AE31002 Aerospace Structural Dynamics
Coordinate Coupling and Beating Phenomenon

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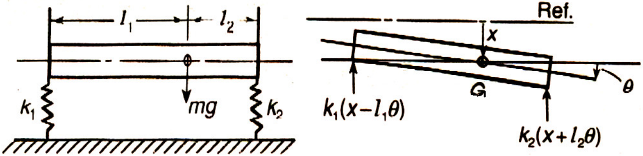
In most general case a undamped coupled 2 degree of freedom system is as follows.

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- **Mass or dynamic coupling** exists if mass matrix is nondiagonal.
- **Stiffness or static coupling** exists if stiffness matrix is nondiagonal.
- It is possible to decouple the equations with help of principle or natural or normal coordinates.
- It is not always possible to decouple the equation of motions for damped systems.

$$\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let us consider a 2DOF system where the mass center does not coincide with the geometric center.

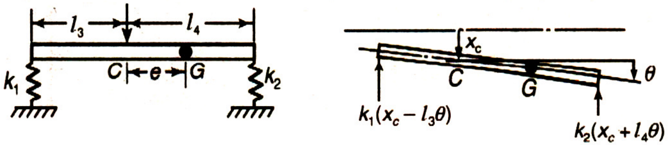


Case of **static coupling** may be observed if we consider the co-ordinates as shown above. The equations of motion becomes

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & (k_2 l_2 - k_1 l_1) \\ (k_2 l_2 - k_1 l_1) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

x is the linear displacement of the center of mass.

If $k_1 l_1 = k_2 l_2$ the equations get decoupled.



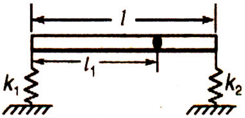
The general displacement behaviour is

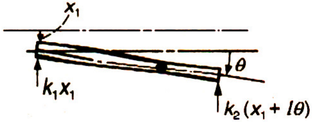
$$\begin{bmatrix} m & me \\ me & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 l_4 - k_1 l_3) \\ (k_2 l_4 - k_1 l_3) & (k_1 l_3^2 + k_2 l_4^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

If we consider the displacement coordinate at the place where the bar produces pure translation, $k_1 l_3 = k_2 l_4$ and that is not the c.g., we get **dynamic coupling** in the equations of motion.

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix} \begin{Bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & 0 \\ 0 & (k_1 l_3^2 + k_2 l_4^2) \end{bmatrix} \begin{Bmatrix} x_c \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

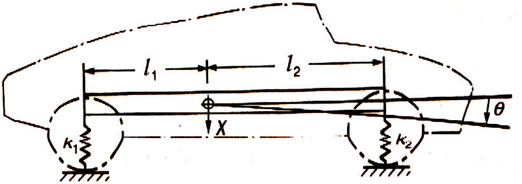
In case of arbitrary selection of the coordinate the equation of motion may lead to a system which is **coupled in both static and dynamic** and the equations of the system becomes





$$\begin{bmatrix} m & ml_1 \\ ml_1 & J_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & k_2 l \\ k_2 l & k_2 l^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Determine the normal modes of vibration of an automobile simulated by the simplified 2DOF system with the following



numerical values.
W= 14500N, l_1= 1.4m, k_1= 35000N/m, $J_c = \frac{W}{g}r^2$, l_2=1.7m, k_2= 38000N/m, r=1.22m, l=3.1m. Assume c.g. is located l_1 distance from the center of the rear wheel of the car.

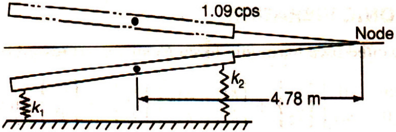
This is a case of static coupling. The equation of motion becomes

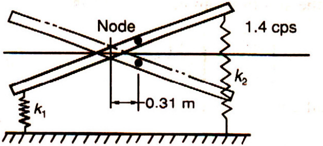
$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & (k_2 l_2 - k_1 l_1) \\ (k_2 l_2 - k_1 l_1) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We need to find out the eigenvalues of the system. Considering the following form of the equations, where ω_i are the eigenvalues or the frequencies of the system, we have.

$$\begin{bmatrix} (k_1+k_2-m\omega^2) & (k_2l_2-k_1l_1) \\ (k_2l_2-k_1l_1) & (k_1l_1^2+k_2l_2^2-J_c\omega^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

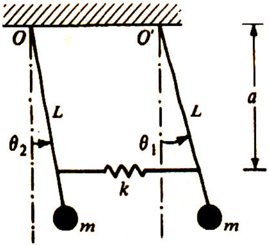
Considering the determinant to be zero we have,
$\omega_1 = 6.87$ rad/s $= 1.09$cps and $\omega_2 = 9.13$ rad/s $= 1.45$ cps.
Amplitude ratio of the modes may be found out from the above equation by substituting the corresponding values of ω.
$\left(\frac{x}{\theta}\right)_{\omega_1} = -4.78\ m/rad = -83.43\ mm/deg$,
$\left(\frac{x}{\theta}\right)_{\omega_2} = 0.31\ m/rad = 5.41\ mm/deg$

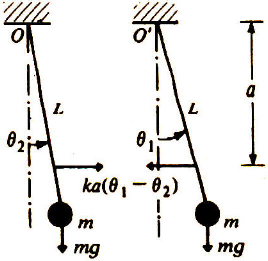




Beating is very interesting experimental/physical observation of the modal superposition assumption where we assume that **the motion of the system at any time can be expressed as a superposition of the natural modes of vibration multiplied by the natural coordinates**

Let us consider two identical pendulums connected by a spring. Let us also assume that the θ_1 and θ_2 are small angles.





The moment equations about the points O and O', respectively, yield the differential equations shown below

$$mL^2\ddot{\theta}_1 + mgL\theta_1 + ka^2(\theta_1 - \theta_2) = 0$$

$$mL^2\ddot{\theta}_2 + mgL\theta_2 - ka^2(\theta_1 - \theta_2) = 0$$

in matrix form

$$\begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} mgL + ka^2 & -ka^2 \\ -ka^2 & mgL + ka^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

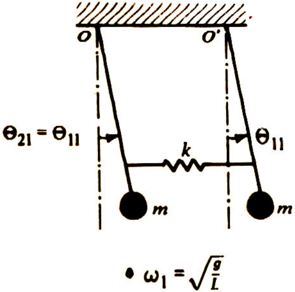
The system is coupled elastically. When k=0 the independent pendulums have individual natural frequencies $= \sqrt{g/L}$. For $k \neq 0$, natural frequencies may be found out from the eigenvalue solution, i.e.,

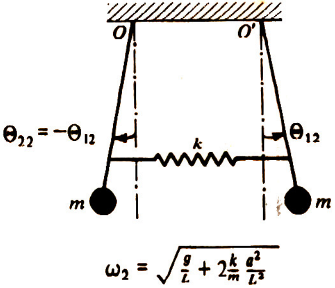
$$\left[-\omega^2 \begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix} + \begin{bmatrix} mgL + ka^2 & -ka^2 \\ -ka^2 & mgL + ka^2 \end{bmatrix} \right] \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The frequencies are

$$\omega_1 = \sqrt{\frac{g}{L}} \; and \; \omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}\frac{a^2}{L^2}}$$

Now from the relation of the previous page the ratio of $\Theta_{21}/\Theta_{11} = 1$ and $\Theta_{22}/\Theta_{12} =$ -1. This implies that the two pendulums move like a single pendulum with the spring unstretched in first mode and they are 180° out of phase in the other mode.





Solution of the problem expressed in terms of superposition of two natural modes multiplied by the associated natural coordinates are

$$\begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}_1 cos\,(\omega_1 - \phi_1) + C_2 \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}_2 cos\,(\omega_2 - \phi_2)$$

Considering $\Theta_{11} = \Theta_{12} = 1$

$$\theta_1(t) = C_1 \; cos\,(\omega_1 - \phi_1) + C_2 \; cos\,(\omega_2 - \phi_2)$$

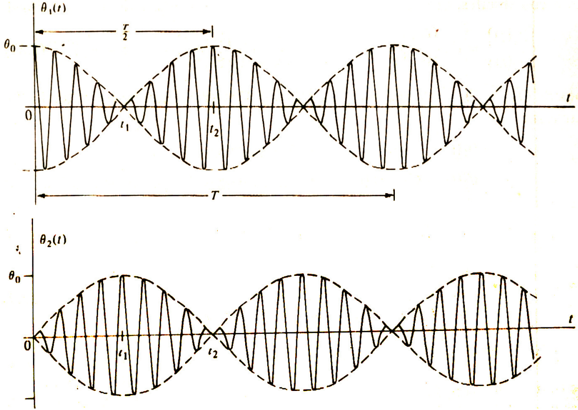
$$\theta_2(t) = C_1 \; cos\,(\omega_1 - \phi_1) - C_2 \; cos\,(\omega_2 - \phi_2)$$

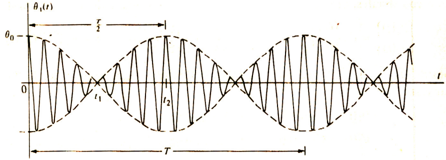
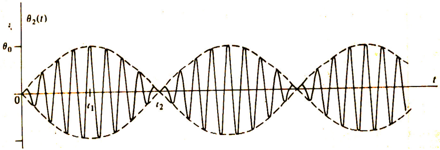
Now with the initial condition $\theta_1(0) = \theta_0$,
$\theta_2(0) = \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$

$$\begin{aligned} \theta_1(t) &= \tfrac{1}{2}\theta_0 \; cos\; \omega_1 t + \tfrac{1}{2}\theta_0 \; cos\; \omega_2 t \\ &= \theta_0 \; cos(\tfrac{\omega_2 - \omega_1}{2} t) \; cos(\tfrac{\omega_2 + \omega_1}{2} t) \\ \theta_2(t) &= \theta_0 \; sin(\tfrac{\omega_2 - \omega_1}{2} t) \; sin(\tfrac{\omega_2 + \omega_1}{2} t) \end{aligned}$$

$$\begin{aligned}
\theta_1(t) &= \theta_0 \ cos(\tfrac{\omega_2-\omega_1}{2}t) \ cos(\tfrac{\omega_2+\omega_1}{2}t) \\
\theta_2(t) &= \theta_0 \ sin(\tfrac{\omega_2-\omega_1}{2}t) \ sin(\tfrac{\omega_2+\omega_1}{2}t)
\end{aligned}$$

If ka^2 is very small compared to the to the mgL (observe the first equation of motion), there exists a weak coupling between the pendulums. The response becomes.



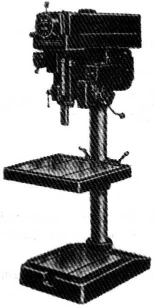



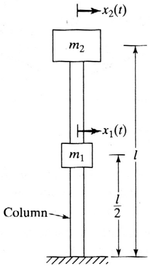
The above phenomenon is known as the **beating phenomenon**. It is purely the result of adding two harmonic functions of equal amplitudes and nearly equal frequencies. In this particular case the first part $(\theta_0\ cos(\frac{\omega_2-\omega_1}{2}t)\ or\ \theta_0\ sin(\frac{\omega_2-\omega_1}{2}t))$ increase or decrease the amplitude of the oscillatory motion whereas the other part contribution is manifested into the frequency part.
The variation of amplitude maintains a frequency, i.e., $\frac{\omega_2-\omega_1}{2}$ is known as the **beating frequency**.

The drilling machine shown can be modeled as a two degree of freedom system as indicated in the figure. Since a transverse force applied to mass m_1 or mass m_2 causes both the masses to deflect, the system exhibit elastic coupling. the bending stiffness of the columns are given by

$$k_{11} = \frac{786EI}{7l^3}, \quad k_{12} = k_{21} = -\frac{240EI}{7l^3}, \quad k_{22} = \frac{96EI}{7l^3}$$

Determine the natural frequencies of the drilling machine.





The frequency or eigen value equation becomes

$$|\left[-\omega^2[m] + [k]\right]| = 0$$

$$\begin{bmatrix} (k_{11} - \omega^2 m_1) & k_{12} \\ k_{21} & (k_{22} - \omega^2 m_2) \end{bmatrix} = 0$$

The expanded equation is

$$(m_1 m_2)\omega^4 - (m_1 k_{22} + m_2 k_{11})\omega^2 + (k_{11} k_{22} - k_{12}^2) = 0$$

$$\omega_1^2, \omega_2^2 = \frac{(m_1 k_{22} + m_2 k_{11}) \pm \sqrt{(m_1 k_{22} - m_2 k_{11})^2 + 4 m_1 m_2 k_{12}^2}}{2 m_1 m_2}$$

$$\omega_1^2, \omega_2^2 = \frac{48}{7} \frac{EI}{m_1 m_2} \left[(m_1 + 8m_2) \pm \sqrt{(m_1 - 8m_2)^2 + 25 m_1 m_2} \right]$$