AE31002 Aerospace Structural Dynamics
Support Motion

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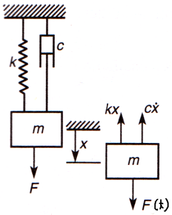


A mass damper spring system (as shown) has been observed to achieve a peak magnification factor D = 5 at the driving frequency $\bar{\omega} = 10$ rad/s. It is required to determine 1) the damping factor, 2) the driving frequencies corresponding to the half-power points and 3) the band width of the system.

As we have observed earlier dynamic magnification factor may be defined as a function of forcing frequency $\bar{\omega}$

$$D(\bar{\omega}) = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} = \frac{1}{\sqrt{\left(1 - \left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\bar{\omega}}{\omega_n}\right)^2}}$$

To find out the maximum value of D($\bar{\omega}$) we differentiate w.r.t $\bar{\omega}$ and ser to zero and it leads to

$$\bar{\omega} = \omega_n \left(1 - 2\xi^2\right)^{1/2}$$

Now substituting the above value for resonance or

$$5 = \frac{1}{\sqrt{(1-(1-2\xi^2))^2 + 4\xi^2(1-2\xi^2)}} \Rightarrow \xi = 0.1005$$

Half power point

$$\frac{5}{\sqrt{2}} = \frac{1}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\bar{\omega}}{\omega_n}\right)^2}}$$

$$\left(\frac{\bar{\omega}}{\omega_n}\right)^4 - 1.9596\left(\frac{\bar{\omega}}{\omega_n}\right)^2 + 0.92 = 0$$

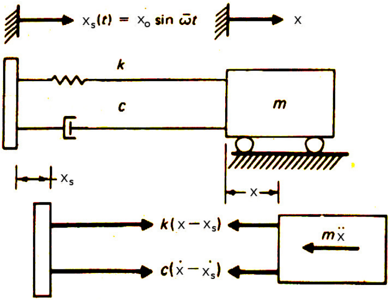
$$\omega_1 = 0.8831\omega_n \quad and \quad \omega_2 = 1.0862\omega_n$$

Now from the relationship $\bar{\omega} = \omega_n\left(1-2\xi^2\right)^{1/2}$, $\omega_n = 10.101$ rad/s
The bandwidth of the system is $\omega_2 - \omega_1 = 2.0515$ rad/s

Support Motion

Let us assume a support motion

$$x_s(t) = x_0 \ sin \ \bar{\omega} t$$



$$m\ddot{x} + c(\dot{x} - \dot{x}_s) + k(x - x_s) = 0$$

$$m\ddot{x} + c\dot{x} + kx = kx_0 \ sin \ \bar{\omega} t + c\bar{\omega} x_0 \ cos \ \bar{\omega} t$$

Now RHS may be modified as

$$m\ddot{x} + c\dot{x} + kx = F_0 \; sin \; (\bar{\omega}t + \beta)$$

Where, $F_0 = x_0\sqrt{k^2 + (c\bar{\omega})^2} = x_0 k\sqrt{(1 + (2r\xi)^2}$
and $tan \; \beta = \frac{c\bar{\omega}}{k} = 2r\xi$
We know the steady state solution for this type of forcing function in case of SDOF system, i.e.,

$$x(t) = \frac{F_0/k \quad sin \; (\bar{\omega}t + \beta - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

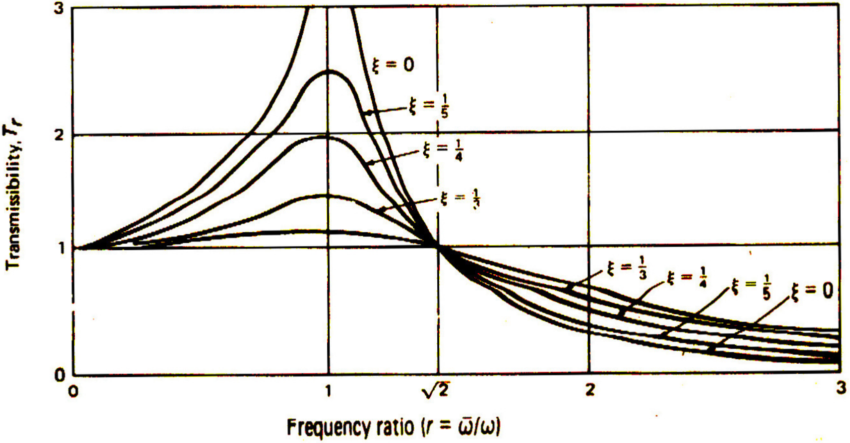
substitution of F_0 gives,

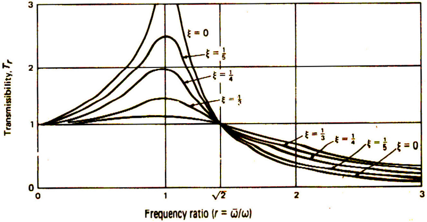
$$\frac{x(t)}{x_0} = \frac{\sqrt{(1 + (2r\xi)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} sin \; (\bar{\omega}t + \beta - \theta)$$

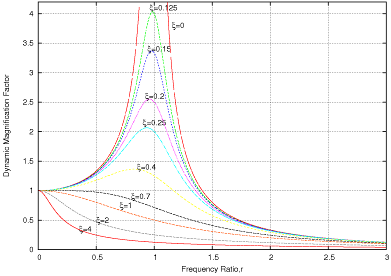
The ratio of the amplitudes of the vibrating mass to the support is known as the **transmissibility**, T_r

$$T_r = \frac{X}{x_0} = \sqrt{\frac{(1 + (2r\xi)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$

relative transmission of the support motion to the oscillating body. This is a measure for the degree of isolation required for important instruments. This is a function of frequency ratio r and damping ration ξ.

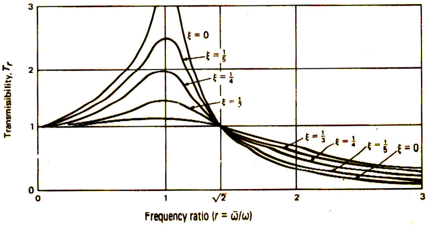

Transmissibility, T_r
3
2
1
0
$\xi = 0$
$\xi = \frac{1}{5}$
$\xi = \frac{1}{4}$
$\xi = \frac{1}{3}$
$\xi = \frac{1}{3}$
$\xi = \frac{1}{4}$
$\xi = \frac{1}{5}$
$\xi = 0$
0
1
$\sqrt{2}$
2
3
Frequency ratio ($r = \bar{\omega}/\omega$)

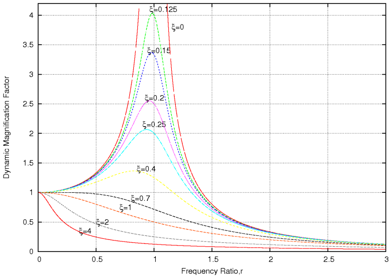




Observation

- It is similar to the frequency response of the damped oscillator.





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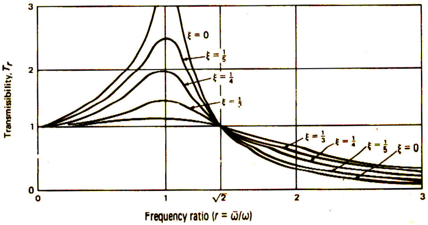
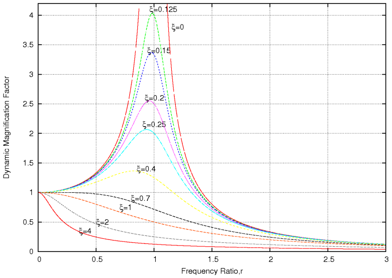
- Here all the curves pass through the point at $r=\sqrt{2}$, in contrast to the previous one.





Observation

- It is similar to the frequency response of the damped oscillator.
- Here all the curves pass through the point at r=$\sqrt{2}$, in contrast to the previous one.
- Damping tends to reduce the effectiveness of vibration isolation for frequencies greater than r=$\sqrt{2}$.

Alternatively if we look for the response in terms of the relative motion between the base and the oscillator, i.e., $u = x - x_s$ the equation $m\ddot{x} + c(\dot{x} - \dot{x_s}) + k(x - x_s) = 0$ changes to

$$m\ddot{u} + c\dot{u} + ku = F_{eff}(t)$$

where $F_{eff}(t) = -m\ddot{x_s}$ is the effective force on the oscillator and displacement is u. Now with the same support excitation, $x_s(t) = x_0 \; sin \; \bar{\omega}t$, we get

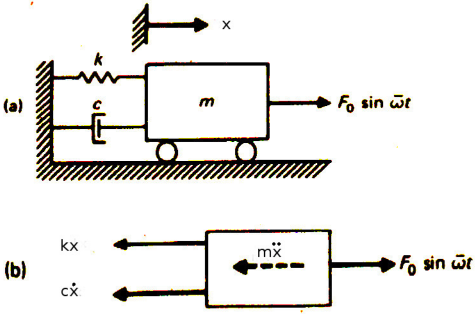
$$m\ddot{u} + c\dot{u} + ku = mx_0\bar{\omega}^2 \; sin \; \bar{\omega}t$$

and the steady state response becomes

$$u(t) = \frac{mx_0\bar{\omega}^2/k \; sin \; (\bar{\omega}t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

$$\frac{u(t)}{x_0} = \frac{r^2 \; sin \; (\bar{\omega}t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \; where \; \frac{\bar{\omega}^2}{k/m} = \frac{\bar{\omega}^2}{\omega^2} = r^2$$

FORCE TRANSMITTED TO THE SUPPORT



General solution, we have found out as

$$m\ddot{x} + c\dot{x} + kx = F_0 \ sin \ \bar{\omega}t$$

$$for \ a \ steady - state \ solution \ x(t) = X \ sin \ (\bar{\omega}t - \theta)$$

$$X = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}, \quad tan \ \theta = \frac{2\xi r}{1-r^2}$$

There are two components of the force transmitted to the support, one is through spring, kx, and through damping, c$\dot{x}$. Total force transmitted is $F_T = kx + c\dot{x}$.

$$F_T = X\{k\ sin\ (\bar{\omega}t - \theta) + c\bar{\omega}\ cos\ (\bar{\omega}t - \theta)\}$$

$$\Rightarrow F_T = X\sqrt{k^2 + c^2\bar{\omega}^2}\ sin\ \{\bar{\omega}t - (\theta - \beta)\}, \quad tan\ \beta = \frac{c\bar{\omega}}{k} = 2\xi r$$

Maximum force transmitted to the support is A_T (amplitude of F_T), i.e.,

$$A_T = F_0\sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$

The transmissibility T_r

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{(1 + (2r\xi)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$

Observations

- It is interesting to note that $T_r = \frac{X}{x_0} = \frac{A_T}{F_0} = \sqrt{\frac{(1+(2r\xi)^2}{(1-r^2)^2+(2\xi r)^2}}$ or

Observations

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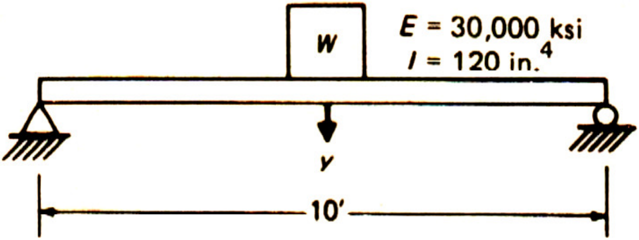
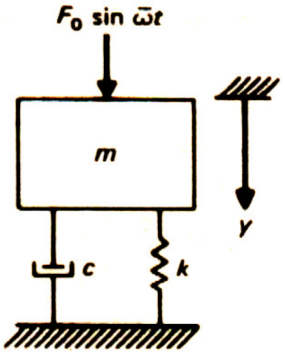
– T_r w.r.t. displacement amplitude from support to the oscillator is same to the T_r w.r.t. force amplitude from oscillator to the support is same.

A machine of weight W=3860 lb is mounted on a simple supported steel beam as shown. A piston that moves up and down in the machine produce a harmonic force of magnitude F_0= 7000 lb and frequency $\bar{\omega} = 60$ rad/sec. Neglecting the weight of the beam and assuming 10% of the critical damping, determine a) the amplitude of the motion of the machine, b) the force transmitted to the beam supports, and c) the corresponding phase angle.





$$k = \frac{48EI}{L^3} = 10^5 \ lb/in$$

$$\omega = \sqrt{\frac{k}{m}} = 100 \ rad/s$$

$$\xi = 0.1$$

$$r = \frac{\bar{\omega}}{\omega} = 0.6$$

$$y_{st} = \frac{F_0}{k} = 0.07 \ in$$

$$a) \ Amplitude \ of \ motion \ Y = \frac{y_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$With \ a \ phase \ angle \ \theta = tan^{-1} \frac{2\xi r}{1-r^2} = 10.6^\circ$$

$$b)\ Transmissibility\ is\ T_r = \frac{A_T}{F_0} = \sqrt{\frac{(1+(2r\xi)^2}{(1-r^2)^2+(2\xi r)^2}} = 1.547$$

Amplitude of force transmitted to the foundation
$A_T = F_0 \times T_r = 10,827\ lb$

c) Corresponding phase angle

$$\phi = \theta - \beta = tan^{-1}\left(\frac{tan\ \theta -\ \ tan\ \beta}{1 + tan\ \theta\ \ tan\ \beta}\right) = \frac{2\xi r^3}{1 - r^2 + (2\xi r)^2} = 3.78^\circ$$