AE31002 Aerospace Structural Dynamics
Forced Vibration of SDOF system

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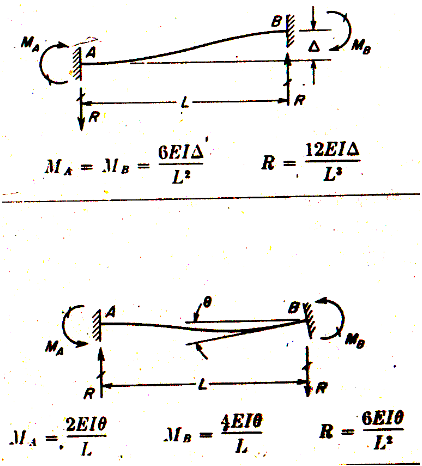
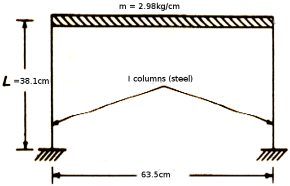


Tutorial Problem 1.1

Here the most important part is the estimation of stiffness of column. In the left hand side figure it appears that for a Δ displacement of any fixed support require $12EI\Delta/L^2$ force. Consequently the stiffness of any one column in this example is



$12EI/L^2$.

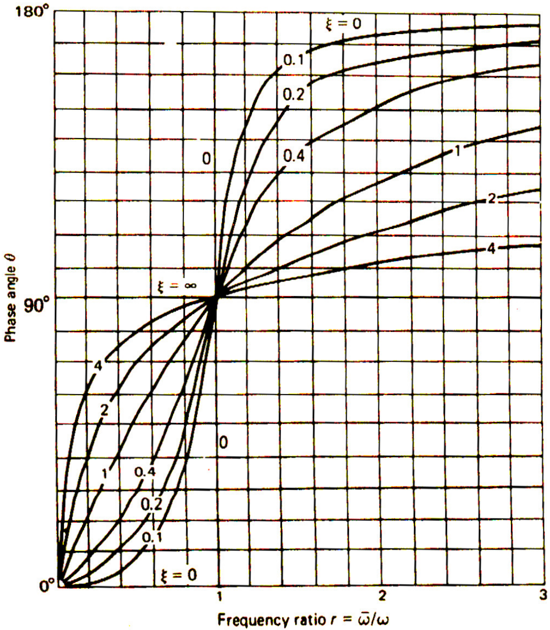
$$\Rightarrow x_p(t) = \frac{F_0 \ e^{i(\bar{\omega} t - \phi)}}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}}, \ where \ tan \ \phi = \frac{c\bar{\omega}}{k - m\bar{\omega}^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}; \ c_c = 2m\omega_n; \ \xi = \frac{c}{c_c}; \ \frac{c\bar{\omega}}{k} = \frac{c}{c_c}\frac{c_c\bar{\omega}}{k} = 2\xi\frac{\bar{\omega}}{\omega_n} = 2\xi r$$

$$x(t) = e^{-\xi\omega t}(A \ cos \ \omega_d t + B \ sin \ \omega_d t) + \frac{x_{st} \ sin(\bar{\omega} t - \phi)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

$$D = \frac{X}{x_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad and \quad tan \ \phi = \frac{2\xi r}{1 - r^2}$$

These indicate that non-dimensional amplitude and the phase angle ϕ are functions of the frequency ratio r and the damping factor ξ

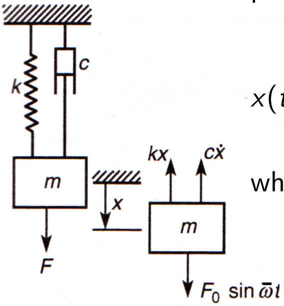


$$tan\ \phi = \frac{c\bar{\omega}}{k - m\bar{\omega}^2}$$

$$tan\ \phi = \frac{2\xi r}{1 - r^2}$$

The total response of the system

$$x(t) = e^{-\xi\omega t}(A\ cos\ \omega_d t + B\ sin\ \omega_d t) + \frac{x_{st}\ sin(\bar{\omega} t - \phi)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$



The steady state response is dependent on the phase angle ϕ

$$x(t) = \frac{F_0\ sin(\bar{\omega} t - \phi)}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}} = X\ sin(\bar{\omega} t - \phi)$$

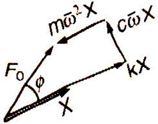
where

$$X = \frac{F_0}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}}$$

is the amplitude of the steady-state motion.

Vector Relationship for Forced Vibration

There are three cases possible: (i) For $r = \bar{\omega}/\omega \ll 1$ Both inertia and damping forces are small, which results in a small phase angle ϕ. The magnitude of the impressed force is then nearly equal to the spring force.



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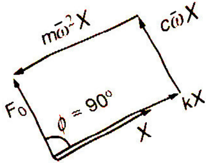
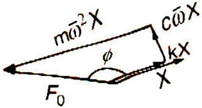
(ii) For $r = \bar{\omega}/\omega = 1$, the phase angle is 90° and the force diagram appears as



. The inertia force, which is now larger, is balanced by the spring force, whereas the impressed force overcomes the damping force.

Vector Relationship for Forced Vibration

(iii) At large value of $r = \bar{\omega}/\omega \gg 1$, ϕ approaches 180° , and the impressed force expanded almost entirely in overcoming the large inertia force as shown in here 

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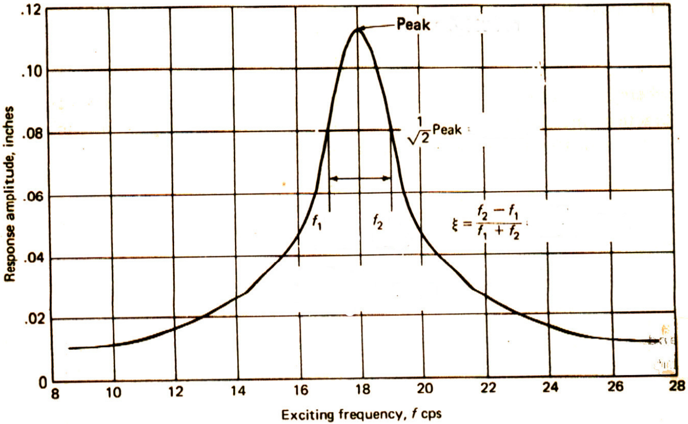
Half Power Method of Damping



In the evaluation of damping, it is convenient to measure the bandwidth at $1/\sqrt{2}$ of the peak amplitude.

Half Power Method of Damping

The frequencies corresponding in this bandwidth f_1 and f_2 are also referred to as half-power points. Now equating the amplitude at first point to $1/\sqrt{2}$ times the resonance amplitude.

$$\frac{x_{st}}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = 1/\sqrt{2} \times D(r=1) = 1/\sqrt{2} \times \frac{x_{st}}{2\xi}$$

$$\Rightarrow r^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1+\xi^2}$$

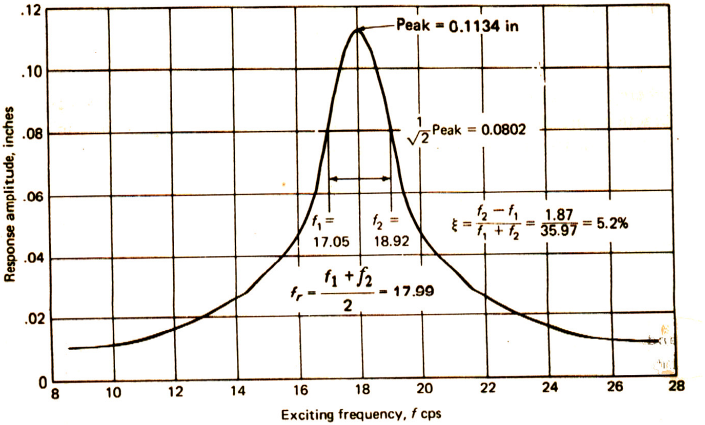
Considering that $\xi \ll 1$, let us neglect ξ^2 terms

$$r^2 \simeq 1 \pm 2\xi \Rightarrow 4\xi = \frac{\bar{\omega_2}^2 - \bar{\omega_1}^2}{\omega_n^2}$$

$$\xi = \frac{1}{2} \times \frac{\bar{\omega_2} - \bar{\omega_1}}{\omega_n} = \frac{f_2 - f_1}{f_2 + f_1}$$

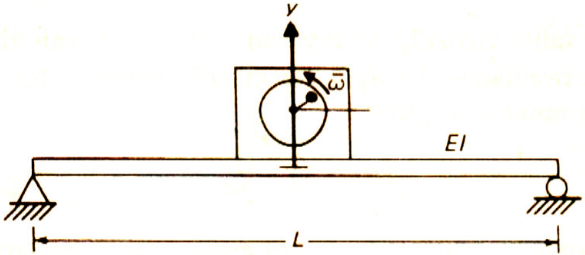
where $r = \bar{\omega}/\omega_n = \bar{f}/f$ and $f \simeq (f_1 + f_2)/2$

Experimental data for the frequency response of a SDOF system are plotted below. Determine the damping ratio of this system.



f_1 and f_2 are determined from the plot and ξ is determined subsequently.

A simple beam supported at its center a machine having a weight $W = 16000$ lb. The beam is made of a standard section having total sectional second moment of area, $I = 128.4$ in^4 with a clear span of 12ft. The motor runs at 300 rpm, and its rotor is out of balance to the extent of $W' = 40$ lb at a radius of $e_0 = 10$ in. What will be the amplitude of the steady-state response if the equivalent viscous damping for the system is assumed 10% of the critical? (Neglect distributed mass of the beam, $E = 30 \times 10^6 lb/in^2$)



The force at the center of the beam necessary to deflect this point an unit is the stiffness in that direction and it is

$$k = \frac{48EI}{L^3} = \frac{48 \times 30 \times 10^6 \times 128.4}{144^3} = 61,920 lb/in$$

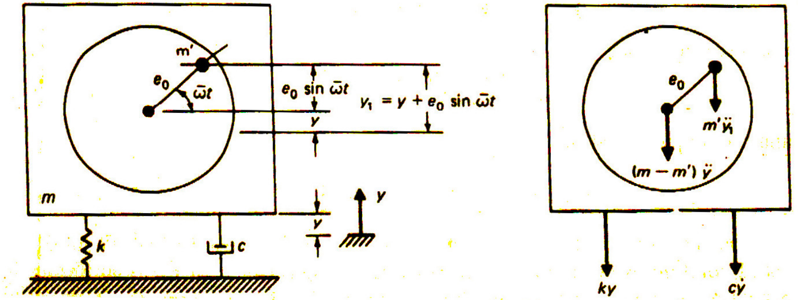
The natural frequency of the system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{61920}{16000/386}} = 38.65 rad/sec$$

$$The\ forcing\ frequency\ \bar{\omega} = \frac{300 \times \pi}{60} = 31.42 rad/sec$$

$$So,\ the\ frequency\ ratio,\ r = \frac{\bar{\omega}}{\omega} = \frac{31.41}{38.65} = 0.813$$

What is the amplitude of the forcing function?



Let m be the total mass of the motor and m' the unbalanced rotating mass. If y is the vertical displacement of the nonrotating mass (m-m') from the equilibrium position, the displacement y_1 of m' is

$$y_1 = y + e_0 \; sin \; \bar{\omega} t$$

The equation of motion from the other diagram

$$(m - m')\ddot{y} + m'\ddot{y}_1 + cy + k\dot{y} = 0$$

By substituting the value of y_1

$$(m - m')\ddot{y} + m'(\ddot{y} - e_0\bar{\omega}^2 sin\ \bar{\omega}t) + c\dot{y} + ky = 0$$

$$\Rightarrow m\ddot{y} + c\dot{y} + ky = m' e_0\bar{\omega}^2 sin\ \bar{\omega}t$$

The above equation may be assumed as a forced SDOF system with force magnitude $F_0 = m' e_0\bar{\omega}^2$

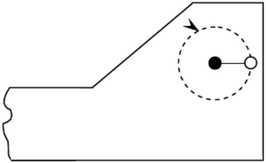
$$F_0 = 40 \times 10 \times 31.41^2/386 = 1022lb$$

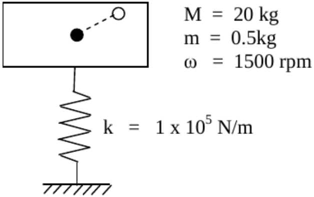
The amplitude of the steady-state resulting motion is

$$Y = \frac{F_0/k}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} = \frac{1022/61,920}{\sqrt{(1 - 0.813^2)^2 + (2 \times 0.813 \times 0.1)^2}}$$

$$Y = 0.044in$$

A tail rotor of a helicopter is supported on a tail section having stiffness, k = 1 X 105 N/m in the vertical direction and effective mass of 20 kg. Calculate the deflection of the tail section when the tail rotor rotates at 1500 rpm with an eccentric mass of 0.5 kg attached to a blade of arm length 15 cm.





$$X = \frac{me\omega^2}{\sqrt{\left(k-\omega^2 M\right)^2 + (C\omega)^2}}$$

$$= \frac{me\omega^2/k}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$= \frac{0.15\times 0.5\times 157.08^2}{(1-4.9348)\times 10^5}$$

$$= 4.703\times 10^{-3}\ m$$

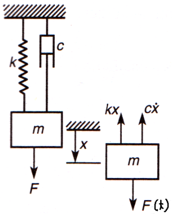
$$C = 0$$

$$e = 15cm = 0.15m$$

$$\omega = \frac{1500\times 2\pi}{60} = 157.08\ rad/s$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\times 10^5}{20}} = 70.7\ rad/s$$

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{157.08}{70.7}\right)^2 = 4.9348$$



A mass damper spring system (as shown) has been obwerved to achieve a peak magnification factor D = 5 at the driving frequency $\bar{\omega} = 10$ rad/s. It is required to determine 1) the damping factor, 2) the driving frequencies corresponding to the half-power points and 3) the band width of the system.

As we have observed earlier dynamic magnification factor may be defined as a function of forcing frequency $\bar{\omega}$

$$D(\bar{\omega}) = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = \frac{1}{\sqrt{\left(1 - \left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\xi \frac{\bar{\omega}}{\omega_n}\right)^2}}$$

To find out the maximum value of D($\bar{\omega}$) we differentiate w.r.t $\bar{\omega}$ and ser to zero and it leads to

$$\bar{\omega} = \omega_n \left(1 - 2\xi^2\right)^{1/2}$$

Now substituting the above value for resonance or

$$5 = \frac{1}{\sqrt{\left(1-\left(1-2\xi^2\right)\right)^2 + 4\xi^2\left(1-2\xi^2\right)}} \Rightarrow \xi = 0.1005$$

Half power point

$$\frac{5}{\sqrt{2}} = \frac{1}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\bar{\omega}}{\omega_n}\right)^2}}$$

$$\left(\frac{\bar{\omega}}{\omega_n}\right)^4 - 1.9596\left(\frac{\bar{\omega}}{\omega_n}\right)^2 + 0.92 = 0$$

$$\omega_1 = 0.8831\omega_n \quad and \quad \omega_2 = 1.0862\omega_n$$

Now from the relationship $\bar{\omega} = \omega_n\left(1-2\xi^2\right)^{1/2}$, $\omega_n = 10.1010$ rad/s