotate about point $\mathbf{z}$. (It is important that $Q(t)$ is orthogonal. In non-orthogonal transformations, distances don't remain the same.) Finally, we move the whole system by a vector $\mathbf{y}(t)$.

Yes, it's a scary transformation. Now let's make a definition. We say that any vector $\mathbf{a}$ which satisfies

$$\mathbf{a}^* = Q(t)\mathbf{a} \tag{3.1.4}$$

is called an **indifferent vector**. A normal position vector $\mathbf{x}$ is not indifferent, since $\mathbf{x}^* \neq Q(t)\mathbf{x}$. However, a difference vector $\mathbf{a} = \mathbf{x_2} - \mathbf{x_1}$ is indifferent. To see why, we could insert it into our transformation

(3.1.3). This would eventually give us

$$\mathbf{x_2^*} - \mathbf{x_1^*} = Q(t)(\mathbf{x_2} - \mathbf{x_2}) \qquad \Rightarrow \qquad \mathbf{a}^* = Q(t)\mathbf{a}. \tag{3.1.5}$$

(The vectors $\mathbf{y}$ and $\mathbf{z}$ cancel out.) However, the velocity vector $\mathbf{v}$ is not indifferent. To see why, we could differentiate equation (3.1.3) with respect to time t. This would then give us

$$\mathbf{v}^* = \dot{\mathbf{x}}^* = Q\mathbf{v} + \dot{Q}(\mathbf{x} - \mathbf{z}) + \dot{\mathbf{y}}. \tag{3.1.6}$$

(For simplicity we have written $Q(t)$ as Q.) So $\mathbf{v}^* \neq Q\mathbf{v}$.

3.1.4 Indifferent tensors

Tensors that map indifferent vectors onto indifferent vectors are called **indifferent tensors**. So if $\mathbf{b}$ and $\mathbf{a}$ are indifferent vectors, and $\mathbf{b} = A\mathbf{a}$, then the tensor A is indifferent. An indifferent tensor A also has a corresponding indifferent cotensor A^* satisfying $\mathbf{b}^* = A^*\mathbf{a}^*$. Of course there is a relation between A and A^*. To find this relation, we use

$$\mathbf{b}^* = Q\mathbf{b} = QA\mathbf{a} = QAQ^T\mathbf{a}^*. \tag{3.1.7}$$

Note that, in the last step, we have used $\mathbf{a} = Q^T\mathbf{a}^*$. Because Q is orthogonal, it satisfies $Q^T = Q^{-1}$. From the above equation follows that

$$A^* = QAQ^T. \tag{3.1.8}$$

3.2 Velocity properties

3.2.1 A vector gradient

Let's examine a vector field $\mathbf{a}$. We now define the **gradient** $\nabla\mathbf{a}$ of a vector field $\mathbf{a}$. It is the tensor which converts a change in position $\mathbf{x}$ (written as $d\mathbf{x}$) to a change in the vector $\mathbf{a}$ (denoted by $d\mathbf{a}$). So we have $d\mathbf{a} = \nabla\mathbf{a}\, d\mathbf{x}$.

If we use a 3D Cartesian coordinate system, we could write this in matrix form, according to

$$\begin{bmatrix} \delta a_1 \\ \delta a_2 \\ \delta a_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}. \tag{3.2.1}$$

3.2.2 The velocity gradient and the rate of strain

The **velocity gradient** is denoted by $\nabla\mathbf{v}$. It satisfies

$$d\mathbf{v} = \nabla\mathbf{v}\, d\mathbf{x} \qquad \text{and} \qquad d\mathbf{v}^* = \nabla\mathbf{v}^* d\mathbf{x}^*. \tag{3.2.2}$$

We could ask ourselves, is $\nabla\mathbf{v}$ an indifferent tensor? Well, it can be shown that

$$\nabla\mathbf{v}^* = Q\ \nabla\mathbf{v}\ Q^T + \dot{Q}Q^T. \tag{3.2.3}$$

Since $\nabla\mathbf{v}^* \neq Q\ \nabla\mathbf{v}\ Q^T$, the velocity gradient is not an indifferent tensor.

Now let's examine the **rate of strain** tensor D. It is defined as

$$D = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right). \tag{3.2.4}$$

In other words, the components of D are

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right). \tag{3.2.5}$$

The diagonal entries of D are called the **normal strain rates**. The non-diagonal entries are the **shear strain rates**. By the way, it can be shown that $\nabla D^* = Q\ \nabla D\ Q^T$. So the rate of strain tensor is indifferent.

3.2.3 Rotation and vorticity

Similar to the rate of strain tensor, we can define the **rate of rotation tensor** Ω. It is given by

$$\Omega = \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T\right), \qquad \text{or} \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right). \tag{3.2.6}$$

We can now also define the **rotation vector** ω as

$$\omega = \omega_{32}\mathbf{e_1} + \omega_{13}\mathbf{e_2} + \omega_{21}\mathbf{e_3} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u^3}{\partial x^2} - \frac{\partial u^2}{\partial x^3}\right) \\ \frac{1}{2}\left(\frac{\partial u^1}{\partial x^3} - \frac{\partial u^3}{\partial x^1}\right) \\ \frac{1}{2}\left(\frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2}\right) \end{bmatrix}^T. \tag{3.2.7}$$

Finally, the **vorticity vector** ξ is defined as $\xi = 2\omega = \text{curl } \mathbf{u}$.

3.3 The stress tensor

3.3.1 The total stress tensor

Now we turn our attention to the important stress tensor. It can be shown that the stress tensor σ is indifferent. (The equation $\sigma^* = Q\sigma Q^T$ holds.) We can split σ up in two parts, being

$$\sigma = -pI + \tau, \tag{3.3.1}$$

where I is the **identity tensor**. (The identity tensor satisfies $I^* = QIQ^T = QQ^T = I$.) The part $-pI$ represents the stresses due to **compression** of the fluid. (In fact, p is the pressure, so $-p$ is the compression.) The part τ is due to viscous stresses.

If we write σ like this, we call σ the **total stress tensor**. τ is the **extra stress tensor** and $-pI$ is the **extra deviatoric part**. There is a rule stating that adding and subtracting indifferent tensors also gives indifferent tensors. Since both σ and I are indifferent, also τ must be indifferent.

3.3.2 The extra stress tensor

We know from experience that viscous stresses depend on the velocity $\mathbf{v}$ of the fluid. So we need to find a constitutive model that relates τ to something related to velocity. We can not relate τ directly to $\mathbf{v}$, since $\mathbf{v}$ is not indifferent. The velocity gradient $\nabla \mathbf{v}$ is not indifferent either. However, the rate of strain tensor D is indifferent. For that reason, we generally say that $g(\tau) = f(D)$, with f and g functions. These functions depend on the type of fluid.

Let's look at the stress-strain behaviour of the fluid. If it is isotropic (the same in all directions) and linear, then τ only depends on two parameters. In this case, the so-called **linear isotropic (Newtonian) stress-strain relation** applies. It states that

$$\tau = \lambda(\text{div } \mathbf{u})I + 2\mu D. \tag{3.3.2}$$

The variable μ is the **viscosity**. This viscosity depends on the composition of the gas, the pressure p and the temperature T. Luckily, for $T < 3000K$, the dependence on pressure is negligible. In that case, **Sutherland's formula** holds. It states that

$$\frac{CT^{3/2}}{T+S}, \tag{3.3.3}$$

where $C = 1.458 \cdot 10^{-6} Pa\, s\, K^{1/2}$ and $S = 110.4K$.

So, we eventually find that the stress tensor can be represented by

$$\sigma = -pI + \lambda(\text{div } \mathbf{u})I + 2\mu D. \tag{3.3.4}$$

4. The conservation laws

Very important in fluid dynamics are the three **conservation laws**, also known as the **Navier Stokes equations**. Let's see how they are derived, and how we can write them.

4.1 Two ways to describe fluids

4.1.1 Describing fluids

When describing fluids, we usually examine a small piece of fluid, called a **fluid packet**. There are, however, two ways to represent a packet.

In the **Lagrangian description**, the fluid properties (like density, temperature, and such) are attached to the packet. On top of that, the packet has a position vector $\mathbf{x}$ to indicate its position.

In the **Eulerian description**, the fluid properties are attached to the position vector. So, if we have a position vector $\mathbf{x}$, then we know the properties of the fluid packet $\mathbf{x}$ points to.

In solid mechanics, the Lagrangian description is more convenient. In fluid mechanics, however, we usually use the Eulerian description.

4.1.2 Transforming between descriptions

There are several conservation laws. There are **conservation of mass**, **conservation of momentum** and **conservation of energy**. In the Lagrangian description, these three laws are described by the equations

$$\frac{dm}{dt} = 0, \qquad \frac{d}{dt}\left(\frac{d\mathbf{x}}{dt}\right) = \mathbf{F} \qquad \text{and} \qquad \frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}. \tag{4.1.1}$$

m denotes mass, $\mathbf{F}$ the external forces, E the energy, Q the added heat and W the work done.

These laws aren't the same in the Eulerian description. This is because in the Lagrangian description, the position vector $\mathbf{x}$ changes with time. To find out how to transform these laws, we examine some property $\Phi(t, x^1, x^2, x^3)$. The time derivative of this property is given by

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial\Phi}{\partial x^2}\frac{dx^2}{dt} + \frac{\partial\Phi}{\partial x^3}\frac{dx^3}{dt} = \frac{\partial\Phi}{\partial t} + u^i\frac{\partial\Phi}{\partial x^i} = \frac{\partial\Phi}{\partial t} + \mathbf{u}\cdot(\nabla\Phi). \tag{4.1.2}$$

So we see that, to transform the laws to the Eulerian description, we have to use

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{4.1.3}$$

This derivative D/Dt is called the **Lagrangian time derivative** or the **material time derivative**. (Do not confuse it with the rate of strain tensor D.)

Now we know what to do with the time-derivative of a normal scalar. But what about time-derivatives of integrals? We often integrate over volumes. However, a volume in Lagrangian space (consisting of a lot of fluid packets) can change shape. This is rather troubling. Luckily, there is the **Reynolds transport theorem**. It states that, for some variable ϕ, we have

$$\frac{d}{dt}\int_V \Phi\, dV = \int_V \left(\frac{D\Phi}{Dt} + \Phi\nabla\mathbf{u}\right) dV. \tag{4.1.4}$$

Now we also know what to do with time-derivatives of integrals.

4.2 Conservation of mass

4.2.1 Derivation of the equation

With what we've just learned, let's derive the conservation laws. First, we examine conservation of mass. It states that the total mass doesn't change. To find the total mass, we integrate over the density. This gives us

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho \, dV = \int_V \left(\frac{D\rho}{Dt} + \rho\nabla\mathbf{u}\right) dV = 0. \tag{4.2.1}$$

This must hold for every volume. Therefore, the part within the integral must be zero. So,

$$\frac{D\rho}{Dt} + \nabla\mathbf{u} = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot(\nabla\rho) + \rho\nabla\mathbf{u} = 0. \tag{4.2.2}$$

This can further be simplified to

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{u}) = 0. \tag{4.2.3}$$

This equation is known as the **conservation of mass equation**.

4.3 Conservation of linear momentum

4.3.1 Derivation of the equation

Now let's consider the law of linear momentum. The time rate of change of linear momentum in a direction $\mathbf{u^i}$ must equal the force applied in that direction. We thus have

$$\frac{d}{dt}\int_V (\rho u^i) dV = \int_V \left(\frac{D(\rho u^i)}{dt} + \rho u^i \nabla\mathbf{u}\right) dV = \int_V \left(\frac{\partial(\rho u^i)}{\partial t} + \mathbf{u}\cdot(\nabla(\rho u^i)) + \rho u^i\nabla\mathbf{u}\right) dV = \int_V f^i \, dV, \tag{4.3.1}$$

Again, this must hold for every volume. So we can get rid of the integral. We can then also rewrite it to

$$\frac{\partial(\rho u^i)}{\partial t} + u^i\mathbf{u}\cdot(\nabla\rho) + \rho\mathbf{u}\cdot(\nabla u^i) + \rho u^i\nabla\mathbf{u} = \frac{\partial(\rho u^i)}{\partial t} + \nabla(\rho u^i\mathbf{u}) = \mathbf{f}. \tag{4.3.2}$$

Working this out further, we get

$$\rho\frac{\partial u^i}{\partial t} + u^i\frac{\partial\rho}{\partial t} + u^i\nabla(\rho\mathbf{u}) + \rho\mathbf{u}\cdot\nabla u^i = f_i. \tag{4.3.3}$$

Thanks to conservation of mass, the two middle terms cancel. We remain with

$$\rho\frac{\partial u^i}{\partial t} + \rho\mathbf{u}\cdot\nabla u^i = f_i. \tag{4.3.4}$$

This equation is known as the **conservation of linear momentum equation**.

4.3.2 The force time split up

The equation of the previous paragraph looked quite simple. However, we haven't looked at the force per unit volume $\mathbf{f}$ yet. (Or equivalently, at its component f_i.) So let's do that now.

The force per unit volume is made up out of a body force (per unit volume $\mathbf{f_b}$ and a surface force (per unit area) $\mathbf{f_s}$. This surface force has to be transformed to a force per unit volume as well, if we want to

apply it in our equation. By using the stress tensor, we find that $\mathbf{f_s} = \sigma\mathbf{n} = (-pI + \tau)\mathbf{n}$. It can then be shown that the force per unit volume due to $\mathbf{f_s}$ is given by $-\nabla p + \nabla\tau$. This eventually gives us

$$\rho\frac{\partial u^i}{\partial t} + \rho\mathbf{u}\cdot\nabla u^i = f_{b_i} - \frac{\partial p}{\partial x^i} + \frac{\partial \tau_{ij}}{\partial x^i}. \tag{4.3.5}$$

Note that we have used the summation convention in the last term. By the way, we have

$$\tau_{ij} = \lambda(\nabla\mathbf{u})\delta_{ij} + \mu\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right). \tag{4.3.6}$$

4.3.3 Variants of the equation

An **incompressible flow** is a flow in which fluid elements don't change in volume. This implies that $\nabla\mathbf{u} = 0$. It is often also assumed that $\rho = \rho_\infty =$ constant as well. If we combine these data with equations (4.3.5) and (4.3.6), we eventually find that

$$\rho_\infty\frac{\partial u^i}{\partial t} + \rho_\infty\mathbf{u}\cdot\nabla u^i = f_{b_i} - \frac{\partial p}{\partial x^i} + \nabla^2 u^i. \tag{4.3.7}$$

The above equation represents only one component of the linear momentum equation. We could put the above equation back into its general vector form. Let's do that, and rewrite it a bit, by bringing in the vorticity vector ξ. (Okay, we rewrite it a lot.) We then get

$$\rho_\infty\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\rho_\infty\mathbf{u}^2\right) = \mathbf{f_b} - \nabla p + \mathbf{u}\times\xi + \mu\nabla^2\mathbf{u}. \tag{4.3.8}$$

This is called the **Lamb-Gromeka equation**. Now let's assume that the flow is **steady** ($\partial\mathbf{u}/\partial t = 0$), there is **conservative body force** $\mathbf{f_b}$ (there is a function ϕ satisfying $\mathbf{f_b} = \nabla\mathbf{f}$) and the flow is **inviscid** ($\mu = 0$). The above equation then simplifies to

$$\nabla H(\mathbf{x}) = \rho_\infty\mathbf{u}\times\xi, \qquad \text{where} \qquad H(\mathbf{x}) = \nabla\left(\frac{1}{2}\rho_\infty\mathbf{u}^2 + p - \phi\right). \tag{4.3.9}$$

Now let's examine a curve C in space, along which $H(\mathbf{x})$ remains constant, and thus $\mathbf{u}\times\xi = \mathbf{0}$. (For example, a curve along which $\mathbf{u} = \mathbf{0}$, $\xi = \mathbf{0}$, or where $\mathbf{u}$ and ξ are parallel.) In this case, along the curve C, the above equation simplifies to

$$\frac{1}{2}\rho_\infty\mathbf{u}^2 + p - \phi = \text{constant}. \tag{4.3.10}$$

You might know this equation. (In case you don't, it's called **Bernoulli's equation**.)

4.4 Conservation of energy

4.4.1 The basic conservation of energy equation

It is time to examine **conservation of energy**. To find a relation for energy, we examine a small fluid packet. This packet has a **kinetic energy per unit mass** of $\frac{1}{2}\mathbf{u}^2$ and an **internal energy per unit mass** of e. So the total energy per unit mass is

$$E = \frac{1}{2}\mathbf{u}^2 + e. \tag{4.4.1}$$

Now let's examine a very small volume. The total energy is given by mE. The change of energy depends on the heat created Q and the work done W. So we have

$$\frac{d(mE)}{dt} = \frac{dQ}{dt} + \frac{dW}{dt} \tag{4.4.2}$$

We know that $dm/dt = 0$. Also, $m = \rho V$. This turns the above equation into

$$\rho V \frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt} \qquad \Rightarrow \qquad \rho \frac{dE}{dt} = \frac{dq}{dt} + \frac{dw}{dt}, \tag{4.4.3}$$

where q is the heat added per unit volume and w is the work done per unit volume. There is just one slight problem. The above equation is in the Lagrangian description. To transform it to the Eulerian description, we need to use the material derivative. We thus have

$$\rho \frac{DE}{Dt} = \frac{Dq}{Dt} + \frac{Dw}{Dt}. \tag{4.4.4}$$

4.4.2 The work done

Let's take a closer look at the work W done. We know that work is force times distance. So $dW = \mathbf{f} \cdot d\mathbf{x}$. It follows that

$$dW/dt = \mathbf{f} \cdot \frac{d\mathbf{x}}{dt} = \mathbf{f} \cdot \mathbf{v}. \tag{4.4.5}$$

The work, however, is done at the boundary of the volume. It is caused by surface forces $\mathbf{f_s} = \sigma \mathbf{n}$. If we transform these surface forces to something having to do with volumes, we eventually find that

$$\frac{Dw}{Dt} = \nabla(\sigma \mathbf{u}). \tag{4.4.6}$$

4.4.3 The heat added

Now let's turn our attention to the heat added. This is a bit more difficult, since there are two ways of adding heat. We can add a certain amount of heat per unit mass q_m over the entire volume. The heat per unit volume added then becomes ρq_m. (Note that q_m is a scalar.)

We can, however, also add heat per unit area $\mathbf{q_s}$, also known as **heat flow**. (This time $\mathbf{q_s}$ is a vector.) This heat flow is likely to depend on the temperature gradient ∇T. A temperature difference is, however, a configuration variable, and is defined along a line. Heat flow is a source variable (it changes the configuration) and is defined through a surface. We thus need a constitutive model connecting the two. The most common method is **Fourier's law**. It states that

$$\dot{\mathbf{q}}_\mathbf{s} = -\kappa \nabla T, \tag{4.4.7}$$

where κ is a tensor. (The minus sign is present, because heat flow is defined to flow along negative temperature gradients.) If the heat flow properties are isotropic (the same in all directions), then $\kappa = \bar{\kappa} I$, with $\bar{\kappa}$ a constant. The relation $\dot{\mathbf{q}}_\mathbf{s} = -\bar{\kappa} \nabla T$ then also holds.

The heat flow per unit area needs to be converted to a quantity per unit volume. Once we have done that, we can find that

$$\frac{Dq}{dt} = \rho \dot{q}_m + \nabla \cdot (\kappa \nabla T). \tag{4.4.8}$$

4.4.4 The final energy equation

Let's put all the terms we found together. We then get that

$$\rho \frac{DE}{Dt} = \rho \dot{q}_m + \nabla \cdot (\kappa \nabla T - p\mathbf{u} + \tau \mathbf{u}). \tag{4.4.9}$$

With quite some manipulation, we could also rewrite this equation to

$$\rho \frac{De}{Dt} = -p \nabla \mathbf{u} + \nabla \cdot (\kappa \nabla T) + \Phi. \tag{4.4.10}$$

The variable Φ is known as the **Rayleigh dissipation function** and is given by

$$\Phi = \nabla \cdot (\tau \mathbf{u}) - \mathbf{u} \cdot (\nabla \tau). \tag{4.4.11}$$

4.4.5 Fixing the gaps

So, now we've derived all three conservation laws. A bit of counting, however, would show that we have more unknowns then equations. So we need a few more equations. One of these equations relates internal energy and temperature. It states that

$$e = c_V T, \tag{4.4.12}$$

where c_V is the **specific heat at constant volume**. The other equation is the famous **equation of state** for ideal gases, and states that

$$p = \rho R T, \tag{4.4.13}$$

with R being the **specific gas constant**. Now we have the same amount of equations as unknowns. So we should be able to solve it.

5. Application of the conservation laws

It is finally time to actually put theory into practice. We have three conservation laws. Let's apply them!

5.1 Two parallel plates

5.1.1 The problem statement

Let's suppose we have two horizontal plates. One is positioned at $y = 0$ and the other at $y = D$. The top plate moves with a velocity u_e to the right. This causes a flow to be present between the plates. There is also a pressure distribution p between the plates, which also causes a flow.

We now make a few assumptions about the flow. We assume that the flow has a **constant density** $\rho = \rho_\infty$, that it is **steady** $\partial/\partial t$, that the flow is parallel to the plates ($v = w = 0$), that it doesn't vary in the z-direction ($\partial/\partial z$), and that there are no body forces $\mathbf{f_b}$.

5.1.2 Deriving the solution

Now let's solve the problem. Since ρ is constant, we can simplify the continuity equation to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5.1.1}$$

Since $v = w = 0$, two terms cancel. We remain with $\partial u/\partial x = 0$. That's our first sub-result.

Now let's examine the momentum equation. If we write down the equation for all three components, we can see that a lot of terms cancel. We remain with

$$\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \qquad \frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = 0. \tag{5.1.2}$$

The two rightmost equations imply that $p = p(x)$. If we look at the leftmost equation, we see that it is separated. Both sides must thus equal a constant c. This implies that

$$u(y) = \frac{c}{2\mu}y^2 + ay + b. \tag{5.1.3}$$

By using boundary conditions $u(0) = 0$ and $u(D) = u_e$ (the so-called **no-slip conditions**), we find that

$$u(y) = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left(y^2 - Dy\right) + u_e\frac{y}{D}. \tag{5.1.4}$$

We have found the velocity distribution. We see that it consists of two parts. The left part originates from pressure differences, while the right part is caused by the moving plates. If there is no pressure difference ($dp/dx = 0$), then we have a so-called **planar Couette flow**. If, on the other hand, the plates are not moving ($u_e = 0$, but $dp/dx \neq 0$), then we have a **planar Poiseuille flow**.

5.2 Analysis the planar Poiseuille flow

5.2.1 Analyzing the velocity

Let's examine the planar Poiseuille flow now. So we assume the plates are not moving. We can now derive a lot of things from the solution. First, we can see that the **maximum velocity** u_{max} occurs at

$y = D/2$, exactly between the plates. Its magnitude is

$$u_{max} = -\frac{D^2}{8\mu}\left(\frac{dp}{dx}\right). \tag{5.2.1}$$

The minus sign makes sense. The velocity flows in the direction of a negative pressure gradient.

Now let's examine the **volume flow** $\dot{Q}$ (also known as the **volumetric flow rate**) flowing through the channel. It is given by

$$\dot{Q} = \int_0^D u(y)\,dy = -\frac{1}{12\mu}\left(\frac{dp}{dx}\right)D^3. \tag{5.2.2}$$

From this, we can derive the **mean velocity** $\bar{u}$. It is given by

$$\bar{u} = \frac{Q}{D} = -\frac{D^2}{12\mu}\left(\frac{dp}{dx}\right) = \frac{2}{3}u_{max}. \tag{5.2.3}$$

5.2.2 Analyzing the forces

The **wall shear stress** τ_w is given by

$$\tau_w = \mu\left(\frac{du}{dy}\right). \tag{5.2.4}$$

If we evaluate this at the walls ($y = 0$ and $y = D$), we find that this flow gives us

$$\tau_w = \frac{1}{2}D\left(\frac{dp}{dx}\right). \tag{5.2.5}$$

So the wall stress is independent of the viscosity μ. From this wall stress, we can derive the **friction coefficient** c_f. It is defined as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{u}^2}. \tag{5.2.6}$$

If we apply this definition to our flow, we can find that

$$c_f = 12\frac{\mu}{\rho\bar{u}D} = \frac{12}{Re_D}. \tag{5.2.7}$$

The variable Re_D is the **Reynolds number**, with length D as the reference length.

5.3 Analysis of the planar Couette flow

5.3.1 Velocity and stress

Now let's examine the Couette flow. So we assume that $\partial p/\partial x = 0$. However, the top plate does move with a velocity u_e. The velocity distribution is thus given by

$$u(y) = u_e\frac{y}{D}. \tag{5.3.1}$$

The shear stress in the flow can be found using

$$\tau_{xy} = \mu\left(\frac{du}{dy}\right) = \mu\frac{u_e}{D}. \tag{5.3.2}$$

This shear stress is the whole reason why the flow is moving. It is caused by the moving top plate, and is, through the fluid, transferred to the bottom plate.

5.3.2 Finding the temperature distribution

It is time to apply the energy equation. After removing the zero terms, we find that

$$\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)=0. \tag{5.3.3}$$

We now have to assume that κ and μ are constant. (We say that $\kappa=\kappa_\infty$ and $\mu=\mu_\infty$.) So we can pull them out of the derivative. We also introduce the **enthalpy** $h=c_pT$, with c_p the **specific heat**. This turns the energy equation into

$$\frac{1}{P_r}\frac{\partial^2 h}{\partial y^2}+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial y}\right)=0, \qquad \text{where} \qquad P_r=\frac{\mu_\infty c_p}{\kappa_\infty}. \tag{5.3.4}$$

The number P_r is known as the **Prandtl number**. Integrating twice will give

$$h+\frac{1}{2}P_ru^2=ay+b, \tag{5.3.5}$$

where a and b are constants. To find them, we use the boundary conditions $u(0)=0$, $h(0)=h_w$ (the enthalpy at the bottom plate), $u(D)=u_e$ and $u(D)=h_e$ (the enthalpy at the top plate). This then gives us an expression for the enthalpy, being

$$h(y)=h_w+(h_e-h_w)\,\frac{y}{D}+\frac{1}{2}P_ru_e^2\left(\frac{y}{D}-\left(\frac{y}{D}\right)^2\right). \tag{5.3.6}$$

And with this, the temperature distribution has been found. Something interesting can be seen here. If $h_e=h_w$, then the enthalpy isn't just constant. It is still parabolically distributed. This is caused by heat creation due to viscous effects.

5.3.3 Analysis of the heat flow

But we can do even more with the temperature (or enthalpy) distribution. From it, we can derive the **heat flow** $\dot{q}$, using

$$\dot{q}=-\kappa\frac{dT}{dy}=-\frac{\kappa}{c_p}\frac{dh}{dy}=-\frac{\mu}{D}\frac{h_e-h_w}{P_r}+\frac{1}{2}\frac{\mu}{D}u_e^2\left(2\frac{y}{D}-1\right). \tag{5.3.7}$$

Again, we see that, if $h_e=h_w$, there is still heat flow. It flows away from the center and goes to the plates, where it is dissipated. In fact, we can even find how much heat is dissipated at the walls. For that, we examine $\dot{q}(0)$ and $\dot{q}(D)$. We find them to be

$$\dot{q}(0)=-\frac{\mu}{D}\frac{h_e-h_w}{P_r}-\frac{1}{2}\frac{\mu}{D}u_e^2 \qquad \text{and} \qquad \dot{q}(D)=-\frac{\mu}{D}\frac{h_e-h_w}{P_r}+\frac{1}{2}\frac{\mu}{D}u_e^2. \tag{5.3.8}$$

In other words, the heat flow through the walls due to the viscous effects is $\frac{1}{2}\frac{\mu}{D}u_e^2$. This means that the total heat creation per second due to viscous effects is equal to $\frac{\mu}{D}u_e^2$ (since there are two walls). And that is interesting to know.