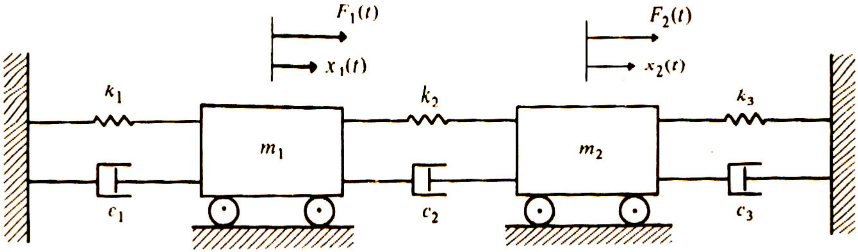
AE31002 Aerospace Structural Dynamics
Two Degrees of Freedom System

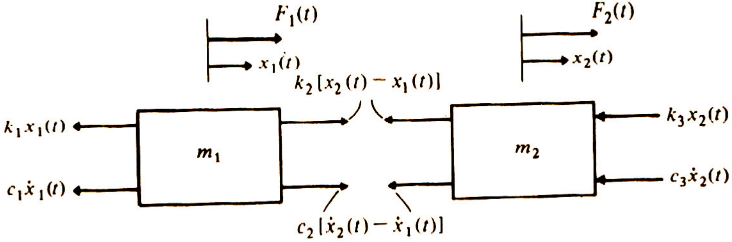
Anup Ghosh

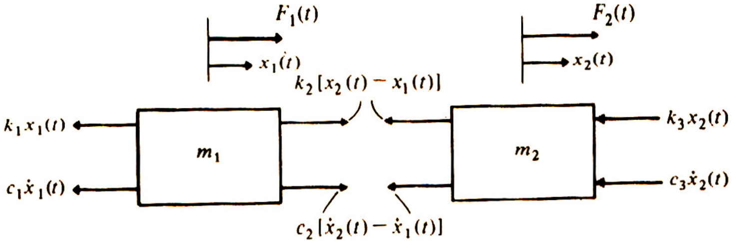
Importance

- Two degree of freedom system is merely a special case of multi degree of freedom system.
- It is an introduction to a more advanced study of discrete systems with an arbitrary large number of degrees of freedom.
- The MDOF system shows as many natural frequencies and associated natural mode shapes as the number of degrees of freedom.
- Natural modes posses a very important property known as orthogonality.
- A proper choice of coordinates, known as the principle or natural coordinates, the system differential equations become independent of each other.
- The motion of the system can be regarded as a superposition of the natural coordinates.


$F_1(t)$
$x_1(t)$
$F_2(t)$
$x_2(t)$
k_1
k_2
k_3
m_1
m_2
c_1
c_2
c_3

(a)


$F_1(t)$
$x_1(t)$
$F_2(t)$
$x_2(t)$
$k_2[x_2(t) - x_1(t)]$
$k_1 x_1(t)$
$k_3 x_2(t)$
m_1
m_2
$c_1 \dot{x}_1(t)$
$c_3 \dot{x}_2(t)$
$c_2[\dot{x}_2(t) - \dot{x}_1(t)]$



$$F_1(t) - c_1\dot{x}_1(t) - k_1x_1(t) + c_2[\dot{x}_2(t) - \dot{x}_1(t)] + k_2[x_2(t) - x_1(t)] = m_1\ddot{x}_1(t)$$

$$F_1(t) - c_2[\dot{x}_2(t) - \dot{x}_1(t)] - k_2[x_2(t) - x_1(t)] - c_3\dot{x}_2(t) - k_3x_2(t) = m_2\ddot{x}_2(t)$$

after simplification

$$m_1\ddot{x}_1(t) + (c_1 + c_2)\dot{x}_1(t) - c_2\dot{x}_2(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = F_1(t)$$

$$m_2\ddot{x}_2(t) - c_2\dot{x}_1(t) + (c_2 + c_3)\dot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = F_2(t)$$

These are coupled equation of motion for 2 degree of freedom system.

In matrix form $[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F(t)\}$, where

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = [m] = \quad mass\ matrix$$

$$\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = [c] = \quad damping\ matrix$$

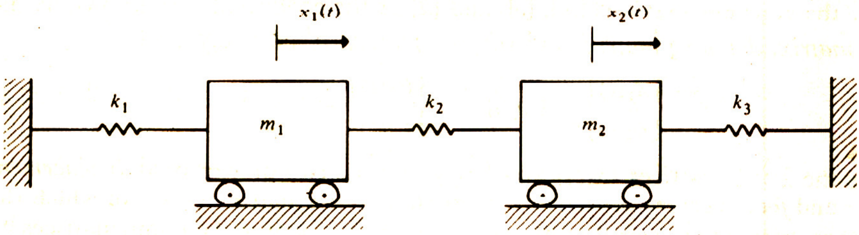
$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = [k] = \quad stiffness\ matrix$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \{x(t)\} = \quad displacement\ vector$$

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \{F(t)\} = \quad force\ vector$$

Observations

- The coupling terms in first equation are $c_2\dot{x}_2(t)$ & $k_2\dot{x}_2(t)$ and in 2nd equation are $c_2\dot{x}_1(t)$ & $k_2\dot{x}_1(t)$
- The matrices [m], [c] and [k] are symmetric in nature with respect to the diagonal of the respective matrices.
- The equations are not independent equations. We need special consideration (decoupling) to solve these.



Equations modify to

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let us consider the special type of solution when the coordinates $x_1(t)$ and $x_2(t)$ increase or decrease in the same proportion as time unfolds – **a synchronous motion**. So, the ratio $x_1(t)/x_2(t)$ is independent of time and the ratio between the two displacements remains constant throughout the motion. Let,

$$x_1(t) = u_1 f(t) \quad and \quad x_2(t) = u_2 f(t)$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} u_1 \ddot{f}(t) \\ u_2 \ddot{f}(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 f(t) \\ u_2 f(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

To posses a solution, we must have,

$$-\frac{\ddot{f}(t)}{f(t)} = \frac{k_{11}u_1 + k_{12}u_2}{m_1 u_1} = \frac{k_{12}u_1 + k_{22}u_2}{m_2 u_2} = \lambda = a\ real\ constant$$

Hence synchronous motion is possible only when,

$$(k_{11} - \lambda m_1)u_1 + k_{12}u_2 = 0$$

$$k_{12}u_1 + (k_{22} - \lambda m_2)u_2 = 0$$

$$\Rightarrow \lambda \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} \Rightarrow \lambda[m]\{u\} = [k]\{u\}$$

The nontrivial solution of this problem is a eigenvalue problem and by setting the determinant equal to zero we get.

$$\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} = \frac{1}{2}\frac{m_1 k_{22} + m_2 k_{11}}{m_1 m_2} \mp \frac{1}{2}\sqrt{\left(\frac{m_1 k_{22} + m_2 k_{11}}{m_1 m_2}\right)^2 - 4\frac{k_{11}k_{22} - k_{12}^2}{m_1 m_2}} \tag{1}$$

This may be proved that $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$ represents the frequencies of the structure when a possible solution of the assumed displacement function $f(t)$ is $f(t) = C\ cos\ (\omega t - \phi)$.
Now, $u_1 =?$ and $u_2 =?$

Now, let,

$$\{u\}_1 = \left\{ \begin{array}{c} u_{11} \\ u_{21} \end{array} \right\} \; and \; \{u\}_2 = \left\{ \begin{array}{c} u_{12} \\ u_{22} \end{array} \right\}$$

the **first suffix** denote the **position** of displacement and the **second suffix** denote the corresponding **frequency**. The quantities $\{u\}_1$ and $\{u\}_2$ are known as the **modal vectors** representing the **shape of deflection** of the system **associated to a natural frequency**.

Since ω_i are found out from the non trivial solution of a set of homogeneous equations, only the ratio u_{21}/u_{11} and u_{22}/u_{12} can be determined uniquely, i.e.,

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_1^2 m_2}$$

$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_2^2 m_2}$$

Orthogonality of Modes and Natural Coordinates

$$\{u\}_1 = \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} \; and \; \{u\}_2 = \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix}$$

$$\{u\}_1 = u_{11} \begin{Bmatrix} 1 \\ -\frac{k_{11}-\omega_1^2 m_1}{k_{12}} \end{Bmatrix} \; and \; \{u\}_2 = u_{12} \begin{Bmatrix} 1 \\ -\frac{k_{11}-\omega_2^2 m_1}{k_{12}} \end{Bmatrix}$$

$$\{u\}_2^T [m] \{u\}_1 = u_{11} u_{12} \begin{Bmatrix} 1 \\ -\frac{k_{11}-\omega_2^2 m_1}{k_{12}} \end{Bmatrix}^T \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ -\frac{k_{11}-\omega_1^2 m_1}{k_{12}} \end{Bmatrix}$$

(Assignment)

The LHS quantity, $\{u\}_2^T [m] \{u\}_1$, becomes zero after substitution of the values of ω_i. **So, the modal vectors $\{u\}_1$ and $\{u\}_2$ shows orthogonality with respect to the mass of the system.**

Now considering the matrix equation for free vibration of 2DOFS, $[k]\{u\} = \omega^2[m]\{u\}$ we have $[k]\{u\}_1 = \omega_1^2[m]\{u\}_1$ and $[k]\{u\}_2 = \omega_2^2[m]\{u\}_2$. Premultiplying the first by $\{u\}_2^T$ we have

$$\{u\}_2^T[k]\{u\}_1 = \omega_1^2\{u\}_2^T[m]\{u\}_1 = 0$$

So, the modal vectors $\{u\}_1$ and $\{u\}_2$ are also orthogonal with respect to the stiffness of the system.

This is also worth noting that

$$\{u\}_i^T[k]\{u\}_i = \omega_i^2\{u\}_i^T[m]\{u\}_i \qquad i = 1, 2$$

This property of orthogonality is used to uncouple the equations both elastically and inertially. Let us rewrite the equation of motion for free vibration in matrix form,

$$[m]\{\ddot{x}(t)\} + [k]\{x(t)\} = 0$$

Let, $\{x(t)\} = \{u\}_1 q_1(t) + \{u\}_2 q_2(t)$, eqn. of motion becomes

$$[m](\{u\}_1 \ddot{q}_1(t) + \{u\}_2 \ddot{q}_2(t)) + [k](\{u\}_1 q_1(t) + \{u\}_2 q_2(t)) = 0$$

Now premultiplying the above equation by $\{u\}_1^T$ and $\{u\}_2^T$ separately and using the relation of orthogonality for same mode, we get

$$\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0$$

$$\ddot{q}_2(t) + \omega_2^2 q_2(t) = 0$$

Two independent equations in terms of $q_1(t), \omega_1$ or $q_2(t), \omega_2$ and the coordinates $q_i(t)$ are known as **natural coordinates or principle coordinates**.

Since q_i represent independent equations and harmonic oscillation, the solution of the equation are

$$q_i(t) = C_i \ cos \ (\omega_i - \phi_i)$$

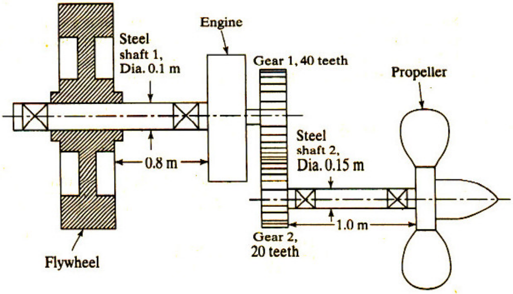
So, **the motion of the system at any time can be expressed as a superposition of the natural modes of vibration multiplied by the natural coordinates**, i.e.,

$$\{x(t)\} = C_1\{u\}_1 \ cos(\omega_1 t - \phi_1) + C_2\{u\}_2 \ cos(\omega_2 t - \phi_2)$$

In the matrix form

$$\left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = C_1 \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}_1 cos \ (\omega_1 - \phi_1) + C_2 \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}_2 cos \ (\omega_2 - \phi_2)$$

The schematic diagram of a engine connected to a propeller through gears is shown in the figure. The mass moments of inertia of the flywheel, engine, gear-1, gear-2 and the propeller (in kg-m^2) are 9000, 1000, 250, 150 and 2000, respectively. Find the natural frequencies and mode shapes of the system.



Obsrvations:

– We need to find the equivalent mass moment of inertia of all rotors.

Assumptions:

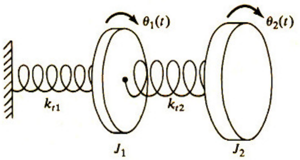
1. The flywheel can be considered as stationary with respect to the other part of the system because of its huge mass moment of inertia.
2. The engine and gears can be replaced by a single equivalent rotor.

The system is having velocity constraint implemented through the two different gear teeth ratio. The mass moment of inertia of gear-2 and the propeller w.r.t. engine are:

$$(J_{G2})_{eq} = 2^2 \times 150 = 600 \ kg - m^2$$
$$(J_P)_{eq} = 2^2 \times 2000 = 8000 \ kg - m^2$$

Considering the flywheel as fixed combined mass moment of inertia of engine and two gears is:

$$\begin{aligned} J_1 &= J_E + J_{G1} + (J_{G2})_{eq} \\ &= 1000 + 250 = 600 = 1850 \ kg - m^2 \end{aligned}$$



With G=80 x 10^9 N/m^2

$$k_{t1} = \frac{G\ I_{01}}{l_1} = \frac{G}{l_1}\left(\frac{\pi d_1^4}{32}\right) = \frac{80\times 10^9\times\pi\times 0.10^4}{0.8\times 32} = 981,750N{-}m/rad$$

$$k_{t2} = \frac{G\ I_{02}}{l_2} = \frac{G}{l_2}\left(\frac{\pi d_2^4}{32}\right) = \frac{80\times 10^9\times\pi\times 0.15^4}{1.0\times 32} = 3,976,087.5N{-}m/ra$$

Following the standard case as derived earlier, i.e.,

$$\begin{matrix}\omega_1^2\\ \omega_2^2\end{matrix} = \frac{1}{2}\frac{m_1k_{22}+m_2k_{11}}{m_1m_2} \mp \frac{1}{2}\sqrt{\left(\frac{m_1k_{22}+m_2k_{11}}{m_1m_2}\right)^2 - 4\frac{k_{11}k_{22}-k_{12}^2}{m_1m_2}}$$

Here $x \equiv \theta$ and $k_1 = k_{t1},\ k_2 = k_{t2},\ k_3 = 0,\ m_1 = J_1\ \ m_2 = J_2$