AE31002 Aerospace Structural Dynamics

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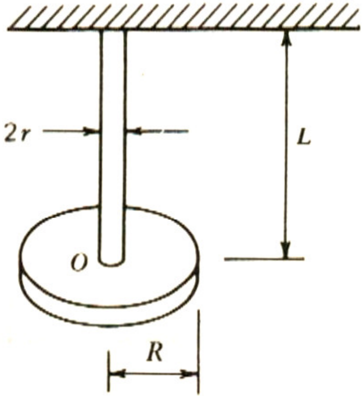
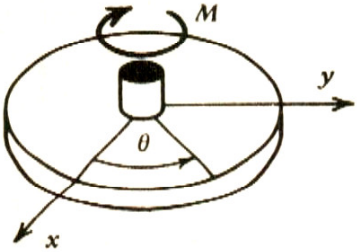
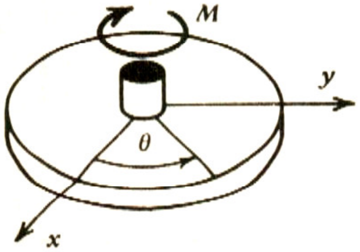
Equation of motion for torsional oscillation of the uniform disk mounted at the end of the shaft.



Assumptions: The shaft is weight less and there is no damping present in the system.
Let us consider θ as any arbitrary position. According to D'Alembert's principle the inertia force involved is $I_0\alpha$, where I_0 is the mass moment of inertia bout the center of gravity and α is the angular acceleration.





The torsional **resisting force** involved is $T = \frac{GJ}{L}\theta$.
The equation of equilibrium becomes

$$M + T = 0$$

$$I_0\ddot{\theta} + \left(\frac{GJ}{L}\right)\theta = 0$$

$$\frac{1}{2}\left(mR^2\right)\ddot{\theta} + \left(\frac{G}{L}\frac{\pi r^4}{2}\right)\theta = 0$$

$$\frac{1}{2}\left(mR^2\right)\ddot{\theta} + \left(\frac{G}{L}\frac{\pi r^4}{2}\right)\theta = 0$$

The above equation may be simplified as

$$I_0\ddot{\theta} + k_\theta\theta = 0$$

Where $I_0 = \frac{1}{2}\left(mR^2\right)$ and $k_\theta = \frac{G}{L}\frac{\pi r^4}{2}$
The natural frequency of the system becomes

$$\omega_n = \sqrt{\frac{k_\theta}{I_0}}$$

With introduction of damping c_θ the equation becomes

$$I_0\ddot{\theta} + c_\theta\dot{\theta} + k_\theta\theta = 0$$

Damping Ratio ξ ?

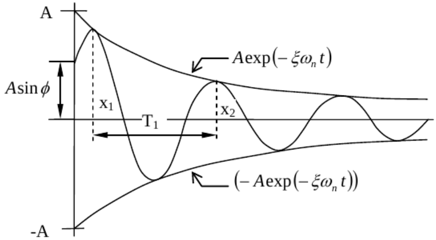
The displacement x of a SDOF system in free vibration is

$$x = e^{(-\xi \omega_n t)} \left[A \, sin(\omega_d t + \phi) \right]$$

where $\omega_d = \omega_n \sqrt{1 - \xi^2}$

ANIMATION

Damping Ratio ξ ?



Let, amplitude of vibration at two consecutive v=0 condition are x_1 and x_2 respectively.
Let us also define the quantity $\delta = \ln \frac{x_1}{x_2}$ as Logarithmic Decrement of the amplitude of vibration in two consecutive peaks.

$$\delta = \ln \frac{A\ e^{-\xi\omega_n t}\ sin(\omega_d t + \phi)}{A\ e^{-\xi\omega_n (t+T)}\ sin\{\omega_d (t+T) + \phi\}}$$

Damping Ratio ξ ?

$$\delta = \ln \frac{A\ e^{-\xi\omega_n t}\ sin(\omega_d t + \phi)}{A\ e^{-\xi\omega_n (t+T)}\ sin(\omega_d t + \phi)}, \quad where T = \frac{2\pi}{\omega_d}$$

$$\delta = \xi\omega_n T = \xi\omega_n \frac{2\pi}{\omega_d} = \xi\omega_n \frac{2\pi}{\omega_n\sqrt{1-\xi^2}} = \xi\frac{2\pi}{\sqrt{1-\xi^2}}$$

Value of ξ is structure is very small, it ranges between 2-20%.
For the highest value of ξ, i.e., $\xi = 0.2$, $\omega_d = 0.98\omega_n$.
So, logarithmic decrement simplifies to $\delta = 2\pi\xi$

$$\xi = \frac{\delta}{2\pi}$$

Damping Ratio ξ ?

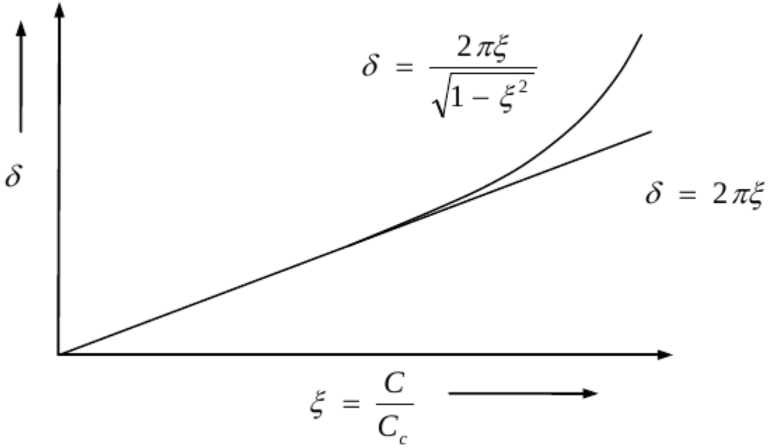
In practice it becomes very difficult to measure the amplitude of two corresponding peaks.

In that case if two peak with j-cycle apart are measured and those are x_1 and x_j, respectively and $t_{j+1} = t_1 + j \times T$

$$\frac{x_1}{x_{j+1}} = \frac{x_1}{x_2}.\frac{x_2}{x_3}.\frac{x_3}{x_4}\dots\frac{x_j}{x_{j+1}} = e^{(\xi\omega_n T)j}$$

$$\delta = \frac{1}{j}\ln\frac{x_1}{x_{j+1}} = \frac{1}{j}\ln\left(e^{(\xi\omega_n T)j}\right) = \xi\omega_n T$$

Damping Ratio ξ ?



Example

A vibrating system consisting of a weight of W=10 lb and a spring with stiffness k=20 lb/in is viciously damped so that the ratio of two consecutive amplitude is 1.00 to 0.85. Determine a) the undamped natural frequency of the damped system, b) the logarithmic decrement, c) the damping ratio, d) the damping coefficient, and e) the damped Natural frequency.

a) The undamped natural frequency of the system in rad/sec is
$\omega = \sqrt{k/m} = \sqrt{(20\ lb/in\ \times\ 386\ in/sec^2\ /10\ lb} = 27.78$ rad/sec
Frequency, in cycles per second, $f = \frac{\omega}{2\pi} = 4.24 cps$

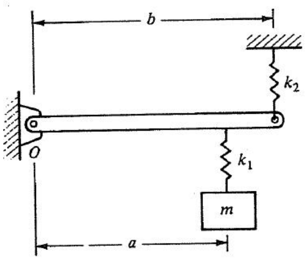
b) The logarithmic decrement is given by
$\delta = \ln \frac{x_1}{x_2} = \ln \frac{1.00}{0.85} = 0.163$

Example

c) The damping ratio is approximately
$\xi \simeq \frac{\delta}{2\pi} = \frac{0.163}{2\pi} = 0.026$

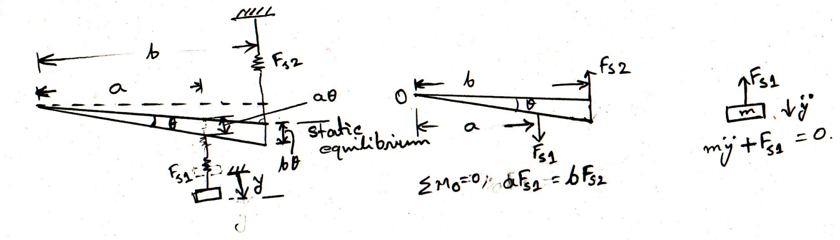
d) The damping coefficient is
$c = \xi \; c_c = \xi \times 2\sqrt{km} = 2 \times 0.026\sqrt{(10 \times 20)/386} = 0.037\frac{lb.sec}{in}$ e)

The natural frequency of the damped system is
$\omega_d = \omega_n\sqrt{1-\xi^2} = 27.78\sqrt{1-0.026^2} = 27.77$



A massless rigid bar is hinged at O, as shown in the figure. Determine the natural frequency of oscillation of the system for the parameters, $k_1 = 2500$ lb/in (4.3782×10^5 N/m), $k_2 = 900$ lb/in (1.5761×10^5 N/m), m = 1 lb$\cdot s^2$/in (175.13 kg), a = 80 in (2.03m), and b= 100 in (2.54 m).

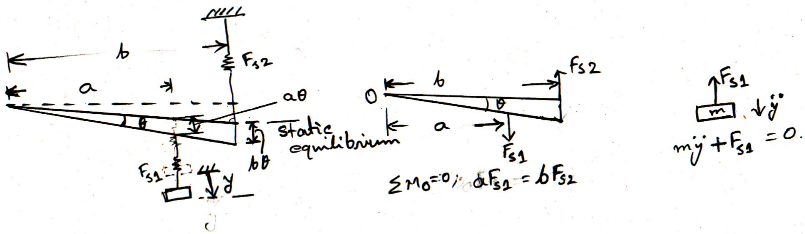
Choose the static equilibrium position of the mass m as the reference for y.



$$\sum F_y = m\ddot{y} = -F_{s1} = -k_1(y - a\theta)$$

$$\sum M_O = bF_{s2} - aF_{s1} = 0, \qquad F_{s2} = k_2 b\theta$$

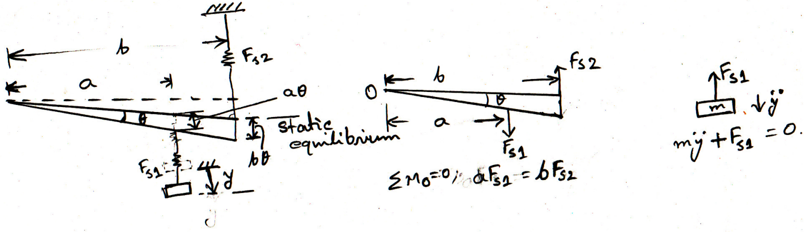
$$k_2 b^2\theta - ak_1(y - a\theta) = 0, \rightarrow \theta = \frac{1}{1 + \frac{k_2}{k_1}\frac{b^2}{a^2}}\frac{y}{a}$$



$$m\ddot{y} + k_1 y - k_1 a \frac{1}{1 + \frac{k_2}{k_1}\frac{b^2}{a^2}} \frac{y}{a} = 0, \quad \rightarrow m\ddot{y} + k_{eq} = 0$$

$$k_{eq} = \frac{k_1}{1 + \frac{k_1}{k_2}\frac{a^2}{b^2}}, \Rightarrow \omega_n = \sqrt{\frac{k_{eq}}{m}}$$

Equivalent spring approach



$$y = y_1 + y_2 \frac{a}{b} = \frac{F_{s1}}{k_1} + \frac{F_{s2}}{k_2}\frac{a}{b}$$

$$\frac{F_{s1}}{k_{eq}} = \frac{F_{s1}}{k_1} + \frac{F_{s2}}{k_2}\frac{a}{b} = \frac{F_{s1}}{k_1} + \frac{F_{s1}}{k_2}\frac{a^2}{b^2}$$

$$k_{eq} = \frac{k_1}{1 + \frac{k_1}{k_2}\frac{a^2}{b^2}}, \Rightarrow \omega_n = \sqrt{\frac{k_{eq}}{m}}$$