AE31002 Aerospace Structural Dynamics
Laplace Transformation in Vibration Analysis

Anup Ghosh

Advantages

- It can handle discontinuous functions.
- It takes into account initial conditions automatically.

Laplace transformation of x(t), is written symbolically as $\bar{x}(s) = \mathcal{L}x(t)$ is defined by definite integral

$$\bar{x}(s) = \mathcal{L}x(t) = \int_0^\infty e^{-st}x(t)\ dt$$

where s is in general a complex quantity (s = x + iy)referred to as a subsidiary variable. The function e^{-st} is known as the kernel of the transformation.

For the SDOF system we need

$$\mathcal{L}\frac{dx(t)}{dt} = \int_0^\infty e^{-st}\ \frac{dx(t)}{dt}\ dt = s\bar{x}(s) - x(0)$$

and

$$\mathcal{L}\frac{d^2x(t)}{dt^2} = \int_0^\infty e^{-st} \ \frac{d^2x(t)}{dt^2} \ dt = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$$

Laplace transformation of the excitation function is

$$\bar{F}(s) = \mathcal{L}F(t) = \int_0^\infty e^{-st} \ F(t) \ dt \tag{1}$$

Applying to the damped SDOF system and rearranging

$$(ms^2 + cs + k)\bar{x}(s) = \bar{F}(s) + m\dot{x}(0) + (ms + c)x(0)$$

Now ignoring the homogeneous solution i.e., letting $x(0) = \dot{x}(0) = 0$, ratio of transformed excitation to transformed response (generalised impedance of the system)

$$\bar{Z}(s) = \frac{\bar{F}(s)}{\bar{x}(s)} = ms^2 + cs + k$$

Similarly the **transfer function** or the system function $\bar{G}(s)$ is expressed as

$$\bar{G}(s) = \frac{\bar{x}(s)}{\bar{F}(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{m(s^2 + 2\xi\omega_n s + \omega_n^2)} \tag{2}$$

The above equation may be written as

$$\bar{x}(s) = \bar{G}(s) \;\; \bar{F}(s) \tag{3}$$

so that the **transfer function can be regarded as an algebraic operator that operates on the transformed excitation** to yield the transformed response.

Steps to follow are, to find out the equations as numbered (1) $\rightarrow$ (2) $\rightarrow$ (3). After finding out the value of $\bar{x}(s)$ an inverse Laplace transformation is required to find out the response in real time t, i.e.,

$$x(t) = \mathcal{L}^{-1}\bar{x}(s) = \mathcal{L}^{-1}\bar{G}(s) \;\; \bar{F}(s)$$

Find out the response of a damped SDOF system under a unit impulse load.

Laplace transformation of unit impulse is

$$\bar{F}(s) = \mathcal{L}F(t) = 1$$

Laplace transformed response, $\bar{x}(s)$, in terms of transfer function $\bar{G}(s)$

$$\bar{x}(s) = \bar{G}(s) = \frac{1}{m(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$\Rightarrow \bar{x}(s) = \frac{1}{2i\omega_d m}\left(\frac{1}{s + \xi\omega_n - i\omega_d} - \frac{1}{s + \xi\omega_n + i\omega_d}\right)$$

Response after the inverse Laplace transformation

$$\Rightarrow x(t) = \mathcal{L}^{-1}\bar{x}(s) = \frac{1}{2i\omega_d m}\left[e^{-(\xi\omega_n - i\omega_d)t} - e^{-(\xi\omega_n + i\omega_d)t}\right]$$

$$\Rightarrow x(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} sin\ \omega_d t$$

Find out the response of a damped SDOF system under a suddenly applied load of unit magnitude (step function with F_0=1).

$$\bar{F}(s) = \mathcal{L}F(t) = \int_0^\infty e^{-st}.\ 1\ .dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

$$\bar{x}(s) = \bar{G}(s)\bar{F}(s) = \frac{\bar{G}(s)}{s} = \frac{1}{ms(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Now following a standard process of inverse Laplace transformation, i.e., by factorising the denominator we get

$$\bar{x}(s) = \frac{1}{m\omega_n^2}\left[\frac{1}{s} - \frac{\xi\omega_n + i\omega_d}{2i\omega_d}\frac{1}{s + \xi\omega_n - i\omega_d} + \frac{\xi\omega_n - i\omega_d}{2i\omega_d}\frac{1}{s + \xi\omega_n + i\omega_d}\right]$$

Since

$$\mathcal{L}^{-1}\frac{1}{s-\alpha} = e^{\alpha t}$$

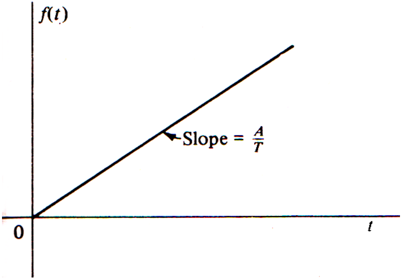
$$\mathcal{L}^{-1}\bar{x}(s) = \frac{1}{k}\left[1 - \frac{\xi\omega_n + i\omega_d}{2i\omega_d}e^{-(\xi\omega_n - i\omega_d)t} + \frac{\xi\omega_n - i\omega_d}{2i\omega_d}e^{-(\xi\omega_n + i\omega_d)t}\right]$$

The above equation can be simplified to

$$x(t) = \mathcal{L}^{-1}\bar{x}(s) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{(1-\xi^2)}}e^{-\xi\omega_n t} cos\ (\omega_d t - \psi)\right]$$

where $tan\ \psi = \frac{\xi}{\sqrt{(1-\xi^2)}}$

Obtain the response of the system $m\ddot{x} + kx = kf(t)$ by the Laplace transform method. Assume no damping and consider f(t) = 0 for t<0.



$$m\ddot{x} + kx = k\frac{A}{T}t$$

$$\bar{F}(s) = \mathcal{L}kf(t) = k\frac{A}{T}\int_0^{\infty} te^{-st}\ dt = k\frac{A}{T}\frac{1}{s^2}$$

$$\bar{G}(s) = \frac{1}{m(s^2 + \omega_n^2)}$$

$$x(t) = \mathcal{L}^{-1}\bar{G}(s)\bar{F}(s)$$

$$x(t) = \frac{kA}{T}\mathcal{L}^{-1}\left[\frac{1}{ms^2+k}\cdot\frac{1}{s^2}\right] = \frac{\omega_n^2 A}{T}\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega_n^2}\cdot\frac{1}{s^2}\right]$$

$$x(t) == \frac{A}{T}\mathcal{L}^{-1}\left[\frac{1}{s^2}-\frac{1}{s^2+\omega_n^2}\right]$$

$$x(t) == \frac{A}{T\omega_n}\left(\omega_n t - \sin\omega_n t\right), \qquad t > 0$$

Applying to the damped SDOF system and rearranging

$$(ms^2 + cs + k)\bar{x}(s) = \bar{F}(s) + m\dot{x}(0) + (ms + c)x(0)$$

Considering the initial boundary condition as x(0) = x_0 and $\dot{x}(0) = v_0$ we obtain the transformed response in the form

$$\bar{x}(s) = \frac{\bar{F}(s)}{m(s^2 + 2\xi\omega_n + \omega_n^2)} + \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n + \omega_n^2} x_0 + \frac{1}{s^2 + 2\xi\omega_n + \omega_n^2} v_0$$

It can be shown by inverse Laplace transform that

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\xi\omega_n(t-\tau)} \sin\omega_d(t-\tau) d\tau$$

$$+ \frac{x_0}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos(\omega_d t - \psi) + \frac{v_0}{\omega_d} e^{-\xi\omega_n t} \sin\omega_d t$$