University of the West of England
BRISTOL

Module UFMEQT-20-1

Stress and Dynamics

Dynamics: Part 2

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Contents

6 Work & Energy

6.1 Introduction 2

In Section 5 of this course, we discussed momentum, and it was shown that the integral of force with respect to time is the momentum of a particle, and that if momentum changes, a force must have acted upon the particle.

If the force acting on a particle causes it to move, then it is possible to integrate the force with respect to *displacement* instead of time, and this results in a value known as **Work**. This section discusses work and energy.

Energy is a property that gives a body **the capacity to do work**. There are many different forms of energy, such as chemical, electrical, mechanical, nuclear, solar and sonar (sound). These are all forms in which energy may be **stored** in a body.

Conversely, **Heat** and **Work**, whilst also being forms energy, are known as **energy transfers** as these are the only forms in which energy may be transferred from one body of stored energy to another.

6.2 Work

6.2.1 Definition of Work

A particle lying in three dimensional space. The work required to move the particle 3
from one arbitrary point, A to another arbitrary point B is defined as:

$$W_{AB} = \int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{6.1}$$

where $\mathbf{F}$ is the force required to move the particle, and $\mathbf{r}$ is the position vector indicating its displacement. This is the area under a force-displacement graph.

Note that this process involves the dot product of the two vectors (see Section 1), and 4
we know that the dot product of two vectors results in a scalar. We know that the dot product is:

$$\mathbf{F} \cdot \mathrm{d}\mathbf{r} = |\mathbf{F}||\mathrm{d}\mathbf{r}| \cos\alpha$$

where α is the angle between the vectors. Moving to a one-dimensional space, such as the x-axis, the force will be inline with the path of the particle, so $\cos\alpha = \cos 0 = 1$. This one dimensional motion is shown in Figure 6.1.

In this one-dimensional problem, the work required to move the particle from A to B is:

$$W_{AB} = \int_A^B F \,\mathrm{d}x \tag{6.2}$$

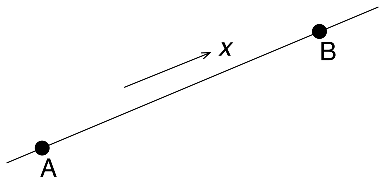


Figure 6.1: Work in a single dimension

By inspection, the units of work are Newton metres (Nm), which are more often called Joules (J). For the time being, we will analyse work in one-dimension only. This will be extended to more than one dimension in Section 6.2.4.

6.2.2 Work and Kinetic Energy

5 Taking Newton's second law, and the definition of acceleration:

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \text{also:} \quad \mathrm{d}x = v\mathrm{d}t$$

Substituting into equation 6.2:

$$W_{AB} = \int_A^B m\frac{\mathrm{d}v}{\mathrm{d}t}v\mathrm{d}t$$

The two dt terms cancel, resulting in:

$$W_{AB} = \int_A^B m\frac{\mathrm{d}v}{\cancel{\mathrm{d}t}}v\cancel{\mathrm{d}t} = \int_{v_A}^{v_B} mv\,\mathrm{d}v = \frac{1}{2}mv^2\bigg|_{v_A}^{v_B} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

6 As discussed in Section 5, the term $\frac{1}{2}mv^2$ is known as the **kinetic energy**. So:

$$\text{Work to go from A to B} = \text{Kinetic Energy at B} - \text{Kinetic Energy at A}$$

or:

$$W_{AB} = U_{kB} - U_{kA} = \Delta U_k \tag{6.3}$$

So equation 6.3 states that the work done moving a particle from A to B is the change in kinetic energy at A and B. This is known as the **work-energy** theorem.

6.2.3 Work Done

Work Done by a Constant Force

7 Taking the definition described in equation 6.2, and if the force is constant:

$$W_{AB} = F\int_A^B dx = F(x_B - x_A)$$

So the work done is:

$$\text{Work Done} = \text{Force} \times \text{Distance moved in the direction of the force} = F\Delta x$$

Work Done Against Gravity

Imagine a particle of mass m lying between A and B, a distance h apart, as shown in 8
Figure 6.2

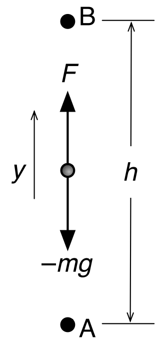


Figure 6.2: Work done against gravity

The force required to raise the particle up is F, and it is clear that this force has to overcome the gravitational force $(-mg)$, for the object to move the vertical distance, h. Thus the work done by the force is:

$$\text{Work done against gravity} = W_{AB} = mgh \tag{6.4}$$

We can also see that the work done by gravity:

$$W_{gr} = -mgh$$

Net Work

This leads to an important concept: the idea of **net work**. Work can be positive or 9
negative. Applying a force to lift an object to a point, and then letting gravity apply work to bring it back to the starting point, the net work is:

$$W_{AB} + W_{gr} = mgh - mgh = 0$$

This means that continuously lifting an object, and letting it fall, then lifting again, then letting it fall results in a net work of zero.

Work Done Against Friction

Consider a force F causing a body of mass m to slide at constant velocity along a 10
horizontal surface for which the coefficient of frcition between the body and the surface is μ, as shown in Figure 6.3. The distance the body is moved by the force is Δx.

The friction force resisting the motion is the coefficient of friction, μ, multiplied by the normal reaction, N:

$$\text{Friction force} = \mu N$$

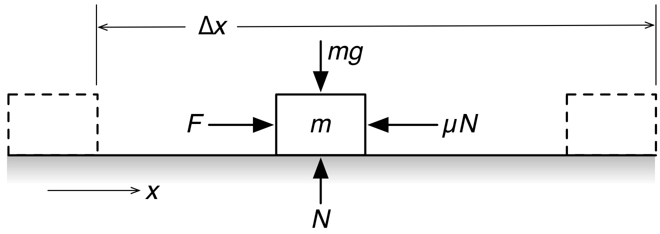


Figure 6.3: Work done against friction

We also know that the normal reaction force, N is the weight:

$$N = mg$$

Therefore the applied force, F is:

$$F = \mu mg$$

and for a constant force, work done is force times distance, hence:

$$\text{Work done against friction} = W_f = \mu mg\Delta x \tag{6.5}$$

Work Done by a Gradually Applied Force

11 A force applied gradually in such a way that its magnitude varies uniformly from zero up to a maximum value of F is shown in Figure 6.4

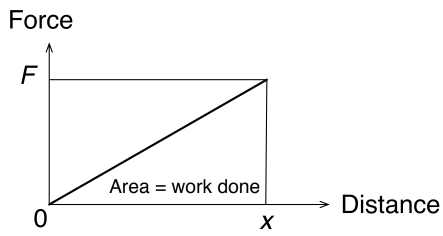


Figure 6.4: Work done by a gradually applied force

When this is the case, the average force can be taken:

$$\text{Average force} = \frac{1}{2}(F_0 + F_x) = \frac{1}{2}F$$

So, the work done:

$$\text{Work done by a gradually applied force} = \text{Average force} \times \text{Distance moved} = \frac{1}{2}Fx$$

Work Done Against a Spring

We have not yet covered springs in details but suffice to say that the force required to 12
extend (or compress) a spring by a length x is:

$$F = kx$$

where k is the stiffness of the spring, known as the **spring rate** or **spring constant**, measured in Newtons per metre (N/m).

When a force is applied to a spring, it is normally applied gradually, the force increasing from zero up to its maximum value F, producing a maximum extension, x. Hence, the work done against a spring is:

$$\text{Work done against a spring} = \text{Average force} \times \text{Extension} = W_s = \frac{1}{2}kx \times x = \frac{1}{2}kx^2 \tag{6.6}$$

6.2.4 Work in 3-dimensions

Work can of course work in more than one dimension, and the original definition stated 13
in equation 6.1 accomodates for this. Figure 6.5 shows the path taken by a particle moving from A to B in three dimension space. At a particular point, the force, $\mathbf{F}$, is being applied over displacement $\mathrm{d}\mathbf{r}$, and this is shown. It is clear that $\mathbf{F}$ is *not* inline with the direction of motion (indicated by the line, and $\mathrm{d}\mathbf{r}$). It is perhaps clearer to

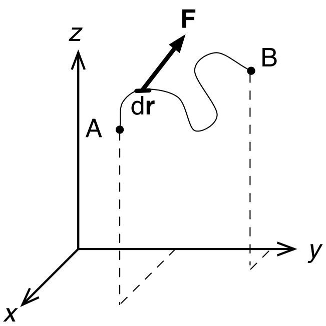


Figure 6.5: Work done on particle in 3-dimensional space

explain in 2-dimensions. In Figure 6.6, representing 2-dimensional space, we have force 14
$\mathbf{F}$ being applied to mass m a distance $\mathbf{r}$ along a surface, where $\mathbf{F}$ is not inline with the surface. In this case, $\mathbf{F}$ and $\mathbf{r}$ are constant. The force doing the work, is the component
of the force that is parallel to the path the mass is moving, i.e.: 15

$$W = F \cos\alpha \mathbf{r} = |\mathbf{F}||\mathbf{r}| \cos\alpha$$

which is the dot product of the vectors $\mathbf{F}$ and $\mathbf{r}$, hence the definition:

$$W_{AB} = \int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

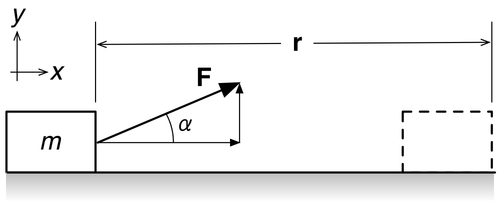


Figure 6.6: Work done on particle in 2-dimensional space

We can decompose the vectors $\mathbf{F}$ and $\mathbf{r}$:

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \qquad \mathrm{d}\mathbf{r} = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j} + \mathrm{d}z\mathbf{k}$$

The derivative of work, $\mathrm{d}W$:

$$\mathrm{d}W = F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z$$

16 So:

$$W_{AB} = \int_A^B \mathrm{d}W = \int_A^B F_x\mathrm{d}x + \int_A^B F_y\mathrm{d}y + \int_A^B F_z\mathrm{d}z$$

Each of these integrals are one dimensional problems, and we already did this in Section 6.3.2.

$$W_{AB} = \frac{1}{2}m(v_{B_x}^2 - v_{A_x}^2) + \frac{1}{2}m(v_{B_y}^2 - v_{A_y}^2) + \frac{1}{2}m(v_{B_z}^2 - v_{A_z}^2)$$

so:

$$W_{AB} = \frac{1}{2}m(v_B^2 - v_A^2) \tag{6.7}$$

which is exactly the same result as we had before, namely that the work done is the difference in kinetic energy.

6.2.5 Power

17 Power is simply the rate of doing work. Written mathematically:

$$\text{Power} = \frac{\text{work done}}{\text{time taken}}$$

This can be continued, knowing that work done is force multiplied by distance:

$$\text{Power} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \text{Force} \times \text{Velocity} = Fv$$

The unit of power is the Watt (W), which is equivalent to $1\,\mathrm{J/s} = 1\,\mathrm{Nm/s}$.

6.2.6 Work and Energy: Example

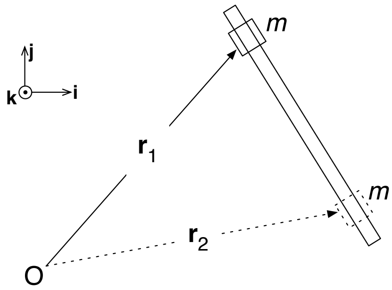


Figure 6.7: Work example

In Figure 6.7, a mass m slides, under the force of gravity, along a rail from position $\mathbf{r}_1$ to position $\mathbf{r}_2$, as shown in the diagram ($\mathbf{j}$ is upwards).

Some data:

$$m = 0.5\,\mathrm{kg} \quad \mathbf{r}_1 = 2\mathbf{i} + 5\mathbf{j}\,\mathrm{m} \quad \mathbf{r}_2 = 5\mathbf{i} + 2\mathbf{j}\,\mathrm{m} \quad v_1 = 0.5\,\mathrm{m/s}$$

Determine the work done by gravity on the mass.

6.3 Energy

18 As stated in the introduction, energy is the **capacity for doing work**. This section only considers mechanical forms of stored energy; we are not considering chemical energy, or electrical energy, etc. The mechanical energies studied here are:

- Gravitational potential energy
- Kinetic energy
- Elastic potential energy

6.3.1 Potential Energy

Gravitational Potential Energy

19 This is the energy stored in a body by virtue of its position, as shown in Figure 6.8

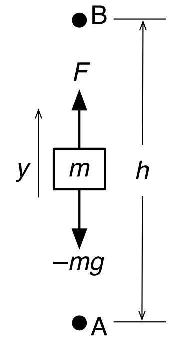


Figure 6.8: Gravitational Potential Energy

The force required to lift a mass m from A to B is equal to the weight mg. Hence the work done in raising the mass through a height h is given by, as given in Section 6.2.3:

$$W_{AB} = mgh$$

However, this work done has gone to increase the store of gravitational potential energy within the body, hence:

$$\text{Gravitational Potential Energy} = U_{pg} = mgh \tag{6.8}$$

Elastic Potential Energy

20 A compressed or extended spring contains **elastic potential energy**. This is also referred to as **strain energy**, as a compressed or extended string has required the material to deform, creating a strain.

It has been shown, in Section 6.2.3, that the work required to compress a spring a distance x is given by:

$$W_s = \frac{1}{2}kx^2$$

As above with gravitational energy, this work done has gone into increasing the store of potential elastic energy within the spring. Hence:

$$\text{Elastic Potential Energy} = U_{ps} = \frac{1}{2}kx^2 \tag{6.9}$$

6.3.2 Kinetic Energy

This is the energy stored in a body by virtue of its velocity. Kinetic energy is a scalar quantity as it is independent of direction. 21

As has been described in Sections 6.2.6 and 6.2.4, the work done to move a body has gone to increase the store of kinetic energy within the body so:

$$\text{Kinetic Energy} = U_k = \frac{1}{2}mv^2 \tag{6.10}$$

6.3.3 Mechanical Energy

The total mechanical energy possessed by a body is the sum of its gravitational potential energy, its elastic potential energy and its kinetic energy: 22

$$\text{Mechanical Energy} = U_{pg} + U_{ps} + U_k$$

6.3.4 Conservation of Energy

The principle of conservation of energy states that energy can neither be created nor destroyed, although it can be converted from one form to another. So all energy must be accounted for (although not necessarily as mechanical energy—mechanical energy can often be converted to thermal energy (heat), for example). 23

The total energy must remain constant

For example, a falling body loses potential energy as it loses height (h is reducing) but it gains an equal amount of kinetic energy as it gains speed (v increases).

Likewise, a mass vibrating on the end of a spring loses velocity and hence kinetic energy as the spring becomes stretched, the kinetic energy of the mass being converted into strain energy in the spring. Then as the motion is reversed, the strain energy of the spring reduces as the spring contracts, the strain energy being converted back into kinetic energy of the mass as it gains velocity.

6.4 Energy Methods

6.4.1 Principles of the Energy Method

Energy (accounting) methods are widely used to avoid elaborate force diagrams and other complications. According to the energy method a mechanical system is considered 24

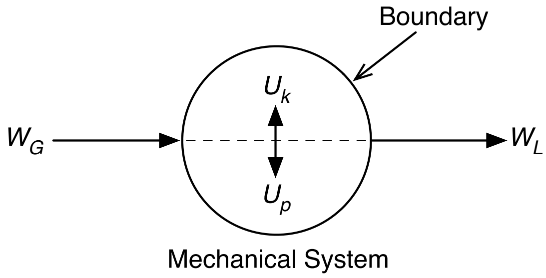


Figure 6.9: Visualisation of Mechanical System

a an entity. The system has a boundary that may be positioned as required to suit the particular problem being considered. Such a system is shown in Figure 6.9.

- W_G = the energy gained by the system as a result of work being done on the system (work gained).
- W_L = the energy lost from the system as a result of work being done by the system on the surrounds (work lost).
- U_k = the kinetic energy stored within the system.
- U_p = the various forms of potential energy stored within the system (Gravitational, spring etc.)

Note that kinetic and potential energy are both forms of stored mechanical energy.

Within the mechanical system, exchanges between mechanical forms of energy can take place. If work is done on the system by a force considered to be external to the system, the system gains energy (W_G).

For example, consider a windup toy car. Energy enters the mechanical system via winding up the spring (W_G) which increases U_p. The spring potential energy is converted to kinetic energy (U_k), and work done by the system (W_L) is used to overcome friction, for example.

6.4.2 The Energy Balance Equation

25 This is based on the conservation of energy, and states that:

$$\text{Energy in state 1} + \text{Energy gained as work} - \text{Energy lost as work} = \text{Energy in state 2}$$

or

$$(U_{k1} + U_{p1}) + W_{G1,2} - W_{L1,2} = (U_{k2} + U_{p2})$$

The advantage of the energy method is that it can be used to solve potentially complex problems that would be laborious using any other method.

6.4.3 Energy Method: Example

A mass of 200 kg (m_1) is connected by a rope over a pulley to a mass of 400 kg (m_2), as shown in Figure 6.10. Initially m_1 is held in position; it is then released and allowed to slide over the horizontal surface. The coefficient of friction is 0.2. What is the velocity of the masses when they move 2 m from their starting at rest? What assumptions have you made?

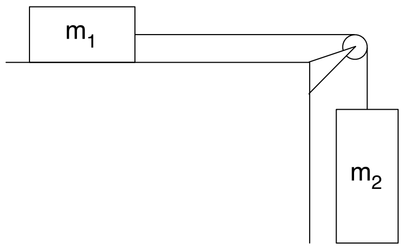


Figure 6.10: Energy Method Example

Summary

Work

26 Units of work are Nm, or more commonly Joules (J)

Definition of work: Work required to move a particle from point A to point B:

$$W_{AB} = \int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Work from A to B is the kinetic energy at B minus the kinetic energy at A:

$$W_{AB} = U_{kB} - U_{kA}$$

Work Done

27 Work done by a **constant force:**

$$W_{AB} = F(x_B - x_A) = \text{Force} \times \text{Distance moved}$$

Work done against **gravity** (where point B is h above point A):

$$W_{AB} = mgh$$

conversely:

$$W_{\text{gravity}} = -mgh$$

Work done against **friction**:

$$W_f = \mu mg\Delta x$$

28 Work done by a **gradually applied force** (for force varying uniformly with time):

$$W = \text{average force} \times \text{distance moved} = \frac{1}{2}Fx$$

This leads to work done against a **spring**:

$$W_s = \frac{1}{2}kx^2$$

Power

29

$$\text{Power} = \frac{\text{work done}}{\text{time taken}} = \frac{W}{\Delta t} = Fv$$

Units of power are Nm/s, or J/s which are more commonly called Watts (W).

Energy

30 Energy is the **capacity to do work**.

Gravitational potential energy:

$$U_{pg} = mgh$$

Elastic potential energy (strain energy):

$$U_{ps} = \frac{1}{2}kx^2$$

Kinetic energy:

$$U_k = \frac{1}{2}mv^2$$

Mechanical energy:

$$U_k + U_{pg} + U_{ps} = \text{Mechanical Energy}$$

Energy Methods

The energy balance equation: 31

$$\underbrace{(U_{k1} + U_{p1})}_{\text{Mech } U \text{ at } 1} + \underbrace{W_{G1,2}}_{\text{Work gained}} - \underbrace{W_{L1,2}}_{\text{Work Lost}} = \underbrace{(U_{k2} + U_{p2})}_{\text{Mech } U \text{ at } 2}$$

Work and Energy: Exercises

Work Done

1. Calculate the work done against gravity when a concrete block of mass 11 kg is raised through a vertical distance of 2.4 m. (Ans. 258.72 J)
2. A man pulls his car a distance of 8 m along a horizontal surface at a constant speed against a total resistance to motion of 300 N. Calculate the work done by the man against the resistance. (Ans. 2400 J)
3. A horizontal force pulls a mass of 3.25 kg a distance of 3.8 m at constant speed across a rough horizontal surface having a coefficient of friction of 0.4. Calculate the work done against friction. (Ans. 48.412 J)
4. A load of mass 38 kg is pulled at constant speed a distance of 26 m up a rough surface inclined at 22.62° to the horizontal, the coefficient of friction being 0.25. Calculate the work done against gravity, the work done against friction, and the total work done. (Ans. 3724 J; 2234.4 J; 5958.4 J)

Work Done: Springs

5. A compound spring assembly is compressed between two plates and the applied force varies linearly from zero to 120 N over the first 40 mm of compression, then from 120 N to 480 N over the next 20 mm. Calculate the work done in compressing the assembly. (Ans. 8.4 J)
6. A spring of stiffness 40 kN/m is compressed by an initial load of 4 kN, gradually applied, and the spring is then further compressed an additional distance of 400 mm by an additional load, again applied gradually. Calculate the total work done on the spring. (Ans. 5000 J)

Energy & Power

7. A pump draws in 15 tonne of water with negligible velocity and discharges the water with a velocity of 3 m/s in a total time of 1min 20s. Calculate:
   a) The kinetic energy imparted to the water;
   b) The total work done by the pump on the water;
   c) The power required to drive the pump assuming the pump to be frictionless.

   (Ans. 67.5 kJ; 67.5 kJ; 843.75 W)
8. A lorry of total mass 38 tonne is driven up an incline having a gradient of 1 in 60 (sine) against a rolling resistance to motion of 55 N per tonne of lorry mass. If the velocity of the lorry is increased from 18 km/h to 72 km/h whilst travelling a distance of 900 m in a time of 1 minute 12 seconds, calculate for the lorry during this period of motion:
   a) The change in potential energy;
   b) The change in kinetic energy;
   c) The work done against the rolling resistance;

   d) The total work done by the engine;
   e) The power output of the engine.

   (Ans. 5.5917 MJ; 7.125 MJ; 1.881 MJ; 14.5977 MJ; 202.75 kW)

9. The maximum speed which a car can attain along a level road when its engine is producing a power of 12 kW is 90 km/h. Calculate the magnitude of the resistance to motion being experienced by the car.
   If the mass of the car is 1100 kg, calculate the acceleration of the car that would be produced at a velocity of 45 km/h if the power output of the engine and the resistance to motion are the same as before. (Ans. 480 N; 0.4364 m/s^2)
10. A trailer of mass 800 kg is towed along a level road by a car of mass 1200 kg. The trailer experiences a resistance to motion of 320 N, whilst the resistance to motion of the car is 180 N.
   If the car engine exerts a forward driving force on the car (i.e. tractive effort) of 1500 N, calculate:
   a) The resultant force acting on the car and trailer combined.
   b) The resulting acceleration of the car and trailer.
   c) The tensile force in the tow-bar.
   d) The power output of the engine at the instant when the velocity of the car is 36 km/h.

   (Ans. 1000 N, 0.5 m/s^2, 720 N, 15 kW)
11. A pump draws in 1200 kg of water with negligible velocity, and discharges it at a height of 6 m above the pump with a velocity of 8 m/s, the total time taken to pump the water being 10 minutes. Calculate for the 1200 kg of water:
   a) Its increase in gravitational potential energy.
   b) Its increase in kinetic energy.
   c) The total work done on the water by the pump.
   d) The power necessary to drive the pump if its efficiency is assumed to be 100%.
   e) The volume of water pumped per second expressed in m^3/s.

   (Ans. 70.6 kJ, 38.4 kJ, 109 kJ, 181.7 W, 0.002 m^3/s)
12. A truck of total mass 16 tonne descends a hill with a gradient of 1 in 10 (i.e. a vertical movement of 1m for each 10m of movement along the slope). During this motion, the truck experiences a rolling resistance to its motion of 800 N.
   Whilst travelling at a velocity of 72 km/h, its brakes are applied, and the truck is brought to rest in a distance of 100 m. During this period of deceleration, it may be assumed that the rolling resistance remains constant at 800 N, this being in addition to the retarding force applied by the brakes. Calculate for the truck during its retardation to rest:
   a) The vertical height through which it descends.
   b) Its decrease in gravitational potential energy.
   c) Its initial velocity in m/s.
   d) Its decrease in kinetic energy.
   e) The work done against the rolling resistance.

f) The energy absorbed by the brakes.
g) Its rate of deceleration.
h) The resultant deceleration force (parallel to the slope) that acts on the truck.

(Ans. 10 m, 1.57 MJ, 20 m/s, 3.2 MJ, 80 kJ, 4.69 MJ, 2 m/s^2, 32 kN)

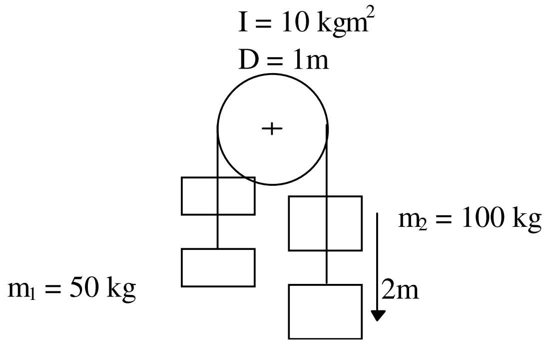
Work & Kinetic Energy

13. A car accelerates from 60 mph to 80 mph in 5 s.
    a) If the mass of the car is 800 kg calculate the total traction force. (Ignore air resistance. 1 mile = 1609.344 m). (1.43×10^3 N)
    b) Calculate the kinetic energy of the car at 60 mph (288 kJ)
    c) Calculate the kinetic energy of the car at 80 mph (512 kJ)
    d) What work is done if the car slows from 80 to 0 mph? (512 kJ)
    e) What work is done if the car slows from 60 to 0 mph? (288 kJ)
    f) What work is done if the car slows from 80 to 60 mph? (224 kJ)
    g) If the braking force is 16 kN find braking distance for case (d) (32 m)
    h) If the braking force is 16 kN find braking distance for case (e) (18 m)
    i) If the braking force is 16 kN find braking distance for case (f) (14 m)
14. The jet engines of an aircraft of mass 4,000 kg give a thrust of $20,000\mathbf{i}$ N while it travels in the $\mathbf{i}$ direction from rest along a runway preparing to take off. If resistance forces of $-2,000\mathbf{i}$ N act and the plane has travelled 200 m, find:
    a) The work done on the aircraft by the force from the engines; (4×10^6 J)
    b) The work done on the aircraft by the force from the resistance forces (= −work done by the aircraft against resistance forces); (-400×10^3 J)
    c) The net work done on the aircraft; (3.6×10^6 J)
    d) The kinetic energy of the aircraft; (3.6×10^6 J)
    e) The speed of the aircraft; (42.43 m/s)
15. An aircraft whose mass is 8000 kg must accelerate to 90 m/s (about 200 mph) over a distance of 100 m to take of from an aircraft carrier. State any assumptions made in the following calculations.
    a) Calculate the momentum of the aircraft at take off. (720×10^3 kgm/s)
    b) Calculate the kinetic energy of the aircraft at take off. (32.4×10^6 J)
    c) Calculate the net horizontal force acting on the aircraft as it accelerates. (324 kN)
    d) What is the force impulse acting on the aircraft as it accelerates? (720×10^3 kgm/s)
    e) How long it takes the aircraft to reach take off speed starting from rest? (2.22 s)
    f) If the landing speed of the aircraft is 60 m/s and the aircraft can be slowed down at a rate of 40 m/s^2, what distance is required for the aircraft to come to halt? (45 m)
    g) During its flight the aircraft launches a rocket. The engine thrust of the

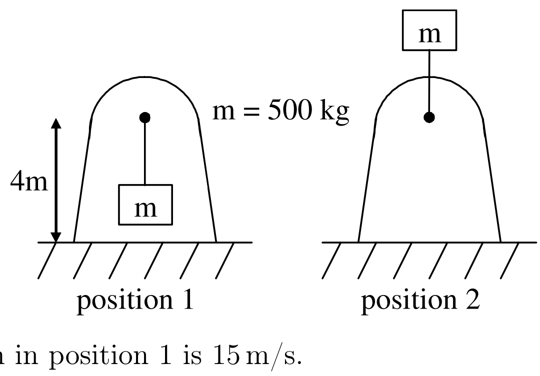
rocket is 20 kN. If the exhaust gas leaves at a speed of 1,000 m/s, estimate the rate at which fuel is consumed (in kg/s). (20 kg/s)

Energy Methods

16. Calculate the work done in compressing a spring of stiffness 20 kN/m through a distance of 17 mm, assuming that the load is applied gradually. [Ans. 2.89 J]
17. A spring of stiffness 32 kN/m is compressed by an initial load of 8 kN, gradually applied, and the spring is then further compressed an additional distance of 420 mm by an additional load, again applied gradually. Calculate the total work done on the spring. [Ans. 7182.4 J]
18. The system shown is initially at rest.



    Calculate, stating any assumptions, the linear speed of m_2 after it has dropped through a distance of 2 m. [Ans. 3.2 m/s]
19. A low emission vehicle has a regenerative braking system that gives a braking force (F) that is proportional to the speed (v) and is given by the relationship: $F = cv$
    a) Sketch speed against time. Assume the motor has been switched off and the braking system is switched on.
    b) If the mass of the vehicle is 1500 kg, the initial speed is 20 m/s and c is 1500 Ns/m find how long it takes for the speed to decrease to 1 m/s. [Ans. 3s]
20. A fairground gondola ride is set in motion then allowed to swing freely



    The speed (v) of m in position 1 is 15 m/s.

a) Calculate for position 1
   i. $\frac{1}{2}mv^2$ [Ans. 56250 J]
   ii. $\frac{1}{2}I\omega^2$ [Ans. 56250 J]

b) Calculate v for position 2. [Ans. 8.25 m/s]

c) State any assumptions.

d) Discuss momentum changes between positions 1 and 2.

21. A rocket has a total mass 100 kg of which 50 kg is fuel. The fuel, including oxidant, is burnt at a constant rate of 5 kg/s, and the exhaust gas "velocity" is 900 m/s.
    a) Calculate the maximum speed achieved by the rocket [525.73 m/s or 1176 mph]
    b) Sketch the graph of speed against time after the rocket motor ignites

7 Rotational Energy and Angular Momentum

7.1 Introduction 2

Up to now we have predominantly dealt with motion in a translational sense along a specific axis or set of axes. Rotational motion must also be considered when studying dynamic systems, and relationships between rotational and linear motion can be made. This Section and Section 8 will deal with rotational motion and the rotational equivalents to mass, force, momentum and energy, along with some specific aspects only pertaining to rotating bodies.

7.2 Review

In your revision notes available on Blackboard, there is a section on angular motion 3
where angular displacement, angular velocity and angular acceleration are discussed, along with the relationships with linear motion and centripetal acceleration.

If we examine the disk shown in Figure 7.1, we can see that:

- Angular displacement = θ [units: rad]
- Angular velocity = ω [units: rad/s]
- Angular acceleration = α [units: rad/s^2]

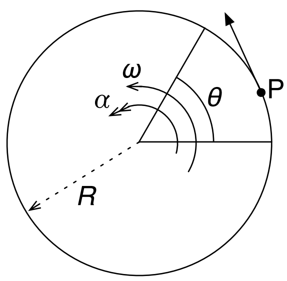


Figure 7.1: A spinning disk

For point P on the edge of the disk, the linear *tangential* dimensions of x, v, and a (indicated by the arrow at P) are related to the angular dimensions by the radius, R:

$$x = R\theta \qquad v = R\omega \qquad a = R\alpha$$

So, while the angular dimensions, θ, ω and α are irrespective of the size of the disk, the tangential equivalents are dependent on the radius. All the equations pertaining to constant acceleration are valid for rotational systems.

7.3 Rotational Energy & Moment of Inertia

7.3.1 Rotational Kinetic Energy

4 We learnt in previous sections that the kinetic energy of a particle travelling in a straight line is:

$$U_k = \frac{1}{2}mv^2$$

Figure 7.2 shows a rotating disk of radius R rotating about its centre of mass, G. The aim is to determine the kinetic energy of the disk.

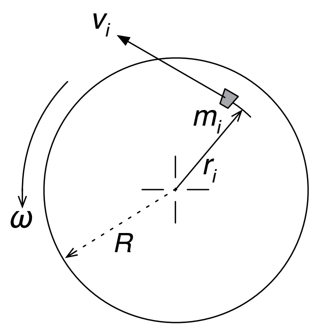


Figure 7.2: A spinning disk, with element m_i

If we take an element of the disk with mass m_i at a radius of r_i, the linear kinetic energy of that element is:

$$U_{k,i} = \frac{1}{2}m_i v_i^2$$

Replacing v_i with $r_i\omega$ gives:

$$U_{k,i} = \frac{1}{2}m_i(r_i\omega)^2$$

Note that ω does not have a subscript i since ω, the angular velocity, is the same for all points on the disk. Now, to calculate the kinetic energy of the disk, we have to sum all
5 of these small elements:

$$U_{k,\text{disk}} = \sum_{i=1}^{m} \frac{1}{2}m_i(r_i\omega)^2 = \frac{\omega^2}{2}\underbrace{\sum m_i r_i^2}_{I_G}$$

The term $\sum m_i r_i^2$ is what we call the *moment of inertia* of an object and is denoted by the letter I. The subscript denotes the axis around which the object rotates, in this case G, so the moment of inertia for this case is I_G. Substituting:

$$U_{k,\text{disk}} = \frac{1}{2}I_G\omega^2 \tag{7.1}$$

which looks very similar to the linear kinetic energy, with I replacing m and ω replacing v. So, we have another analogy that in a linear motion, **where you have m you can replace with I.**

7.3.2 Moments of Inertia

Calculating the equations for the moment of inertia for any object is a rather arduous 6
mathematics challenge, involving plenty of potentially difficult integration. Instead, you can refer to many dynamics text books which have tables that list the equations for moments of inertia for common objects, some of which are listed below.

Note: the units of moments of inertia are kgm^2.

- For a solid disk, mass m, radius R: 7

$$I_G = \frac{1}{2}mR^2$$

- For a sphere, rotating about its centre of mass, G with mass m and radius R: 8

$$I_G = \frac{2}{5}mR^2$$

- A thin ring with radius R and mass m (thickness is much smaller than radius): 9

$$I_G = mR^2$$

- A mass m on the end of a massless arm of length R: 10

$$I = mR^2$$

- A rod, with length l and mass m, rotating about an axis perpendicular to the rod 11
  passing through its centre point, G:

$$I_G = \frac{1}{12}ml^2$$

If an object comprises multiple components for which equations for moments of inertia exist, then the effective moment of inertia is the sum of the individual moments of inertia.

Parallel Axis Theorem

The parallel axis theorem is a theorem that helps when attempting to determine the 12
moment of inertia of an object that has perhaps a known moment of inertia about its centre of mass, but is not actually rotating about that point.

Taking the example shown in Figure 7.3, we have a rotating disk. When it is rotating about axis z, we know its moment of inertia is:

$$I_z = \frac{1}{2}mR^2$$

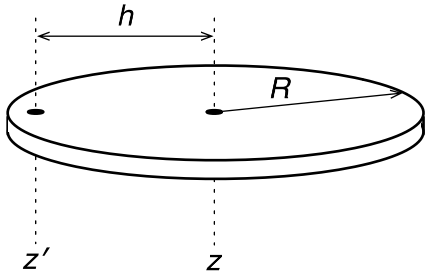


Figure 7.3: A spinning disk

Let us now say that we want to rotate the disk about the axis z', an axis parallel to axis z a distance of h away, but *not* passing through its centre of mass. The parallel axis theorem says that the moment of inertia of rotation about z', provided that z' is parallel to z, is the moment of inertia of the object when rotating about axis z, through the centre of mass, plus the mass of the disk times the distance h squared.

$$I_{z'} = I_z + mh^2$$

So, if we know the moment of inertia of rotation of an object about its centre of mass, we can determine the moment of inertia of rotation of that same object about *any* axis parallel to the original axis.

Radius of Gyration

13 For objects such as pulleys, which could have a form that is part solid disk, part ring, or spools, it is often difficult to determine the moment of inertia based purely on the object's radius and mass. What is often used however is the *radius of gyration*, which is the radius at which the total mass of a body may be considered to be concentrated without affecting its moment of inertia. The radius of gyration is represented by k_G and is defined such that:

$$k_G^2 = \frac{\text{moment of inertia}}{\text{mass}} = \frac{I_G}{m}$$

so:

$$I_G = mk_G^2 \tag{7.2}$$

In your problems, if you are asked to determine a quantity which requires the moment of inertia, and the object is not one listed in section 7.3.2, then the radius of gyration will generally be given. In these cases, equation 7.2 can be used.

7.3.3 Moment of Inertia: Examples

Q1. The rim of a steel pulley-wheel is 120 mm wide and 20 mm thick, with a mean diameter of 1.4 m. Considering the pulley as a thin ring, and neglecting the mass of the hub and the spokes, calculate the moment of inertia of the pulley. ($\rho_{\text{steel}} = 7850\,\text{kg/m}^3$).

Q2. If the steel ring calculated in Q1 is spun at 3000 rpm, what is its rotational kinetic energy?

Q3. An aluminium ($\rho = 2700\,\text{kg/m}^3$) bicycle wheel is constructed from a wheel rim, spokes and a hub. Assume the hub is modelled as a solid disk of width 100 mm and diameter 50 mm and the rim is modelled as a thin ring of width 30 mm, depth 10 mm and a mean diameter of 700 mm. Neglecting the spokes, what is its radius of gyration?

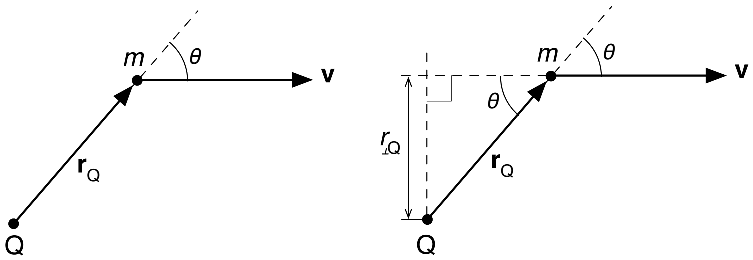
7.4 Angular Momentum

7.4.1 Definition

14 If an object has a mass m and a velocity $\mathbf{v}$, then clearly it has a momentum $\mathbf{p}$:

$$\mathbf{p} = m\mathbf{v}$$

At an arbitrary point, indicated by Q, the position vector of the mass is $\mathbf{r}_Q$, as shown in Figure 7.4(a). The momentum of the mass is $m\mathbf{v}$.



(a) (b)

Figure 7.4: Mass m moving at velocity $\mathbf{v}$ at position $\mathbf{r}$ from point Q

15 The angular momentum of the mass, relative to point Q, is:

$$\mathbf{H}_Q = \mathbf{r}_Q \times \mathbf{p} = (\mathbf{r}_Q \times \mathbf{v})m \tag{7.3}$$

The the angular momentum is a cross product (covered in Section 1). As such, the magnitude of the angular momentum is:

$$\begin{aligned}|\mathbf{H}| &= m|\mathbf{v}||\mathbf{r}_Q|\sin\theta \\ &= mv\underbrace{r\sin\theta}_{r_{\perp Q}}\end{aligned}$$

where $r_{\perp Q}$ is the perpendicular distance from the projected velocity to point Q, as shown in Figure 7.4(b). From our knowledge of cross products, the direction of the angle between $\mathbf{r}_Q$ and $\mathbf{p}$ is clockwise, indicating that the direction of the momentum is into the page, following the right-hand rule.

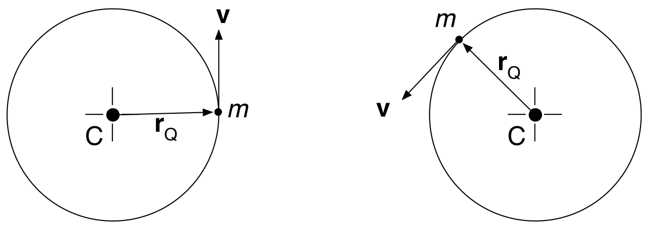
The above calculations are all taken with respect to our arbitrary point Q. If we now choose another point, C say, which is *inline* with the vector $\mathbf{p}$, then the *angular momentum* is zero. This is clear, as the position vector $\mathbf{r}_C$ would be either 0° or 180° from $\mathbf{p}$, and the sine of those angles is zero. So, *angular* momentum is not an intrinsic property of a moving object, *unlike* linear momentum. If a mass is moving linearly, and we know its mass and velocity, we know its momentum. Angular momentum depends on the point you choose as your point of origin.

16

Based on the definition:

$$\mathbf{H}_Q = (\mathbf{r}_Q \times \mathbf{v})m$$

you may reasonably presume that if the direction or the magnitude of the velocity is changing, then the angular momentum will also change. This is true, with one exception. Take, for example, the earth rotating about the sun. Figure 7.5(a) shows the sun at point C, with the earth of mass m rotating about it.



(a) (b)

Figure 7.5: The earth rotating around the sun

In Figure 7.5(a), the earth is in one position, with position vector $\mathbf{r}_C$ and tangential velocity $\mathbf{v}$ at right angles to the position vector. The magnitude for the angular momentum for this first case is:

$$|\mathbf{H}_C| = m|\mathbf{r}_C \times \mathbf{v}| = mrv$$

The angle between the vectors is a right angle, so sine of 90° is one, so the cross product simply becomes the multiplication of two scalars.

In the second case, as shown in Figure 7.5(b), the velocity has changed in direction, but the angular momentum is exactly the same. Only relative to point C, is angular momentum is conserved in this special case.

7.4.2 Angular Momentum of a Disk

Now that we follow the definition of angular momentum, what is the angular momentum 17
of a spinning disk? Figure 7.6 shows a rotating disk of radius R and mass M, rotating about point C, an axis perpendicular to the disk, with a small element of the disk highlighted, with mass m_i and velocity v_i.

The direction of the angular momentum is relatively trivial— $\mathbf{r} \times \mathbf{p}$ is coming out of the page. For the magnitude of the element:

$$H_{i_C} = mr_{i_C}v_i = mr_{i_C}^2\omega \qquad \text{(knowing } v_i = \omega r_i\text{)}$$

The entire angular momentum of the entire disk about point C is the summation of these elements:

$$H_{\text{disk}_C} = \omega \sum_{i=1}^{n} m_i r_{i_C}^2$$

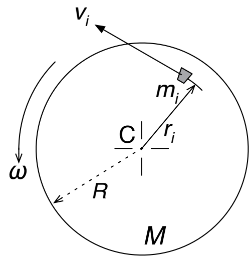


Figure 7.6: Rotating disk about C with element highlighted

Notice, referring to section 7.3.1, that:

$$\sum m_i r_{i_C}^2 = I$$

so, the magnitude of the angular momentum of the entire disk is:

$$H_{\text{disk}_C} = I\omega \tag{7.4}$$

Now, the interesting part of this is that if you choose to determine the angular momentum of the disk *rotating about its centre of mass* relative to an arbitrary point that is *not* at the centre of mass, the angular momentum is the same; this is why the value for moment of inertia in equation 7.4 does *not* have a subscript. In other words, the angular momentum $I\omega$ is an *intrinsic* property of the disk. This is called the **spin angular momentum**, and is only valid if the object is spinning about its centre of mass. The units of angular momentum are kgm^2/s which are generally written using the less cumbersome Nms.

Like linear momentum, angular momentum must be conserved in the absence of external torques (more on this in the next section). So from equation 7.4, we can see that any changing the moment of inertia of an object has a direct consequence on the angular velocity. If the moment of inertia is reduced, the angular frequency must increase.

18 A useful illustration of the conservation of angular momentum is the spinning ice skater.

(a) (b)

Figure 7.7: Ice-skater demonstrating conservation of angular momentum

When the skater spins with her arms held out, as shown in Figure 7.7(a), the moment of inertia is higher than when her arms are held in tight to her body (Figure 7.7(b)). This increase or decrease in moment of inertia has a direct impact on the angular velocity of the skater. The lower the moment of inertia, the greater the angular velocity.

7.4.3 Angular Momentum: Example

The Porsche GT3 R Hybrid uses a flywheel to store energy recouped from braking during endurance races. The flywheel spins at 40,000 rpm and can deliver 120 kW for 8 second bursts. Determine the angular momentum when the flywheel is spinning and its radius of gyration, assuming the flywheel weighs 10 kg.

Summary

19 Rotational kinetic energy & Moment of Inertia:

For a disk rotating about centre of mass, G:

$$U_{k,\text{disk}} = \frac{1}{2} I_G \omega^2 \tag{7.5}$$

Moments of inertia:

Item	Solid Disk	Sphere	Thing Ring	Mass on arm	Rod
Dimensions	m, R	m, R	m, R	m, R	m, l
I	$\frac{1}{2}mR^2$	$\frac{2}{5}mR^2$	mR^2	mR^2	$\frac{1}{12}ml^2$

Parallel Axis Theorem:

20 To calculate moment of inertia rotating about z', where z' is axis parallel to axis through centre of mass, z a distance h away.

$$I_{z'} = I_z + mh^2$$

Radius of gyration:

$$k_G^2 = \frac{I_G}{m} \quad \rightarrow \quad I_G = mk_G^2$$

Angular Momentum

21 Definition: Angular momentum of an object with respect to arbitrary point Q.

$$\mathbf{H}_Q = (\mathbf{r}_Q \times \mathbf{v})m$$

If Q is centre of rotation:

$$|\mathbf{H}_Q| = mrv = mr^2\omega$$

For a disk:

$$H_{\text{disk}} = I\omega$$

Like linear momentum, angular momentum for a disk must be conserved if no external forces are acting upon it.

Rotational Motion: Exercises

The rotational motion exercises are contained within the exercises after Section 8. For the meantime, continue with problems on pages 95–100 on Work and Energy.

8 Torque and Centrifugal Force

8.1 Moments and Torque 2

8.1.1 Moments

A turning moment must be applied to shaft or disk in order to make it rotate, in the 3
same way that a force must be applied to a body in order to make it move. This turning moment is the application of a force in such a way to make an object rotate about a specific point. An example is the crank on a bicycle, pushing a door, turning a door handle or key, a spanner on a nut etc.

We briefly discussed moments in sections 2 and 3 when studying Newton's laws, specifically when dealing with the motion of a rigid body. In addition, moments were dealt with considerably during the stress part of this course.

In Figure 8.1, a force is applied at 90° to a bar at a distance r from a pivot O, about which the bar can rotate, then the force will tend to move along the circumference of a circle radius, r.

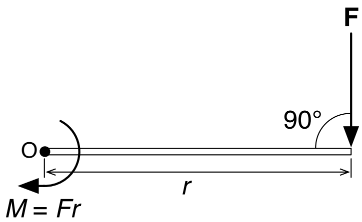


Figure 8.1: Force being applied to a bar

The moment can be calculated as force times radius:

$$M = Fr$$

The radius is always the distance *perpendicular* from the line of action of the force to the pivot point.

8.1.2 Torque

We know that angular momentum relative to an arbitrary point Q is: 4

$$\mathbf{H}_Q = \mathbf{r}_Q \times \mathbf{p}$$

Taking the derivative with respect to time (using differentiation by parts):

$$\frac{\mathrm{d}\mathbf{H}_Q}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}_Q}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r}_Q \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

5 The term $\frac{\mathrm{d}\mathbf{r}_Q}{\mathrm{d}t}$ is the first derivative of the position, which is the velocity, which is in the same direction as the momentum, so the first term $\left(\frac{\mathrm{d}\mathbf{r}_Q}{\mathrm{d}t} \times \mathbf{p}\right)$ is zero. We also know that the first derivative with respect to time of momentum, $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$ is the *force* on the object. So the derivative of angular momentum with respect to time is:

$$\frac{\mathrm{d}\mathbf{H}_Q}{\mathrm{d}t} = \mathbf{r}_Q \times \mathbf{F} = \mathbf{T}_Q \tag{8.1}$$

where $\mathbf{T}_Q$ is called **torque**.

Equation 8.1 is saying that if there is a torque on an object, then its angular momentum is changing. Likewise, if an object's angular momentum is changing, a torque must be acting on it; exactly the same is said of linear momentum. Going back to the
6 example the earth rotating around the sun studied in Section 7, as shown in Figure 8.2, the gravitational force, $\mathbf{F}$ exerted on the earth is in the opposite direction of the position vector (angle is 180°), so according to equation 8.1, $\mathbf{T}_C = 0$, i.e. there is no torque relative to point C.

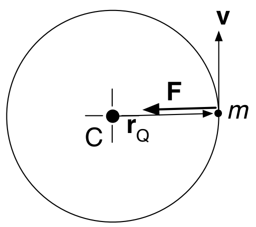


Figure 8.2: The earth rotating about the sun, with gravitational force identified

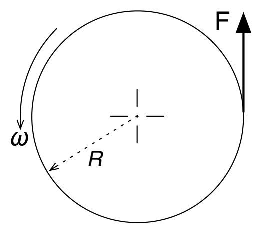


Figure 8.3: Torque in a disk

7 However, when a force is applied tangentially to the disk or shaft to cause rotation about its centre, as shown in Figure 8.3, the magnitude of the cross product is simply

the scalar multiplication of $\mathbf{r}$ and $\mathbf{F}$, so:

$$\mathbf{T} = Fr$$

which is equivalent to the turning moment, M.

The direction of the torque vector follows the right hand rule. A clockwise rotation is into the page, while an anti-clockwise direction is out of it.

You may have heard of torque when referring to the engine of a vehicle. The combustion of fuel inside the cylinder of a engine applies a tangential force to the crank shaft via connecting rod, as shown in Figure 8.4, producing this *torque*.

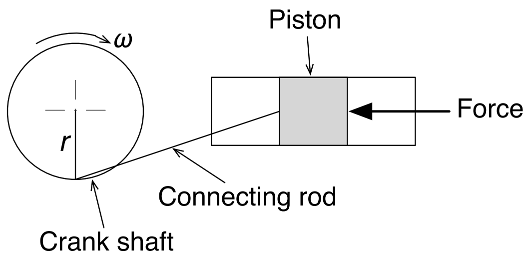


Figure 8.4: Piston and crank producing torque

8.1.3 Newton's Second Law for Rotating Bodies

Above, we derived the equation for torque from taking the derivative of the angular 8
momentum with respect to time:

$$\frac{\mathrm{d}\mathbf{H}_Q}{\mathrm{d}t} = \mathbf{T}_Q$$

We also know that the angular momentum for a disk spinning about its centre of mass is:

$$H_Q = I_Q \omega \tag{8.2}$$

so taking the derivative of equation 8.2 results in:

$$\frac{\mathrm{d}\mathbf{H}_Q}{\mathrm{d}t} = \mathbf{T}_Q = I_Q \alpha \tag{8.3}$$

where α is the angular acceleration. **This is the angular equivalent of Newton's Second Law, $\mathbf{F} = m\mathbf{a}$.** So, we can say that: 9

- A **torque** is angular equivalent of a **force**: $\mathbf{T} \rightarrow \mathbf{F}$
- The **moment of inertia** is the angular equivalent to **mass**: $I \rightarrow m$
- **Angular acceleration** is equivalent to **linear acceleration**: $\alpha \rightarrow a$

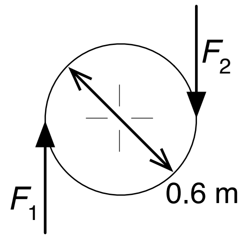


Figure 8.5: Puck

8.1.4 Torque: Example

Considering a puck of mass 0.15 kg and 0.6 m diameter, as shown in Figure 8.5. Determine the instantaneous linear acceleration of the puck and the angular acceleration of the puck, stating any assumptions.

8.2 Torque Impulse, Work and Power

Now that we have identified equivalent quantities for linear and angular motion, much of the analysis used to derive linear properties of impulse, work and power can be used for angular motion.

8.2.1 Torque Impulse

You will recall that the force impulse in linear motion was given by: 10

$$\text{Force impulse} = \int_{t_1}^{t_2} \mathbf{F}\,\mathrm{d}t$$

which had units of Ns.

Similarly, by definition, the impulse of a torque is the the torque being applied multiplied by the time for which it acts. For a varying torque:

$$\text{Torque impulse} = \int_{t_1}^{t_2} \mathbf{T}\,\mathrm{d}t$$

and hence for a constant torque:

$$\text{Torque impulse} = \mathbf{T}\Delta t$$

which both have units of Nms.

We also noted that force impulse with the change in linear momentum. In Section 8.1.3, equation 8.3 states that: 11

$$\mathbf{T} = I\alpha$$

Substituting the definition of angular acceleration:

$$\mathbf{T} = I\frac{\mathrm{d}\omega}{\mathrm{d}t} \quad \rightarrow \quad \mathbf{T}\,\mathrm{d}t = I\,\mathrm{d}\omega \quad \rightarrow \quad \int_{t_1}^{t_2} \mathbf{T}\,\mathrm{d}t = I\int_{\omega_1}^{\omega_2} \mathrm{d}\omega$$

Hence:

$$\int_{t_1}^{t_2} \mathbf{T}\,\mathrm{d}t = I\omega_2 - I\omega_1 = \mathbf{H}_2 - \mathbf{H}_1 = \Delta\mathbf{H} \tag{8.4}$$

where:

$$\mathbf{H} = I\omega = \text{angular momentum}$$

8.2.2 Work Done by a Torque

12 In Section 6, we discussed work done by a force which was:

$$W = \int_A^B F \,\mathrm{d}x$$

Similarly, by definition:

$$\text{Work done by a torque} = \int_{\theta_1}^{\theta_2} \mathbf{T} \,\mathrm{d}\theta \tag{8.5}$$

and for a constant torque:

$$\text{Work done by a constant torque} = \mathbf{T} \int_{\theta_1}^{\theta_2} \mathrm{d}\theta$$

The units for work done are J, as for linear motion.

13 Much like section 6.2.2, we can take Newton's second law (for rotation):

$$\mathbf{T} = I\alpha = I\frac{\mathrm{d}\omega}{\mathrm{d}t} \qquad \text{also:} \quad \mathrm{d}\theta = \omega \mathrm{d}t$$

So, substituting this into equation 8.5:

$$\text{Work done by a torque} = \int_{\theta_1}^{\theta_2} I\frac{\mathrm{d}\omega}{\cancel{\mathrm{d}t}}\omega \cancel{\mathrm{d}t} = \int_{\omega_1}^{\omega_2} I\omega \,\mathrm{d}\omega = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$$

which is the change in angular kinetic energy produced by the torque.

8.2.3 Power Transmitted by a Torque

14 As said in section 6.2.9, power is the rate of doing work:

$$\text{Power} = \frac{\text{work done}}{\text{time taken}}$$

which for linear motion was:

$$\text{Power} = Fv$$

For torque, the equivalent terms can be used:

$$\text{Power} = T\omega$$

8.3 Linear and Angular Dynamics Equivalents

Table 8.1 lays out the relationships between linear and angular equivalent dynamics equations.

Description	Linear	Angular
Resistance to acceleration	Mass (inertia)	Moment of inertia $= I = mk^2$
Produces acceleration	Force	Moment of force or Torque
Momentum	mv	$I\omega$
Newton's 2nd Law	$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv)$	$F = \frac{\mathrm{d}}{\mathrm{d}t}(I\omega)$
Newton's 2nd Law	$F = ma$	$T = I\alpha$
Kinetic Energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
Work (constant force/torque)	$W = Fx = \frac{1}{2}m(v_2^2 - v_1^2)$	$W = T\theta = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$
Work (varying force/torque)	$W = \int F\,\mathrm{d}x = \frac{1}{2}m(v_2^2 - v_1^2)$	$W = \int T\,\mathrm{d}\theta = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$
Impulse (constant force/torque)	Ft	Tt
Impulse (varying force/torque)	$\int F\,\mathrm{d}t = mv_2 - mv_1$	$\int T\,\mathrm{d}t = I\omega_2 - I\omega_1$

Table 8.1: Linear and Angular Dynamics Equivalents

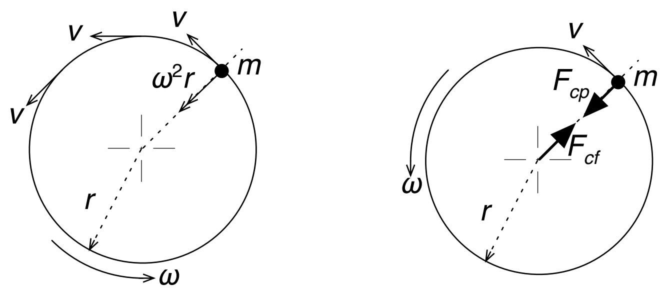
8.3.1 Torque Impulse, Work and Power: Example

A flywheel in the form of disk 1 m in diameter and mass of 100 kg is accelerated from rest by a torque of 1 Nm. How long and how many revolutions does it take for the flywheel to reach a speed of 60 rpm?

8.4 Centrifugal Force

It has been shown (in your revision notes) that when a body moves along a circular 17
path, it experiences an acceleration directed towards the centre of the circular path, even when the body is moving with constant tangential speed. Figure 8.6(a) illustrates this:

The velocity is varying with direction, not magnitude, so this change in velocity needs to be accounted for by an acceleration. This acceleration is known as **centripetal acceleration**, and arises as a result in this *change in direction of velocity* of the body.



(a) (b)

Figure 8.6: Centripetal acceleration (a); Centrifugal force (b)

If a body following the circular path shown in Figure 8.6(a) has a mass, then the centripetal force is:

$$F_{cp} = ma_c$$

where

$$a_c = \frac{v^2}{r} = \omega^2 r$$

From Newton's third law, for every force there is an equal and opposite reaction force, 18
so the **centrifugal force**, as shown in Figure 8.6(b), is equal and and opposite to this centripetal force.

Hence:

$$F_{cf} = -m\frac{v^2}{r} = -m\omega^2 r$$

8.4.1 Centrifugal Force: Example

A motorcycle rides of a level camber-less road and executes a turn with a radius of 75 m. The coefficient of friction between the road and the tyre is $\mu = 0.64$.

(a) Calculate the angle θ that the motorcyclist makes with the vertical when travelling at 15 m/s.
(b) Calculate the maximum speed at which the motorcyclist may take the bend if sliding is not to occur.
(c) Calculate the angle θ that corresponds to part (b).

Summary

Moments and Torque

Moment applied to an arm by force, F distance r from pivot point: 19

$$M = Fr$$

Torque

$$\mathbf{T}_Q = \mathbf{r}_Q \times \mathbf{F}$$ 20

which is the the first derivative of momentum with respect to time:

$$\mathbf{T}_Q = \frac{\mathrm{d}\mathbf{H}_Q}{\mathrm{d}t}$$

If force is tangential, the cross product becomes a multiplication of scalars:

$$T = Fr$$

Newton's 2nd Law for Rotation Bodies

$$\mathbf{T}_Q = I_Q \alpha$$ 21

Torque Impulse, Work and Power

Torque impulse: $\displaystyle\int_{t_1}^{t_2} \mathbf{T}\,\mathrm{d}t = \underbrace{T\Delta t}_{\text{For a constant torque}} = \Delta\mathbf{H}$ 22

Work done by torque: $\displaystyle\int_{t_1}^{t_2} \mathbf{T}\,\mathrm{d}\theta = \underbrace{T\Delta\theta}_{\text{For a constant torque}} = \Delta\mathbf{U}_{k,\text{disk}}$

Power transmitted by a torque: $P = T\omega$

Linear and Angular Dynamics Equivalents

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24

Description	Linear	Angular
Resistance to acceleration	Mass (inertia)	Moment of inertia $= I = mk^2$
Produces acceleration	Force	Moment of force or Torque
Momentum	mv	$I\omega$
Newton's 2nd Law	$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv)$	$F = \frac{\mathrm{d}}{\mathrm{d}t}(I\omega)$
Newton's 2nd Law	$F = ma$	$T = I\alpha$
Kinetic Energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
Work (constant force/torque)	$W = Fx = \frac{1}{2}m(v_2^2 - v_1^2)$	$W = T\theta = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$
Work (varying force/torque)	$W = \int F\,\mathrm{d}x = \frac{1}{2}m(v_2^2 - v_1^2)$	$W = \int T\,\mathrm{d}\theta = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$
Impulse (constant force/torque)	Ft	Tt
Impulse (varying force/torque)	$\int F\,\mathrm{d}t = mv_2 - mv_1$	$\int T\,\mathrm{d}t = I\omega_2 - I\omega_1$

Table 8.2: Linear and Angular Dynamics Equivalents

Centrifugal Force

25 *Centrifugal* force is *centripetal reaction force*:

$$F_{cf} = -F_{cp} = -m\frac{v^2}{r} = -m\omega^2 r$$

Torque and Centrifugal Force: Exercises

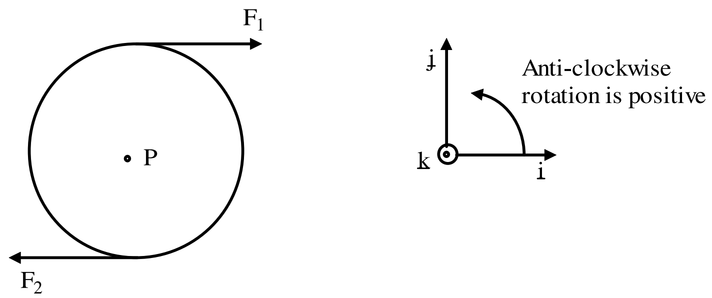
Torque

1. The rim of a steel pulley-wheel is 120 mm wide and 20 mm thick, with a mean diameter of 1.4 m. Considering the pulley as a thin ring, and neglecting the mass of the hub and the spokes, calculate for the pulley
   a) its moment of inertia;
   b) the torque which must be applied to the pulley to give it a speed of 21 rev/s in a time of 20 s. Take the density of steel to be 7850 kg/m^3.

   [Ans. 40.6 kgm^2; 267.9 Nm]
2. A mass of 500 g is mounted on the end of a light arm which is 300 mm long. The arm is accelerated uniformly from rest to 2000 rev/min in 15 s. Calculate for the system
   a) its moment of inertia;
   b) its angular acceleration;
   c) the torque required.

   [Ans. 0.045 kgm^2; 13.963 rad/s^2; 0.6283 Nm]
3. A light arm 600 mm long is pivoted at its centre, and carries a 12 kg mass at each end. Calculate the resulting angular acceleration of the arm when a couple of 4 Nm is applied to it. [Ans. 1.852rad/s^2]
4. The rim of a cast-iron flywheel is 25 mm thick and 160 mm wide, with a mean diameter of 1.2 m. Considering the rim of the flywheel as a thin ring and neglecting the mass of the hub and the spokes, calculate for the flywheel
   a) its moment of inertia;
   b) its rate of deceleration when it slows down under the action of a friction couple of 12 Nm;
   c) the time taken for the flywheel to come to rest from a speed of 20 rev/min due to the friction couple. Take the density of cast-iron to be 7200 kg/m^3.

   [Ans. 39.086 kgm^2; 0.307 rad/s^2; 6.82 s]
5. An ice puck is in the form of a solid circular disc and has a mass of 0.1 kg and a diameter of 180mm, as shown in the figure below.



The puck is stationary on the ice when it is struck simultaneously by two horizontal forces, one of magnitude $F_1 = 8.6\,\mathrm{N}$ in the positive **i** direction, and the other

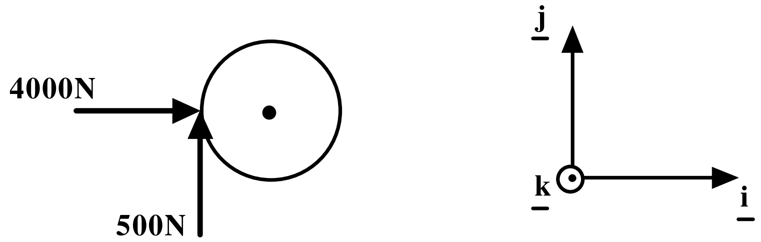
of magnitude $F_2 = 2.8\,\mathrm{N}$ in the negative $\mathbf{i}$ direction, both of these forces being tangential to the puck as shown in the figure above.
Calculate, giving your answers in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form:

a) The instantaneous linear acceleration of the centre of the puck, P. [Ans. $58\mathbf{i}\,\mathrm{m/s^2}$]
b) The angular acceleration of the puck. [Ans. $-2533.3\mathbf{k}\,\mathrm{rad/s^2}$]

6. A solid ball of mass $m = 0.3\,\mathrm{kg}$ and diameter $d = 50\,\mathrm{mm}$ has a force given by:

$$F = 4000\mathbf{i} + 500\mathbf{j}\,\mathrm{N}$$

applied to it as shown in the figure below.



Determine the linear acceleration of the ball in $\mathbf{i}$,$\mathbf{j}$ component form, and also its angular acceleration given that the moment of inertia of a solid ball about its centre is given by:

$$I_0 = \frac{md^2}{10}$$

where d is the diameter. [Ans. $a = 13.33 \times 10^3\mathbf{i} + 1.67 \times 10^3\mathbf{j}\,\mathrm{m/s^2}$, $\alpha = -166.67 \times 10^3\mathbf{k}\,\mathrm{rad/s^2}$]

Centrifugal Force

7. A body of mass 700 g moves in a horizontal circle of radius 1.4 m at a rate of 50 rev/min. Calculate the force that must be acting radially inwards on the body. (Ans. 26.87 N)
8. A car is travelling at 72 km/h and has wheels with an effective rolling diameter of 540 mm. If one of the wheels is out of balance to the extent of 13 g at a radius 80 mm, calculate the magnitude of the unbalanced force acting on the wheel. (Ans. 5.71 N)
9. A train of total mass 20 tonne travels around a horizontal curved track of radius 250 m at a speed of 80 km/h. Calculate the horizontal force acting on the rails. (Ans. 39506 N)
10. A car of mass 1.2 tonne travels around a horizontal un-banked curved track of radius 70 m at a speed of 85 km/h. Calculate the side thrust on the tyres. (Ans. 9557 N)

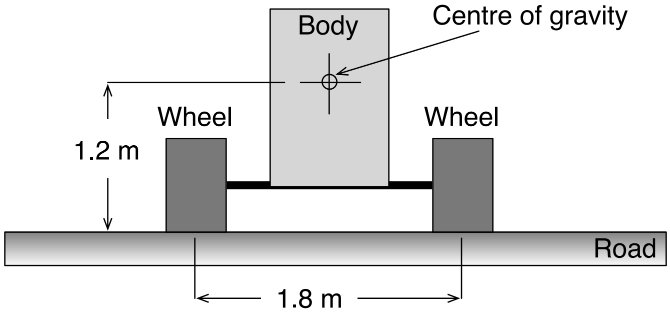
11. Calculate the maximum speed at which a motorcycle may travel over a hump-backed bridge of radius 12 m without leaving the ground. (Ans. 10.85 m/s)
12. A mass of 4 kg is whirled in a vertical circle on the end of a cord 900 mm long. Calculate the maximum angular velocity at which the tension in the cord is zero. Calculate also the maximum tension in the cord when the rotational speed of the mass is 1.2 rev/s. (Ans. 3.3 rad/s; 243.9 N)
13. A trolley of mass 8 kg travels around the inside of a vertical track of radius 3 m. Calculate the least velocity that the trolley must have in order not to fall from the track at the highest point. Calculate also the maximum force exerted on the track when the trolley has a speed of 36 km/h. (Ans. 5.42 m/s; 345.1 N)
14. A motorcyclist rides on a level camber-less road and executes a turn having a radius of 50 m. The coefficient of friction between the tyres and the road is 0.7. Draw a free body diagram showing all the forces acting on the bike and their relationships to the accelerations, and calculate:

    a) The angle β that the motorcyclist makes with the ground if the speed is 10 m/s
    b) The maximum speed if no sliding is to occur, and the corresponding value of β.

    [Ans: 78.5°; 18.5 m/s; 55°]
15. A motor vehicle travels around a level, un-banked and camber-less track of 75 m radius. The centre of gravity of the vehicle is 700 mm above the ground and the wheel track width is 1.6 m. The coefficient of friction between the tyres and the road is 0.7. Calculate the maximum speed at which the vehicle may travel in miles/hour without either over-turning or slipping sideways. [Ans. overturns at 64.9 mph; slips at 50.8 mph]
16. A motor-cyclist rides on a level, un-banked and camber-less road around a bend of radius 50 m, the coefficient of friction between the tyre and the road being 0.65. Calculate:

    a) the angle the motorcyclist makes with the ground when travelling at a speed of 15 m/s;
    b) the maximum speed at which the bend may be taken if sliding is not to occur, and the corresponding angle of inclination of the bike to the ground.

    [Ans. 65.36°; 17.86 m/s; 57°]
17. The figure below represents the rear view of a motor vehicle that is travelling at constant speed around a bend of radius 80 m on a horizontal un-banked road. The vehicle has a total mass of 1400 kg and its centre of gravity is 1.2 m above the road. The wheel track width is 1.8 m and the coefficient of friction between the tyres and the road is $\mu = 0.65$.



Calculate:

a) The weight of the vehicle [13734 N]

b) The maximum value of the friction force that can occur between the tyres and the road, and hence the centrifugal force that would have to act on the vehicle in order to cause it to be on the point of skidding sideways [8927.1 N]

c) The centrifugal force that would have to act on the vehicle in order to cause it to be on the point of overturning [10300.5 N]

d) State whether the vehicle is more likely to skid or overturn if it is driven too quickly around the bend. Explain your answer.

e) The maximum safe speed at which the vehicle may be driven around the bend. Give your answer in m/s, km/h and mph. (Note: 1 mile = 1.609 km) [22.586 m/s, 81.31 km/h, 50.53 mph]

f) The angular velocity of the wheels of the vehicle after it has negotiated the bend, if it is then moving in a straight line at a speed of 80 km/h, given that the rolling radius of the wheels is 0.4 m. [55.56 rad/s]

9 Springs and Mechanical Oscillation

9.1 Oscillations 2

9.1.1 Natural Vibrations

Mechanical oscillations or vibration can be both useful (watches, ultrasound etc) or 3
a nuisance (engine vibration, fatigue failure of materials). The simplest example of a system that can oscillate is a *single degree of freedom* un-damped single point mass attached to a single spring, as shown in Figure 9.1

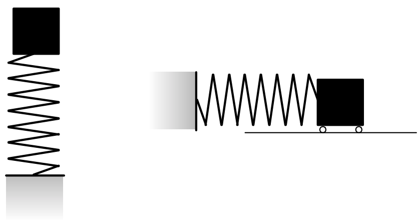


Figure 9.1: A simple oscillatory system

If the position of the mass in both of the illustrations above is the equilibrium position, then the system will not move. The masses will be at rest. If, however, a deflection in the position is made to the mass, then the spring in the system will be deformed from its normal equilibrium position, and this deformation in the material of the spring will produce *internal forces* which will attempt to restore the system back to its equilibrium position.

The work done by deforming the spring is stored within the spring as *elastic potential energy*. If the deforming force is then removed, the internal elastic restoring force cause the body to accelerate back towards its normal equilibrium position in accordance with Newton's Second Law. During this acceleration, the stored elastic energy is converted into kinetic energy as the velocity of the body increases and the deformation reduces.

Assuming that the system is a *conservative* system, i.e. there is *no friction* and hence *no resistance to motion*, then the whole of the original elastic energy is converted into kinetic energy at the instant the body reaches its original equilibrium position, at which point the velocity (and consequently the momentum) of the body will be at their maximum values.

When the body regains its equilibrium position, the momentum of the body causes it to *overshoot*, and, in doing so, produces a restoring force in the opposite direction which gradually overcomes the inertia of the body and brings it to rest. During this retardation between the equilibrium position and the position of maximum displacement, the kinetic energy is converted back into elastic potential energy. The whole cycle of events will then occur again, and in the absence of friction, would recur indefinitely.

An oscillation or vibration of this type, in which after the initial displacement, **no external forces act** and the motion is maintained completely by the *internal elastic restoring forces*, is known as **a free or natural displacement**.

4 In such an oscillation or vibration of a conservative system (no friction), application of the principle of conservation of energy gives the relationship:

$$\text{Elastic (spring) potential energy} + \text{kinetic energy} = \text{constant}$$

A vibration may thus be regarded as a continuous energy conversion process, and the above equation may be used to give an energy method of analysis.

9.1.2 Simple Harmonic Motion

5 The vibrations discussed in this section follow what is known as **Simple Harmonic Motion** or **Simple Harmonic Oscillation** and the spring and pendulum discussed below are **Simple Harmonic Oscillators**. A simple harmonic oscillation is one that is periodic (i.e. repeating) that is neither driven (by an external force) nor damped (i.e. does not decay over time).

A simple harmonic oscillation may be considered as being produced by a vector OQ rotating about centre O with a constant angular velocity ω_n rad/s, as illustrated in Figure 9.2.

6

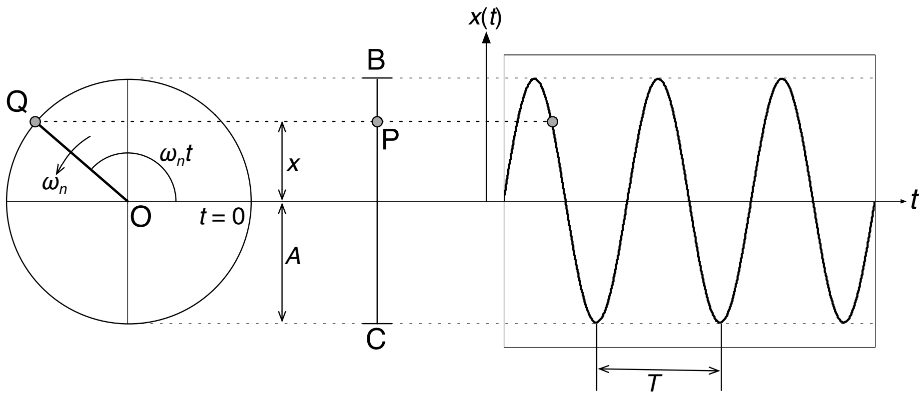


Figure 9.2: Simple Harmonic Motion

Point P is the projection of Q on to diameter BC, and the displacement of point P represents simple harmonic motion. The projection of point Q onto the t-$x(t)$ axes shows the position of point P with respect to time, and it is clear to see that the oscillation follows a sinusoidal wave form with a period of T seconds. One whole revolution of OQ produces one oscillation of P and one cycle of the sinusoidal function.

7

The equation for a sinusoidal function is of the form:

$$x(t) = A\cos(\omega_n t + \varphi)$$

where:

- A is the amplitude of oscillation
- ω_n is the angular frequency and the units are rad/s. Since we are talking about simple harmonic motion, undriven, this frequency is the *natural angular frequency*, signified by the subscript n.
- φ is the phase angle (rad).

The choice of using cosine in this equation is arbitrary. Another valid formulation is:

$$x(t) = A\sin(\omega t + \varphi)$$

since:

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

Characteristics of Simple Harmonic Motion

Table 9.1 lists position, velocity and acceleration of point P at the centre of oscillation 8
and at the extremities of oscillation for simple harmonic motion.

At the centre of oscillation	
$x = 0$ $v = \omega_n A$ $a = 0$	This is the maximum velocity

At the extremities of oscillation (B & C)	
$x = A$ $v = 0$ $a = -\omega_n^2 A$	This is the maximum acceleration

Table 9.1: Characteristics of Simple Harmonic Motion

One of the most important things to remember is that if a system is undergoing simple harmonic motion, **the angular frequency at which it naturally oscillates will be its natural angular frequency**, ω_n. Only if a system is driven (i.e. a external force being applied) will the system *not* oscillate at its natural frequency.

Period, Frequency and Angular Frequency

With the sinusoidal function representing simple harmonic motion, the relationship 9
between the properties are valuable to know. The properties are:

- Period, T which is the time for one cycle of oscillation [s]
- Frequency, f which is the number of cycles per second [Hz]

- Angular frequency, ω which is the angular (or circular) frequency [rad/s]

The relationship between *period* and *frequency* is:

$$T = \frac{1}{f} \quad \longleftrightarrow \quad f = \frac{1}{T}$$

The relationship bewteen *frequency* and *angular frequency* is:

$$\omega = 2\pi f \quad \longleftrightarrow \quad f = \frac{\omega}{2\pi}$$

Consequently, the *period* and *angular frequency* are related by:

$$T = \frac{2\pi}{\omega} \quad \longleftrightarrow \quad \omega = \frac{2\pi}{T}$$

9.2 Springs

9.2.1 Stiffness

10 Consider the spring, illustrated in Figure 9.3 which has a relaxed length of x_1. If we extend the spring to position x_2, the spring produces a force which wants to drive this string back to equilibrium.

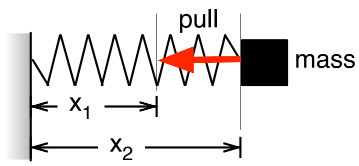


Figure 9.3: Spring accelerating a mass

It is an experimental fact that with many springs, called ideal springs, this force is proportional to the displacement $x_2 - x_1 = \Delta x$.

$$|\mathbf{F}| \propto |\mathbf{\Delta x}|$$

If you make Δx three times larger, then the restoring force will be three times larger.

Figure 9.3 is a one dimensional problem, so we can avoid vector notation, and we can simply say that the force is:

$$F = -kx$$

where x is now the extension and k is the spring constant, which has the units N/m. The minus sign takes care of the direction: when x is positive (extension in this case), the force is in the negative direction, and when x is negative (compression of a spring), the force is in the positive direction. It is a *restoring* force, so **opposes** motion.

Whenever this linear relation between F and x holds, this is referred to as **Hooke's Law**.

Determining the spring constant involves a simple experiment whereby you can hang different masses from the bottom of the spring, measure the extension, and by plotting a line, similar to that shown in Figure 9.4 of restoring force (which are the weights of the masses) with spring extension the slope of which will be the spring constant. Hence 11
the value for k is:

$$k = \frac{\Delta F}{\Delta x}$$

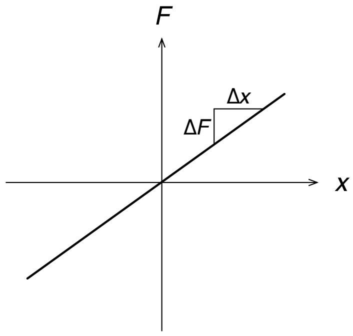


Figure 9.4: Plot of Force and Extension of a spring to determine k

Another way to determine the spring constant is to examine the dynamics of the 12
system. If we have an experimental setup, as shown in Figure 9.5, where the relaxed length of the spring is $x = 0$. if mass m is attached to the spring, and displaced to a value x and released, assuming no friction is occurring, the spring will oscillate back and forth around $x = 0$.

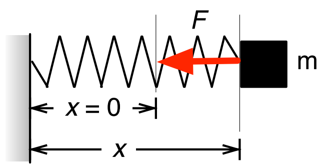


Figure 9.5: Measuring spring constant dynamically

The period of oscillation is:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{9.1}$$

where m is the mass, and k is the spring constant. So, in an experiment, if we measure the period, and we know the mass, we can determine k.

The interesting, and non-intuitive characteristic of equation 9.1 is that the period is independent of the extension of the spring (so long as Hooke's law holds).

9.2.2 Combined Stiffness of Springs

Springs in Parallel

13 Figure 9.6(a) shows two systems in which the springs with stiffnesses k_1 and k_2 are in parallel.

Adding springs in parallel *increases* the overall stiffness. The stiffnesses are thus additive. Hence the combined stiffness, or *effective stiffness*, k_e springs in parallel is:

$$k_e = k_1 + k_2 + \cdots + k_n$$

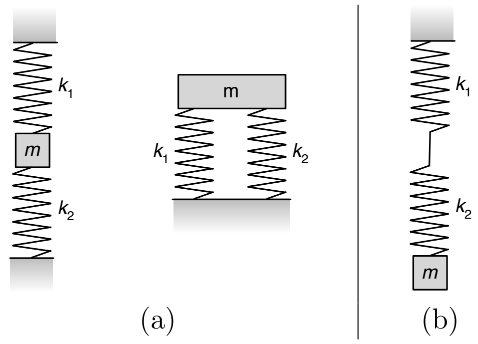


Figure 9.6: Springs in parallel (a); Springs in series (b)

Springs in Series

14 Figure 9.6(b) shows a system where springs of stiffnesses k_1 and k_2 are connected in series with one another.

Adding springs in series *reduces* the overall stiffness. The combined stiffness is therefore *less* stiff, and hence more flexible. Flexibility is defined as the inverse of stiffness:

$$\text{Flexibility} = \frac{1}{\text{Stiffness}}$$

Hence the *flexibilities*, i.e. the inverse of stiffnesses are additive. This results in:

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \quad \rightarrow \quad k_e = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}} = \left(\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}\right)^{-1}$$

9.2.3 Oscillation of a Spring

15 To investigate the motion of a spring, we start by using Newton's second law:

$$m\ddot{x} = -kx \quad \rightarrow \quad m\ddot{x} + kx = 0$$

which results in:

$$\ddot{x} + \frac{k}{m}x = 0 \tag{9.2}$$

Equation 9.2 is one of the most important equations when dealing with mechanical oscillations. It is second order differential equation, which you should be able to solve. If we set up an experiment like that shown in Figure 9.7, it is clear that the mass would behave as a simple harmonic oscillation around $x = 0$.

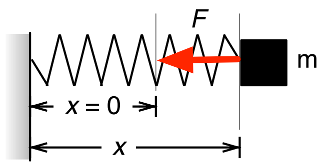


Figure 9.7: Measuring spring constant dynamically

Since simple harmonic oscillations are sinusoidal, a trial function that would satisfy 16
equation 9.2 is:

$$x(t) = A\cos(\omega t + \varphi) \tag{9.3}$$

as shown above in Section 9.1.2

In this case, A is the farthest distance from x to $x = 0$, measured in metres. If the time, t is advanced by:

$$T = \frac{2\pi}{\omega} \tag{9.4}$$

the oscillation will repeat itself (2π is one cycle, or 360°).

To substitute equation 9.3 in to equation 9.2, we need to find the second derivative 17
of $x(t)$:

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ \dot{x}(t) &= -A\omega\sin(\omega t + \phi) \\ \ddot{x}(t) &= -A\omega^2\cos(\omega t + \phi) = -\omega^2 x(t) \end{aligned} \tag{9.5}$$

Substituting equation 9.5 into equation 9.2 results in:

$$-\omega^2 x(t) + \frac{k}{m}x(t) = 0 \tag{9.6}$$

Dividing both sides by $x(t)$ and rearranging results in: 18

$$\omega^2 = \frac{k}{m} \quad \rightarrow \quad \omega = \sqrt{\frac{k}{m}} \tag{9.7}$$

Since this system is free to oscillate (a free or natural oscillation), this angular frequency is known as the **natural angular frequency** and is often denoted by the subscript 0 (zero) or n, i.e. ω_0 or ω_n. Therefore, plugging equation 9.7 into 9.4 gives us 19 equation 9.1:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

So we can see that the period is independent of the extension, x and also independent on the phase angle, φ, and A and φ are determined based on the initial conditions.

Example

Using the system illustrated in Figure 9.5, we have the following initial conditions:

$$x = 0 \text{ at } t = 0 \qquad v = -3\,\mathrm{m/s} \qquad k = 10\,\mathrm{N/m} \qquad m = 0.1\,\mathrm{kg}$$

Determine the equation of motion, $x(t)$.

9.3 Oscillation of a Pendulum

Simple pendulums are another example of an oscillating system. They consist of a 20
light, inextensible string carrying a concentrated mass at one end, while its other end is attached to a fixed point. Figure 9.8 illustrates such a pendulum, with massless string length l and mass m, that oscillates through the arc shown with the dotted line. The only forces acting on the mass are its weight, mg, and the tension in the string T. T can of course be decomposed into its x and y components, and these are shown in the figure.

21

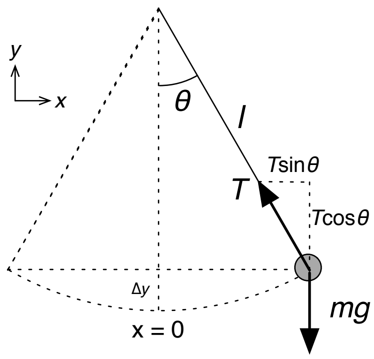


Figure 9.8: Simple Pendulum

We can write down the equations of motion using Newton's Second Law. In the x-direction:

$$m\ddot{x} = -T\sin\theta = -T\left(\frac{x}{l}\right) \tag{9.8}$$

In the y-direction:

$$m\ddot{y} = T\cos\theta - mg \tag{9.9}$$

Now, we have to solve two coupled differential equations, which is somewhat of a 22
challenge. To simplify this, we can make some approximations: small angle approximations, i.e. $\theta << 1$. If that is true, there are two consequences that are relevant to this problem:

- $\cos\theta \approx 1$ (for $\theta << 1$)
- We can also say that the excursion in the y direction (indicated by Δy in Figure 9.8) is negligible for small angles, hence $\ddot{y} \approx 0$.

So, taking equation 9.9 and applying small angle approximations:

$$0 = T - mg \quad \rightarrow \quad T = mg$$

Substituting this back into equation 9.8:

$$m\ddot{x} - mg\left(\frac{x}{l}\right) \quad \rightarrow \quad m\ddot{x} + mg\left(\frac{x}{l}\right)$$

23 Rearranging:

$$\ddot{x} + \frac{g}{l}x = 0$$

Again, this is very similar to the equation we found for the spring. Instead of $\frac{k}{m}$, we know have $\frac{g}{l}$. So we can find the solution by inspection:

$$x(t) = A\cos(\omega t + \varphi)$$

where:

$$\omega = \sqrt{\frac{g}{l}} \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{l}{g}}$$

both of which are valid for *small angles*. Again, since this system is free to oscillate (a free or natural oscillation), this angular frequency, ω is known as the **natural angular frequency** and is often denoted by the subscript 0 (zero) or n, i.e. ω_0 or ω_n.

The interesting characteristic of the period of oscillation for a pendulum is that is completely independent of the mass.

9.4 Other Considerations

9.4.1 Damping

24 All *real* systems are deemed *non-conservative*, i.e. they involve friction. Thus the energy possessed by a vibrating system is gradually dissipated in overcoming internal and external resistances to the motion, and the body eventually comes to rest in its original equilibrium position.

If the oscillating system is damped, then the oscillation will decay exponentially, as shown in Figure 9.9.

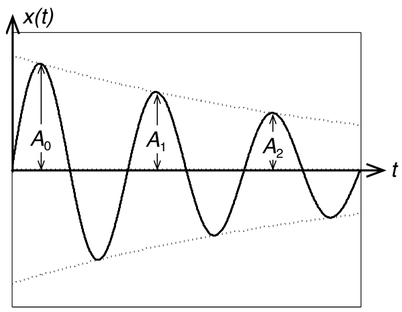


Figure 9.9: Oscillation experiencing exponential decay

The ratio of amplitudes dictates the *logarithmic decrement*, λ:

$$\frac{A_1}{A_0} = \frac{A_2}{A_1} = e^{-\lambda}$$

This effect is known as damping, and so all real vibrations are damped to a certain extent. In some cases, damping is introduced deliberately, such as in the case of shock absorbers on motor vehicles.

9.4.2 Resonance

If every time an oscillating body reaches its point of maximum displacement it receives 25
an external impulse, the amplitude will increase and build up to a maximum value depending on what forces are acting to damp down the oscillation. If no damping forces are present the amplitude will continue to increase until eventually failure of the system will occur.

The frequency at which the external impulse is applied is equal to the natural frequency of the oscillating body, and the effect of the impulse being in unison with the oscillation is known as resonance.

Take the example of someone is pushing child on a swing in a playground. If the person pushing the child times their input forces (which are like impulses) with the natural frequency of the swing, the amplitude of the swing will increase.

Examples of resonance occur in many engineering situations, such as in aircraft wings, motor vehicles, machine tools, and also in other fields such as musical instruments and electronics. For example, severe vibrations of a drilling machine may occur if it is operated at or close to the natural frequency of free vibration of the drill and its fixture. In addition, vibrations of the machine foundations could produce resonance effects on the drill.

An astonishing example of resonance and the problems that it can cause is the first 26
Tacoma Narrows bridge in the U.S. state of Washington. This bridge, a suspension bridge, was constructed between 1938 and 1940 and its design was different from previous suspension bridges. The designers, however, did not fully understand the effects of the crosswinds travelling along the valley, and from the time the road deck was built, it would oscillate vertically in light winds. Four months after opening, however, a new vibrational mode was evident: that of torsion, and the bridge collapsed.

Figure 9.10 shows two photographs of the bridge, the first showing the torsional mode of vibration, and the second the collapse. It is argued that the wind in the valley was such that it excited the resonance mode of the bridge, causing its natural amplitude to increase.

A situation where resonance is required is a wave energy converter, where the energy extraction device is at its most efficient when the waves interacting with the machine are at a frequency that matches the natural frequency of oscillation of the wave energy converter. The main issue, however, is that waves can vary in frequency, so the wave energy converter needs to be able to adapt to different natural frequencies.

Rotating machinery will have critical speeds that correspond with its natural frequencies. A turbine, for example, when starting up from rest, may have to pass through one of its natural frequencies before reaching its normal operating speed. In these circumstances, care would have to be taken to pass through such speeds as quickly as

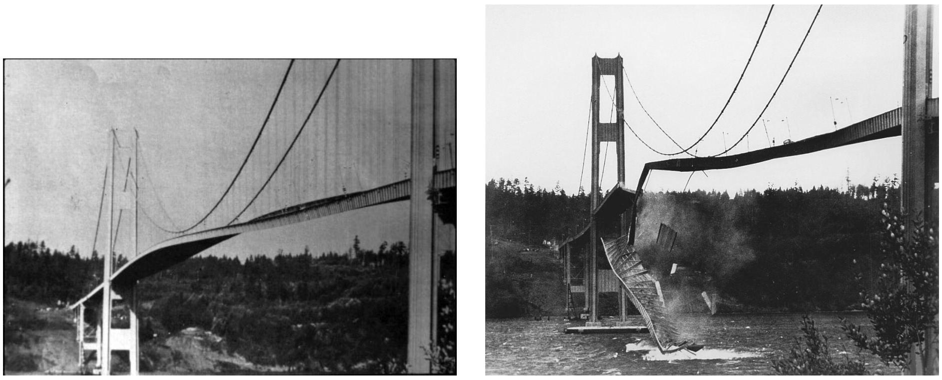


Figure 9.10: Tacoma Narrows Bridge Collapse

possible.

Obviously, unwanted vibrations should be reduced if possible. This may be done by balancing any out-of-balance forces, fitting heavy spring mountings to isolate machines from their foundations, and the use of rubber engine mountings and shock absorbers in motor vehicles.

Note that the **frequency during resonance is equal to the natural frequency of the system**.

Summary

Simple Harmonic Motion

A single-degree of freedom un-damped un-driven oscillation represents an example of 27
Simple Harmonic Motion.

SHM is a sinusoidal motion where the frequency of oscillation is called the undamped **natural frequency**, and is the frequency at which the system will oscillate in the absence of external forces. THe equation of motion is:

$$x(t) = A\cos(\omega_n t + \varphi)$$

where

- A is the amplitude
- ω_n is the angular natural frequency
- φ is the phase angle

Characteristics of SHM

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At the centre of oscillation	
$x = 0$ $v = \omega_n A$ $a = 0$	This is the maximum velocity

At the extremities of oscillation (B & C)	
$x = A$ $v = 0$ $a = -\omega_n^2 A$	This is the maximum acceleration

Table 9.2: Characteristics of Simple Harmonic Motion

Period, Frequency and Angular Frequency

The relationship between *period* and *frequency* is: 29

$$T = \frac{1}{f} \quad \longleftrightarrow \quad f = \frac{1}{T}$$

The relationship bewteen *frequency* and *angular frequency* is:

$$\omega = 2\pi f \quad \longleftrightarrow \quad f = \frac{\omega}{2\pi}$$

Consequently, the *period* and *angular frequency* are related by:

$$T = \frac{2\pi}{\omega} \quad \longleftrightarrow \quad \omega = \frac{2\pi}{T}$$

Springs

30 Force produce by a spring:

$$F = -kx$$

where k is spring stiffness

Springs in parallel:

$$k_e = k_1 + k_2 + \cdots + k_n$$

Springs in series:

$$k_e = \left(\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}\right)^{-1}$$

31 Oscillation of a spring:

$$\ddot{x} + \frac{k}{m}x = 0$$

Solved by using function for SHM as trial function. Values for A and ω and φ found by knowing initial conditions.

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}}$$

Period:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Pendulum

32 Oscillation of a pendulum (for small angles only):

$$\ddot{x} + \frac{g}{l}x = 0$$

where l is length of pendulum.

Natural frequency:

$$\omega_n = \sqrt{\frac{g}{l}}$$

Period:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Damping

33 All *real* systems experience damping due to the **non-conservative** nature of friction. Oscillation will decay exponentially, described by the logarithmic decrement.

Resonance

Resonance is when an oscillating object is excited at its natural frequency. If not con- 34
sidered during the design phase of a system, failure can occur. For resonance to occur, the frequency of the force being applied must be equal to the natural frequency of the system.

Springs and Mechanical Oscillations: Exercises

1. A slider in a mechanism moves with a Simple Harmonic Motion in which the periodic time is 0.5 s and the amplitude is 60 mm. Calculate for the motion:
   a) The natural circular frequency;
   b) The natural frequency;
   c) The maximum velocity of the slider;
   d) The maximum acceleration of the slider;
   e) The velocity and acceleration of the slider when it is 20mm from the mid-position of the oscillation.

   Answers: (a) 12.566 rad/s; (b) 2 Hz; (c) 0.754 m/s; (d) 9.475 m/s^2; (e) 0.711 m/s; 3.158 m/s^2
2. On a packaging machine mechanism, a slider of mass 0.2 kg moves in a straight guide with simple harmonic motion. At distances of 125 mm and 200 mm respectively from its mean position, the slider has velocities of 6m/s and 3 m/s respectively. Determine for the slider:
   a) The amplitude of the motion;
   b) The natural circular frequency;
   c) The periodic time;
   d) The maximum velocity;
   e) The maximum acceleration.

   Answers (a) 0.2194 m; (b) 33.28 rad/s; (c) 0.1888 s; (d) 7.3 m/s (e) 243m/s^2
3. A ship is pitching 10° above and 10° below the horizontal. Assuming the motion to be an angular simple harmonic motion having a period of 12 s, calculate for he ship:
   a) The natural circular frequency [Ans. 0.5236 rad/s]
   b) The maximum angular velocity [Ans. 0.0914 rad/s]
   c) The maximum angular acceleration [Ans. 0.0478 rad/s^2]
4. A mass of 50 kg is suspended from a spring having a stiffness of 28 kN/m and vibrates freely with an amplitude of 30 mm. Calculate:
   a) The natural circular frequency; [Ans. 23.66 rad/s]
   b) The natural frequency; [Ans. 3.766 Hz]
   c) The velocity and acceleration when the mass is at 20 mm from its equilibrium position. [Ans. 0.529 m/s; 11.2 m/s^2]
   d) The maximum inertia force acting on the mass; [Ans. 840 N]
5. The natural frequency of vibration of a mass of 2 kg suspended from a spring is 2 Hz. Calculate:
   a) The spring stiffness; [Ans. 315.8 N/m]
   b) The natural frequency of a 3kg mass on the same spring; [Ans. 1.633 Hz]
   c) The maximum spring tension produced when the 3 kg mass vibrates with an amplitude of 80 mm. [Ans. 54.7 N]

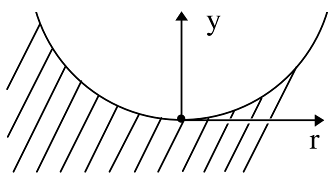
6. A mass of 100 kg is suspended from a spring having a stiffness of 56 kN/m, and vibrates freely with an amplitude of 30 mm. Calculate:

   a) The natural radian frequency; [Ans. 23.66 rad/s]
   b) The natural frequency; [Ans. 3.766 Hz]
   c) The velocity and acceleration when the mass is at 20 mm from its equilibrium position [Ans. 0.529 m/s; 11.2 m/s^2]

7. A mass of 5 kg is supported on two springs, the stiffness of the springs being 2000 N/m and 4000 N/m respectively. Sketch the arrangement and calculate the natural frequency of vibration when the springs are connected:

   a) In series; [Ans. 2.6 Hz]
   b) In parallel. [Ans. 5.513 Hz]

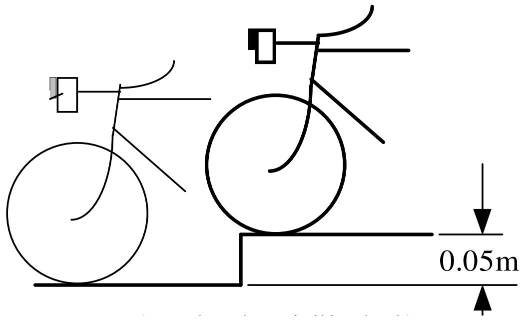
8. A building has an effective moving mass of 400 tonne and an effective stiffness of 160 MN/m in simple flexure. In free oscillation, after a gust of wind, the maximum velocity of the building mass in oscillation is 0.4 m/s. Calculate the amplitude of the oscillation and the maximum force acting on the building due to flexure. State any assumptions made. [Ans. 0.02 m; 3.2 MN]
9. Plans are made to support a Californian building whose mass is 800 tonne on 10 PTFE sliders, each supported by a smooth stainless steel parabolic cup ($y = 0.25r^2$). The purpose of this arrangement is to provide horizontal earthquake "base isolation" of the building.



   Derive an expression for the equivalent elastic stiffness of the support system, and hence calculate for the earthquake "isolated" building:

   a) The equivalent stiffness of each support [Ans. 392.4 kN]
   b) The natural circular frequency [Ans. 2.215 rad/s]
   c) The natural frequency [Ans. 0.352 Hz]

   State any assumptions made.
10. A bicycle lamp of mass 0.5 kg is fixed to the handlebars by a bracket which can be regarded as a cantilever of stiffness 5 kN/m. The bicycle suddenly drops down a step 0.05 m deep.

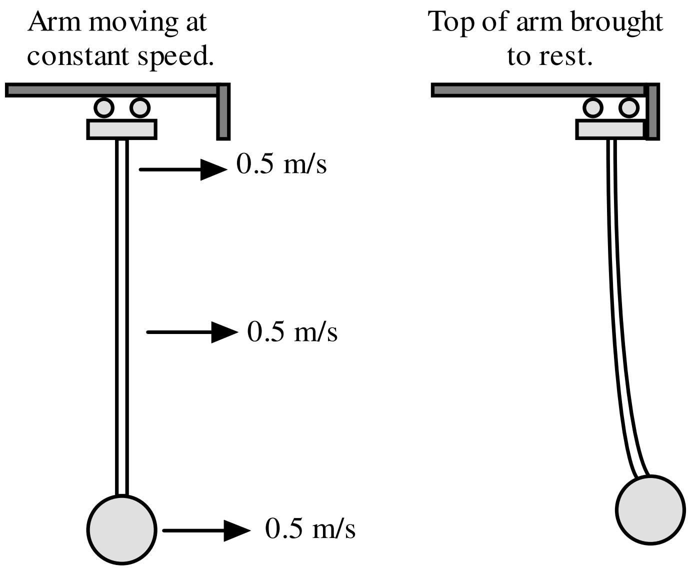


Calculate:

a) The maximum elastic stored energy in the bracket [Ans. 0.24525 J]
b) The displacement amplitude of the subsequent vibration [Ans. 9.9045 mm]
c) The frequency vibration [Ans. 15.92 Hz]
d) The maximum force in the bracket [Ans. 49.52 N]

State any assumptions made.

11. A gantry robot arm is vertical and has an effective mass of 8 kg concentrated at its lower end. The effective stiffness of the arm is 3 MN/m. Initially the arm is moving at 0.5 m/s in a horizontal direction when the upper end of the arm is suddenly brought to rest.



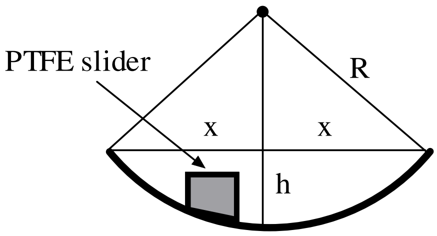
Calculate:

a) The maximum elastic stored energy in the arm [Ans.1 J]
b) The displacement amplitude of the subsequent vibration [Ans. 0.8165 mm]
c) The frequency of the vibration [Ans. 97.46 Hz]
d) The maximum force in the arm [Ans. 2449.5 N]

State any assumptions made.

12. Plans are made to support a Californian building whose mass is 720 tonne on 8

PTFE sliders, each supported by a smooth stainless steel spherical "cup" of radius 4 m. The purpose of this arrangement is to provide horizontal earthquake "base isolation" of the building.

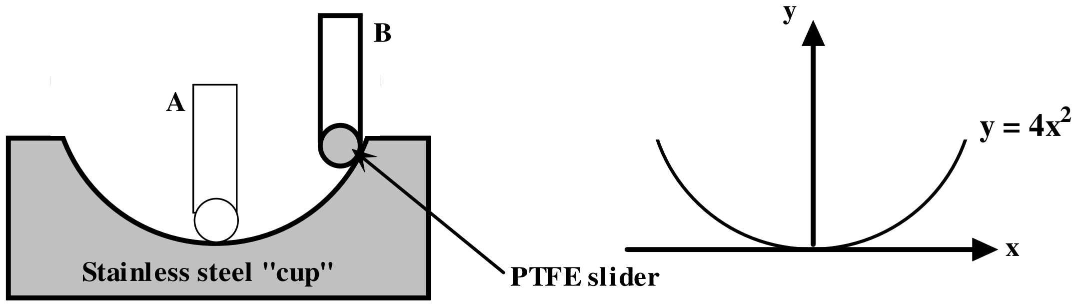


Treating each slider and cup like a simple pendulum, derive an expression for the equivalent elastic stiffness of the support system, and hence calculate for the earthquake "isolated" building:

a) The equivalent stiffness of each support; (Ans. mg/R; 220.725 kN/m)
b) The natural circular frequency; (Ans. 1.566 rad/s)
c) The natural frequency. (Ans. 0.249 Hz)

State any assumptions made.

13. A building having a mass of 50,000 kg is supported on 4 bearings. Each bearing consists of a PTFE slider attached to a supporting leg and resting in a smooth stainless steel spherical "cup", as shown in the figure below.
The purpose of this arrangement is to provide the building with horizontal earthquake "base isolation"
The curvature of the "cup" is given by the equation $y = 4x2$



In the figure, A represents one of the supporting legs of the building when it is in its rest position, while B is the same leg when the building has been displaced horizontally sideways.

a) Obtain an expression for the equivalent stiffness of the supports and hence calculate the natural frequency of transverse vibration of the building. (Hint! Consider the gain in gravitational potential energy of the building when it is displaced as if it were a gain in spring energy, and equate the two expressions) [Ans. 8mg; 1.41 Hz]

b) The natural frequency of vibration of another building supported on similar bearings is 0.25 Hz. During free oscillation of this building, the maximum speed of the building relative to the ground is 2 m/s. Calculate for this oscillation:

   i. The displacement amplitude [Ans. 1.27 m]

   ii. The maximum acceleration of the building. [Ans. 3.14 m/s^2]

c) Name and briefly discuss some ways in which damping can be applied to an oscillating system. [Ans. Dash-pot; eddy currents; air resistance; friction]