AE31002 Aerospace Structural Dynamics
Forced Vibration

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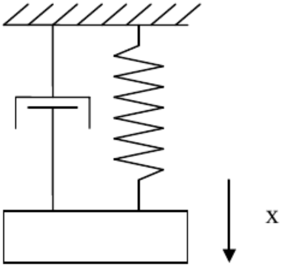
Illustration of the three cases –

Initial condition

$$t = 0\ ;\ x = 0\ ;\ \dot{x} = v_0$$

$$\xi = 2.5, 1, 0.1$$

Find out the response



There are three cases

1) Over damped
2) Critically damped
3) Under damped

Case I $\xi = 2.5$

$$x = A_1 \exp\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t + A_2 \exp\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t$$

$$x = A_1\, e^{-0.209\,\omega_n t} + A_2\, e^{-4.79\,\omega_n t}$$

To find A_1, A_2 use initial condition

$$\Rightarrow A_1 = -A_2 = \frac{v_0}{\omega_n}(0.218)$$

$$x = 0.218\frac{v_0}{\omega_n}\left(e^{-0.209\,\omega_n t} + e^{-4.79\,\omega_n t}\right)$$

Case II

$$\xi = 1$$

$$x = \left(A_1 + A_2 t\right)e^{-\omega_n t}$$

Appling B. C. $\Rightarrow \quad A_1 = 0 \; ; \quad A_2 = v_0$

$$x = V_0 \, t \, e^{\omega_n t}$$

Case III

$$\xi = 0.1$$

$$x = e^{-0.1\omega_n t}\left(A_1 \sin 0.995\omega_n t + B\cos 0.995\omega_n t\right)$$

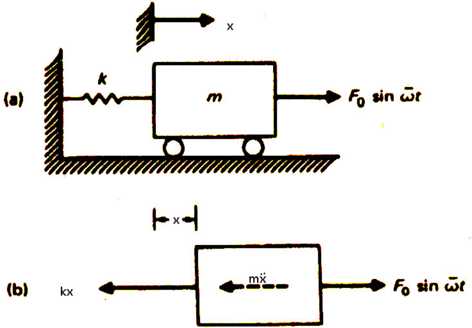
From the initial conditions

$$x = 0 \rightarrow B = 0$$

$$x = v_0 \rightarrow A = 0.995\frac{v_0}{\omega_n}$$

$$x = 0.995\frac{v_0}{\omega_n}e^{-0.1\omega_n t}\sin 0.995\omega_n t$$

UNDAMPED HARMONIC EXCITATION



$$m\ddot{x} + kx = F_0 \ sin \ \bar{\omega}t$$

Where the imposed dynamic force F(t) is assumed as $F_0 \ sin \ \bar{\omega}t$

$$x(t) = x_c(t) + x_p(t)$$

$$x_c(t) = A \ cos \ \omega t + B \ sin \ \omega t$$

The particular solution may be assumed as

$$x_p(t) = X\ sin\ \bar{\omega}t$$

$$\Rightarrow -m\bar{\omega}^2 X + kX = F_0$$

$$\Rightarrow X = \frac{F_0}{k - m\bar{\omega}^2} = \frac{F_0/k}{1 - r^2}$$

Where $r = \frac{\bar{\omega}}{\omega}$ is the frequency ratio and the total response becomes

$$x(t) = A\ cos\ \omega t + B\ sin\ \omega t + \frac{F_0/k}{1 - r^2}\ sin\ \bar{\omega}t$$

If the initial conditions at t=0 are taken as zero (x_0 =0, v_0=0), then, A = 0 and $B = -r\frac{F_0/k}{1-r^2}$ and

$$x(t) = \frac{F_0/k}{1 - r^2}(sin\ \bar{\omega}t - r\ sin\ \omega t)$$

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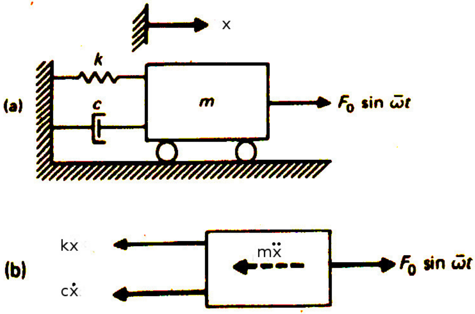
- This terms with ω_d, are known as the **transient response**.
- The terms with forcing frequency $\bar{\omega}$ terms are known as **steady-state response**.

DAMPED HARMONIC EXCITATION



Considering the damping force in the system the equilibrium equation becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \ sin \ \bar{\omega}t$$

Similarly, the complementary solution is

$$x_c(t) = e^{-\xi\omega t}(A \ cos \ \omega_d t + B \ sin \ \omega_d t)$$

The particular solution may be obtained with help of the Euler's equation

$$e^{i\bar{\omega}t} = cos\ \bar{\omega}t + i\ sin\ \bar{\omega}t$$

with the understanding that only the imaginary component of $F_0 e^{i\bar{\omega}t}$, i.e., the force component of the $F_0\ sin\ \bar{\omega}t$ is acting and, consequently, the response will then consist only of the imaginary part of the total solution. Let, $x_p(t) = C\ e^{i\bar{\omega}t}$
$\Rightarrow -m\bar{\omega}^2 C + ic\bar{\omega}C + kC = F_0$
$\Rightarrow C = \frac{F_0}{k - m\bar{\omega}^2 + ic\bar{\omega}}$

$$x_p(t) = \frac{F_0\ e^{i\bar{\omega}t}}{k - m\bar{\omega}^2 + ic\bar{\omega}}$$

By using polar coordinate form

$$\Rightarrow x_p(t) = \frac{F_0\ e^{i\bar{\omega}t}}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}\ e^{i\theta}}$$

$$\Rightarrow x_p(t) = \frac{F_0 \ e^{i(\bar{\omega}t - \theta)}}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}}, \ where \ tan \ \theta = \frac{c\bar{\omega}}{k - m\bar{\omega}^2}$$

The response to the force $F_0 \ sin \ \bar{\omega}t$ (the imaginary component of $F_0 \ e^{i\bar{\omega}t}$) is then the imaginary component of the above equation, i.e.,

$$\Rightarrow x_p(t) = \frac{F_0 \ sin(\bar{\omega}t - \theta)}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}}$$

or

$$\Rightarrow x_p(t) = X \ sin(\bar{\omega}t - \theta)$$

where

$$X = \frac{F_0}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}}$$

is the amplitude of the steady-state motion.

If we consider the static deflection of the spring acted upon by the force F_0 is $x_{st} = F_0/k$

$$\Rightarrow x_p(t) = \frac{x_{st}\ sin(\bar{\omega}t - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

The total response of the system

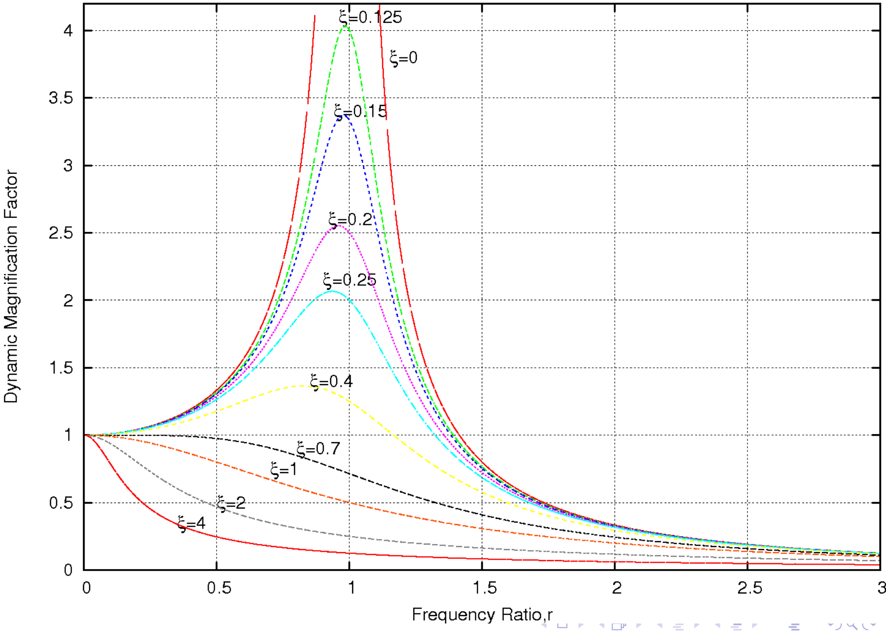
$$x(t) = e^{-\xi\omega t}(A\ cos\ \omega_d t + B\ sin\ \omega_d t) + \frac{x_{st}\ sin(\bar{\omega}t - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

Where, $\xi = c/c_c$ is the damping ration and $r = \bar{\omega}/\omega$ is the frequency ratio.

The terms with ω_d, are known as the **transient response**. The terms with forcing frequency $\bar{\omega}$ are known as **steady-state response**.

The ratio of the steady-state amplitude of $x_p(t)$ to the static deflection x_{st} defined above is known as the **dynamic magnification factor, D**, and is given as

$$D = \frac{X}{x_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$


ξ=0.125
ξ=0
ξ=0.15
ξ=0.2
ξ=0.25
ξ=0.4
ξ=0.7
ξ=1
ξ=2
ξ=4
Dynamic Magnification Factor
Frequency Ratio,r
0
0.5
1
1.5
2
2.5
3
3.5
4

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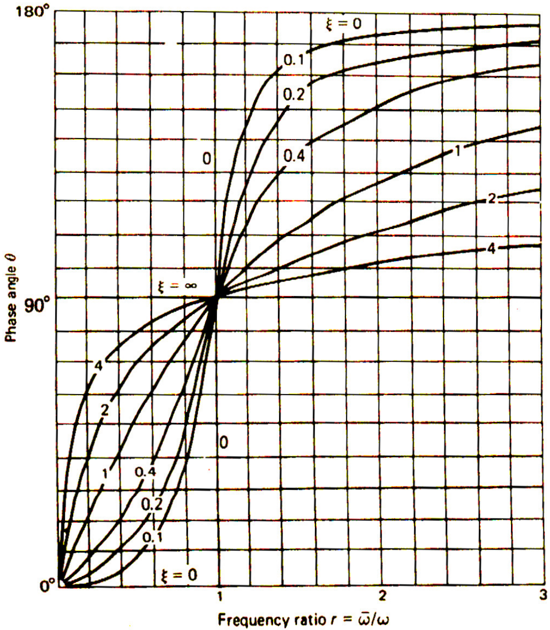
- It is inversely proportional to the damping ratio, i.e., $D(r=1) = \frac{1}{2\xi}$
- D, evaluated at resonance is close to its maximum value $(\omega \neq \omega_d)$. However for moderate damping the difference is negligible.


180°
90°
0°
Phase angle θ
ξ = 0
0.1
0.2
0.4
0
1
2
4
ξ = ∞
0
0.4
0.2
0.1
ξ = 0
1
2
3
Frequency ratio r = ω̄/ω