AE31002 Aerospace Structural Dynamics
Damped SDOF System

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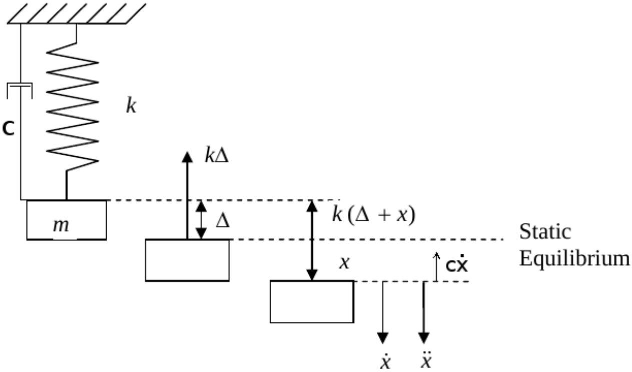
Viscous Damping – the type of damping force that could be developed in a body restrained in its motion by a surrounding viscous fluid.



The summation of forces following D'Alembert's principle
$m\ddot{x} + c\dot{x} + kx = 0$

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The trial solutions $x = A\ sin\ \omega t$ or $x = B\ cos\ \omega t$ does not satisfy the above equation where as $x = Ae^{\lambda t}$ satisfies.

$\Rightarrow\ m\lambda^2 + c\lambda + k = 0$
$\Rightarrow\ \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

$$\lambda = -\left(\frac{c}{2m}\right) \pm \left[\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)\right]^{\frac{1}{2}}$$

$$x = e^{-\left(\frac{c}{2m}\right)t}\left[Ae^{\sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\ t} + Be^{-\sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}\ t}\right]$$

There may be three possible cases where the portion under the radical may be positive or negative or zero for the cases over-damped or under-damped or critically-damped cases, respectively.

Case I $\left(\frac{c}{2m}\right)^2 > \left(\frac{k}{m}\right)$ **– Over Damped Case**

The solution becomes $x = A_1 e^{-\lambda_1 t} + A_2 e^{-\lambda_2 t}$ where λ_1 *and* λ_2 are two real roots of the equation. The system displacement will decay without any oscillation. This equation shows that there is a usual creeping back of the mass to the equilibrium position. This is a case of over-damped situation and no oscillation is possible.

ANIMATION OVER DAMPED

Case II $\left(\frac{c}{2m}\right)^2 < \left(\frac{k}{m}\right)$ **– Under Damped Case**

Roots of the equation are imaginary.

Oscillation continues for a considerable time period depending upon the damping present in the system.

$$\lambda = -\left(\frac{c}{2m}\right) \pm i \left[\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2\right]^{\frac{1}{2}}$$

Now using exponential series,
$e^{i\theta} = cos\theta + i\ sin\theta$
$e^{-i\theta} = cos\theta - i\ sin\theta$

Substituting the expressions similar to the above we get,
considering, $\omega_d = \left[\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2\right]^{\frac{1}{2}}$

$$
\begin{aligned}
x &= e^{-\frac{c}{2m}t}\left[A_1' e^{i\omega_d t} + A_2' e^{-i\omega_d t}\right] \\
&= e^{-\frac{c}{2m}t}\left[A_1'(\cos\omega_d t + i\sin\omega_d t) + A_2'(\cos\omega_d t - \sin\omega_d t)\right] \\
&= e^{-\frac{c}{2m}t}\left[(A_1' + A_2')\cos\omega_d t + i(A_1' - A_2')\sin\omega_d t\right]
\end{aligned}
$$

Let,

$$A_1' + A_2' = A\sin\phi$$

$$i(A_1' + A_2') = A\cos\phi$$

$$x = e^{\left(-\frac{c}{2m}t\right)}\left[A\ sin(\omega_d t + \phi)\right]$$

where,

$$\begin{aligned} A^2 &= (A_1' + A_2')^2 - (A_1' - A_2')^2 \\ &= A_1'^2 + A_2'^2 + 2A_1'A_2' - A_1'^2 - A_2'^2 + 2A_1'A_2' \\ &= 4A_1'A_2' \\ A &= 2(A_1'A_2')^{1/2} \end{aligned}$$

,

$$\tan\phi = \frac{A_1' + A_2'}{i(A_1' - A_2')}$$

So, roots of the equation are,
$\lambda_1 = -\frac{c}{2m} + i\omega_d$
$\lambda_2 = -\frac{c}{2m} - i\omega_d$
With a frequency, $\omega_1 = \omega_d = \left[\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2\right]^{\frac{1}{2}}$
The frequency is not same as of the case of undamped case. It is some amount less than the previous value, i.e., $\sqrt{\frac{k}{m}}$. The change is dependent on the damping parameter c and the mass of the system.
Therefore the equation of damped single degree of freedom system is

$$x = e^{\left(-\frac{c}{2m}t\right)}\left[A\ sin(\omega_d t + \phi)\right]$$

ANIMATION UNDER DAMPED

Case III $\left(\frac{c}{2m}\right)^2 = \left(\frac{k}{m}\right)$ – **Critically Damped Case**

If c_c is the coefficient for critical damping.

$c_c^2 = 4km \Rightarrow c_c = 2\sqrt{km}$

$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow c_c = 2m\omega_n \Rightarrow c_c = \frac{2k}{\omega_n}$

Damping ratio $\xi = \frac{c}{c_c} = \frac{Actual\ \ Damping\ \ Coefficient}{Critical\ \ Damping\ \ Coefficient}$

$\frac{c}{2m} = \frac{\xi c_c}{2m}$ substituting the above relation

$$\frac{c}{2m} = \xi\omega_n$$

The equation for free damped single degree freedom system modifies to

$$x = e^{(-\xi\omega_n t)}\left[A\ sin(\omega_d t + \phi)\right]$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

$T_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$,
$\lambda_{1,2} = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n$
This is the case of $\xi < 1$, i.e., All damped free vibrations.
when $\xi = 1$ It is a case of critically damped one.
$\xi > 1$ It is a case of over-damped one.

In case of under-damped or damped case the constants may be found out solving a initial value problem.
Let, at $t = 0$, $x = x_0$, and $\dot{x} = v_0$ and if we substitute it to
$x = e^{(-\xi\omega_n t)}\left[A\ sin(\omega_d t + \phi)\right]$

We get

$$x = e^{(-\xi \omega_n t)} \left[x_0 \; cos \; \omega_d t + \frac{v_0 + x_0 \xi \omega_n}{\omega_d} \; sin \; \omega_d t \right]$$

$$\Rightarrow x(t) = Ce^{(-\xi \omega_n t)} \; cos(\omega_d t - \alpha)$$

where $C = \sqrt{x_0^2 + \left(\frac{v_0 + x_0 \xi \omega_n}{\omega_d} \right)^2}$
and $tan \; \alpha = \frac{v_0 + x_0 \xi \omega_n}{\omega_d x_0}$
ANIMATION 2

In case of Over-Damped System, we get from the general solution

$$x = Ae^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + Be^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}$$

where $A = \frac{\dot{x}_0+\left(\xi+\sqrt{\xi^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\xi^2-1}}$ and $B = \frac{-\dot{x}_0-\left(\xi-\sqrt{\xi^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\xi^2-1}}$

ANIMATION OVER DAMPED

In case of Critically-Damped System, we get from the general solution

$$x = (A + Bt)e^{\omega_n t}$$

With implementation of initial conditions

$$x(t) = \left[x_0 + \{\dot{x}_0 + \omega_n x_0\} t\right] e^{\omega_n t}$$

ANIMATION of CRITICALLY DAMPED SYSTEM