AE31002 Aerospace Structural Dynamics
Forced Response for SDOF system

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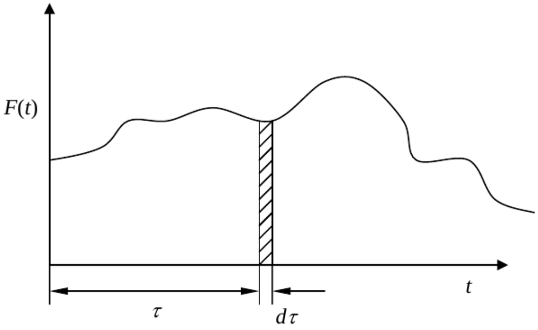
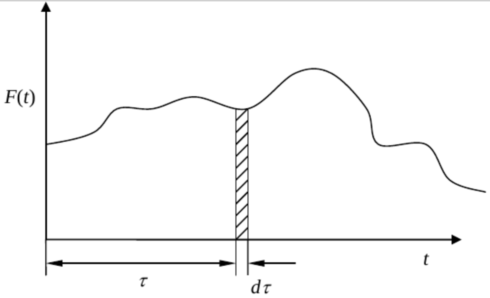
Response of single dergree freedom with general type loading function (Forcing) – Duhamal's Integral, Convolution/Superposition Integral

The dynamic loading resulting from blast, gust of wind or seismic forces is generally not harmonic. In such cases the equation of motions has to be solved numerically solution in exact form can be obtain only for some idealized loading conditions.





Consider an undamped, SDF system subject to a disturbing force $F(t)$ which is a function of time. The solution of problem can be obtained by the superposition of individual results. We shall first consider the effect of a single response subjected to a force $F(\tau)$. From impulse momentum equation i.e. $m\dot{x} = F(\tau)d\tau$, the impulse acting on the mass will result in a sudden change in its velocity equal to $\dot{x} = \frac{F(\tau)d\tau}{m}$ without an appreciable change in displacement.

Under free vibration we found that undamped spring-mass system with initial conditions $x(0)$ and $\dot{x}(0)$:

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

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The displacement at any time t with initial velocity can be written as

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin \omega_n t = \frac{F(\tau)\sin \omega_n t}{m\,\omega_n} d\tau$$

Having obtained the displacement under the effect of an impulse load, we may consider the displacement after application of the impulse given by the above equation except the increase of t we must put $(t-\tau)$ then

$$x(t) = \frac{F(\tau)\sin\omega_n(t-\tau)d\tau}{m\omega_n}$$

$\therefore$ Total displacement due to the arbitrary load applied to the SDF can be written as

$$x(t) = \int_0^t \frac{F(\tau)}{m\omega_n}\sin\omega_n(t-\tau)d\tau$$

Duhamal's integral
Convolution integral
Green's Function

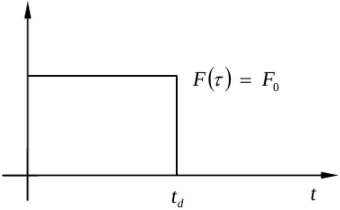
When damping is present we can start with the free vibration equation, with $x(0) = 0$ and displacement with single impulse:

$$x(t) = \frac{\dot{x}(0)\, e^{-\zeta \omega_n t}}{\omega_n \sqrt{1-\zeta^2}} \sin \sqrt{1-\zeta^2}\, \omega_n t$$

Total displacement if damping is considered

$$x(t) = \frac{1}{m\omega_n \sqrt{1-\zeta^2}} \int_0^t e^{-\zeta \omega_n (t-\tau)} F(\tau) \sin \omega_n \sqrt{1-\zeta^2}\,(t-\tau)\, d\tau$$

The above expression can be used directly for computing the response of SDF system of any forcing function provided the integral is to be evaluated.



Rectangular pulse

$t > t_d$ free vibration part

$t < t_d$ forced vibtation part

$$
\begin{aligned}
x(t) &= \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)d\tau \\
&= \frac{1}{m\omega_n}\int_0^t F_0\sin\omega_n(t-\tau)d\tau \\
&= \frac{1}{m\omega_n}\left[+\frac{F_0}{\omega_n}\cos\omega_n(t-\tau)\right]_0^t \\
&= \frac{1}{m{\omega_n}^2}\left[1-\cos\omega_n t\right] \\
&= \frac{F_0}{k}\left(1-\cos\omega_n t\right)
\end{aligned}
$$

For $t > t_d$ the free vibration response can be obtained after substituting velocity displacement at $t = t_d$

$$x(t) = x_{td} \cos(t - t_d)\omega_n + \frac{\dot{x}_{td}}{\omega_n} \sin \omega_n (t - t_d)$$

$t \le t_d$

$$x(t) = \frac{F_0}{k}(1 - \cos \omega_n t)$$

$$\dot{x}(t) = \frac{F_0}{k} \omega_n \sin \omega_n t$$

$t = t_d$

$$x(t_d) = \frac{F_0}{k}(1 - \cos \omega_n t_d)$$

$$\dot{x}(t_d) = \frac{F_0}{k} \omega_n \sin \omega_n t_d$$

$t > t_d$ the response can be written as

$$\begin{aligned} x(t) &= x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \\ &= \frac{F_0}{k}(1 - \cos \omega_n t_d) \cos \omega_n (t - t_d) + \frac{F_0}{k} \sin \omega_n t_d \sin \omega_n (t - t_d) \\ &= \frac{F_0}{k}\left[\cos \omega_n (t - t_d) - \cos \omega_n t\right] \rightarrow \text{simplifing} \end{aligned}$$

$$\text{Dynamic Load Factor (DLF)} = \frac{\text{Displacement at any time } t}{\text{Static displacement}\left(x_{st} = \dfrac{F_0}{k}\right)}$$

$$= 1 - \cos\omega_n t \qquad t \leq t_d$$

$$= \cos\omega_n (t - t_d) - \cos\omega_n t_d \qquad t > t_d$$