AE31002 Aerospace Structural Dynamics
Forced Response for undamped SDOF system

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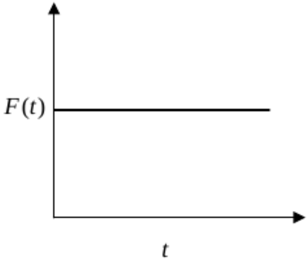
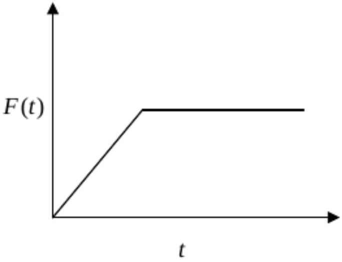
February, 2014

Transient Response

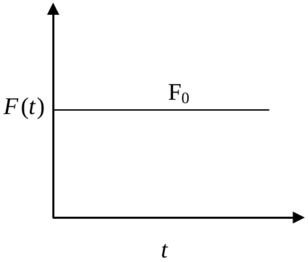
Let us try to find out the transient response of a SDOF system due to the loads shown below.



Suddenly applied step load



Gradually applied step load



Eq. of motion

$$m\ddot{x} + kx = F_0$$
$$\Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F_0}{m}$$
$$\Rightarrow \ddot{x} + \omega_n^2 x = \frac{F_0}{m}$$

The complementary part of the solution is
$x_c = A\ sin\ \omega t + B\ cos\ \omega t$ and the particular integral may be

$$\left(D^2 + \omega^2\right)x = \frac{F_0}{m}$$

$$x = \frac{F_0}{m}\cdot\frac{1}{\left(D^2 + \omega^2\right)} = \frac{F_0}{\omega^2 m}\left(1 + \frac{D^2}{\omega^2}\right)^{-1}$$
$$= \frac{F_0}{\omega^2 m}\left(1 - \frac{D^2}{\omega^2} + \ldots\ldots\right)$$
$$= \frac{F_0}{\omega^2 m} = \frac{F_0}{k}$$

$$\therefore \text{ Complete solution}; \quad x = A\sin\omega t + B\cos\omega t + \frac{F_0}{k}$$

$$\dot{x} = A\omega\cos\omega t - B\omega\sin\omega t$$

B.C. at $t = 0 \quad x = 0 \quad \dot{x} = 0$

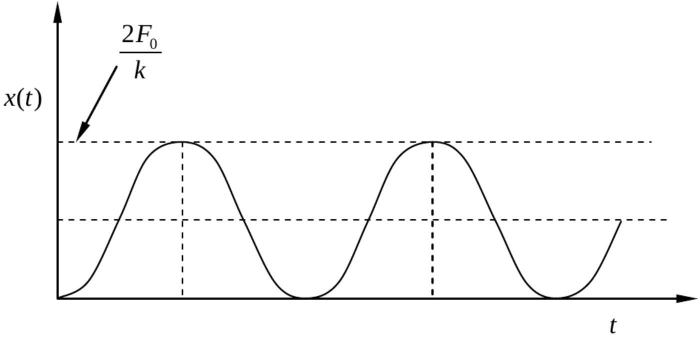
$$0 = A.0 + B.1 + \frac{F_0}{k} \quad \Rightarrow \quad B = -\frac{F_0}{k}$$

$$0 = A\omega.1 - B.0 \quad \Rightarrow \quad A = 0$$

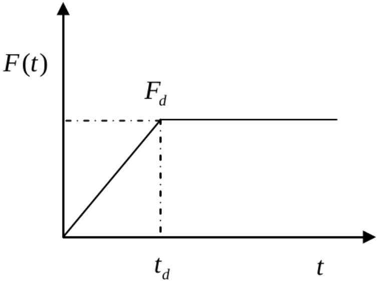
$$\therefore \quad x = -\frac{F_0}{k}\cos\omega t + \frac{F_0}{k} = \frac{F_0}{k}(1 - \cos\omega t) = x_{st}(1 - \cos\omega t)$$

$$\frac{x}{x_{st}} = \text{dynamic amplification} = (1 - \cos\omega t)$$

$$x_{max} = \frac{2F_0}{k} \quad \text{at} \quad \omega t = \pi$$



Response of SDOF System



$$t \le t_d \quad F(t) = F_0 \frac{t}{t_d}$$

$$t \ge t_d \quad F(t) = F_0$$

Eq. of motion

$$m\ddot{x} + kx = F_0 \frac{t}{t_d}$$

$$\Rightarrow \ddot{x} + \omega_n^2 x = \frac{F_0}{m} \cdot \frac{t}{t_d}$$

C. F.

$$x = A\sin\omega t + B\cos\omega t$$

P. I.

$$x = \frac{F_0}{m} \cdot \frac{1}{D^2 + \omega_n^2} \cdot \frac{t}{t_d}$$

$$= \frac{F_0}{m\omega_n^2}\left(1 + \frac{D^2}{\omega_n^2}\right)^{-1} \frac{t}{t_d}$$

$$= \frac{F_0}{k} \cdot \frac{t}{t_d}$$

Complete Solution :

$$x = A\sin\omega t + B\cos\omega t + \frac{F_0}{k}\cdot\frac{t}{t_d}$$

B.C. $t = 0 \quad x = 0 \quad \dot{x}(0) = 0 \quad B = 0$

$$\dot{x} = A\omega\cos\omega t - B\sin\omega t + \frac{F_0}{k}\cdot\frac{1}{t_d}$$

$$\dot{x}(0) = 0 = A\omega\cdot 1 - B\omega\cdot 0 + \frac{F_0}{k}\cdot\frac{1}{t_d}$$

$$\Rightarrow A = -\frac{F_0}{k\omega}\cdot\frac{1}{t_d}$$

$$\therefore \quad x(t) = -\frac{F_0}{k\omega t_d}\cdot\frac{1}{t_d}\sin\omega t + \frac{F_0}{k}\cdot\frac{t}{t_d}$$

$$= \frac{F_0}{k\omega t_d}\left[\omega t - \sin\omega t\right] = \frac{x_{st}}{\omega t_d}\left[\omega t - \sin\omega t\right]$$

$$\frac{x}{x_{st}} = \text{Dynamic Magnification Factor } \frac{1}{\omega t_d}\left[\omega t - \sin \omega t\right]$$

$$\dot{x} = \frac{F_0}{k\,\omega t_d}\left[\omega - \omega \cos \omega t\right] = \frac{F_0}{k\,t_d}\left(1 - \cos \omega t\right)$$

at $t = t_d$

$$x(t) = \frac{F_0}{k\,\omega t_d}\left[\omega t_d - \sin \omega t_d\right]$$

$$\dot{x}(t) = \frac{F_0}{k\,t_d}\left[1 - \cos \omega t_d\right]$$

when $t > t_d$ $\quad m\ddot{x} + kx = F_0 \rightarrow$ earlier case

$$x = A \sin \omega t + B \cos \omega t + \frac{F_0}{k}$$

$$\dot{x} = A\omega \cos \omega t - B\omega \sin \omega t$$

at $t = t_d$ initial conditions

Substititing initial conditions and solve for A & B to get the complete solution.

Initial conditions are

$$x(t_d) = A \sin \omega t_d + B \cos \omega t_d + \frac{F_0}{k} = \frac{F_0}{k\,\omega t_d}\left(\omega t_d - \sin \omega t_d\right) \ldots\ldots. (1)$$

and the velocity becomes

$$\dot{x}(t_d) = A\omega\cos\omega t_d - B\omega\sin\omega t_d = \frac{F_0}{k\,t_d}(1 - \cos\omega t_d) \;.........(2)$$

$$(2) \rightarrow \; A\cos\omega t_d - B\sin\omega t_d = \frac{F_0}{k\omega t_d}(1 - \cos\omega t_d)$$

$$(1) \times \sin\omega t_d \Rightarrow A\sin^2\omega t_d + B\sin\omega t_d\cos\omega t_d = \frac{F_0\sin\omega t_d}{k\omega t_d}(\omega t_d - \sin\omega t_d) - \frac{F_0\sin\omega t_d}{k}$$

$$(2) \times \cos\omega t_d \Rightarrow A\cos^2\omega t_d - B\sin\omega t_d\cos\omega t_d = \frac{F_0\cos\omega t_d}{k\omega t_d}(1 - \cos\omega t_d)$$

Adding above two equation we get

$$A(\sin^2\omega t_d + \cos^2\omega t_d) = \frac{F_0}{k\omega t_d}(\omega t_d\sin\omega t_d - \sin^2\omega t_d + \cos\omega t_d - \cos^2\omega t_d) - \frac{F_0}{k}\sin\omega t_d$$

$$\Rightarrow A = \frac{F_0}{k\,\omega t_d}(\cos\omega t_d + \omega t_d\sin\omega t_d - 1) - \frac{F_0}{k}\sin\omega t_d$$

Similarly, multiplying equation (1) by $\cos\omega t_d$ & (2) by $\sin\omega t_d$ we get,

$$1)\times\cos\omega t_d \Rightarrow A\sin\omega t_d\cos\omega t_d + B\cos^2\omega t_d = \frac{F_0\cos\omega t_d}{k\omega t_d}(\omega t_d - \sin\omega t_d) - \frac{F_0\cos\omega t_d}{k}$$

$$(2)\times\sin\omega t_d \Rightarrow A\cos\omega t_d\sin\omega t_d - B\sin^2\omega t_d = \frac{F_0\sin\omega t_d}{k t_d}(1 - \cos\omega t_d)$$

substracting (2) from (1) we get,

$$B(\sin^2\omega t_d + \cos^2\omega t_d)$$

$$= \frac{F_0}{k\omega t_d}(\omega t_d\cos\omega t_d - \sin\omega t_d\cos\omega t_d - \sin\omega t_d + \sin\omega t_d\cos\omega t_d) - \frac{F_0}{k}\cos\omega t$$

$$\Rightarrow B = \frac{F_0}{k\omega t_d}(\omega t_d\cos\omega t_d - \sin\omega t_d) - \frac{F_0}{k}\cos\omega t$$

$\therefore$ Response at any time $t > t_d$ we get ,

$$x(t) = \left\{\frac{F_0}{k\,\omega t_d}(\cos\omega t_d + \omega t_d\sin\omega t_d - 1) - \frac{F_0}{k}\sin\omega t_d\right\}\sin\omega t$$

$$+ \left\{\frac{F_0}{k\omega t_d}(\omega t_d\cos\omega t_d - \sin\omega t_d) - \frac{F_0}{k}\cos\omega t\right\}\cos\omega t + \frac{F_0}{k}$$