AE31002 Aerospace Structural Dynamics
Vibration Absorbers

Anup Ghosh

Let us consider the most generalized case of 2DOFS in matrix form $[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F(t)\}$, where

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = [m] = \quad mass \;\; matrix$$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = [c] = \quad damping \;\; matrix$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = [k] = \quad stiffness \;\; matrix$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \{x(t)\} = \quad displacement \;\; vector$$

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \{F(t)\} = \quad force \;\; vector$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let us also consider the following harmonic excitation:

$$F_1(t) = F_1 e^{i\omega t} \qquad F_2(t) = F_2 e^{i\omega t}$$

and the steady state response as

$$x_1(t) = X_1 e^{i\omega t} \qquad x_2(t) = X_2 e^{i\omega t}$$

where X_1 and X_2 are in general complex quantities depending on the driving frequency ω and the system parameters. So we get

$$\left[-\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + i\omega \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\Rightarrow [Z(\omega)]\{X\} = \{F\}, \qquad Z_{ij} = -\omega^2 m_{ij} + i\omega c_{ij} + k_{ij} \quad i,j = 1,2$$

$[Z(\omega)]$ is known as the **impedance matrix**

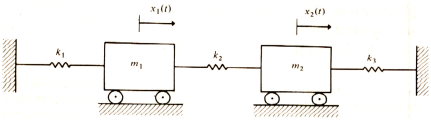
Now multiplying both side by $[Z(\omega)]^{-1}$

$$\Rightarrow \{X\} = [Z(\omega)]^{-1}\{F\}$$

$$[Z(\omega)]^{-1} = \frac{1}{det\,[Z(\omega)]}\begin{bmatrix} Z_{22}(\omega) & -Z_{12}(\omega) \\ -Z_{21}(\omega) & Z_{11}(\omega) \end{bmatrix}$$

$$X_1(\omega) = \frac{Z_{22}(\omega)F_1 - Z_{12}(\omega)F_2}{Z_{11}(\omega)Z_{22}(\omega) - Z_{12}^2(\omega)}, \quad X_2(\omega) = \frac{-Z_{12}(\omega)F_1 + Z_{11}(\omega)F_2}{Z_{11}(\omega)Z_{22}(\omega) - Z_{12}^2(\omega)}$$

The functions $X_1(\omega)$ and $X_2(\omega)$ are **analogous to the frequency response of SDOF system**.



Let us consider the above example with $F_2 = 0$.
After substitution of corresponding values, $Z_{11}(\omega) = k_{11} - \omega^2 m_1$, $Z_{22}(\omega) = k_{22} - \omega^2 m_2$ and $Z_{12}(\omega) = k_{12}$,

$$X_1(\omega) = \frac{(k_{22} - \omega^2 m_2)F_1}{(k_{11} - \omega^2 m_1)(k_{22} - \omega^2 m_2) - k_{12}^2}$$

$$X_2(\omega) = \frac{-k_{12} F_1}{(k_{11} - \omega^2 m_1)(k_{22} - \omega^2 m_2) - k_{12}^2}$$

Let us consider $m_1 = m$, $m_2 = 2m$, $k_1 = k_2 = k$, and $k_3 = 2k$ for the system shown above.

$$k_{11} = k_1 + k_2 = 2k \quad k_{22} = k_2 + k_3 = 3k \quad k_{12} - k_2 = -k$$

Following usual procedure

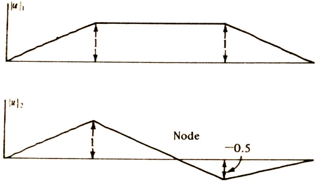
$$\begin{matrix} \omega_1^2 \\ \omega_2^2 \end{matrix} = \left[\frac{7}{4} \mp \sqrt{\left(\frac{7}{4}\right)^2 - \frac{5}{2}} \right] \frac{k}{m} = \left\{ \begin{matrix} \frac{k}{m} \\ \frac{5k}{2m} \end{matrix} \right.$$

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{2k - (k/m)m}{-k} = 1$$

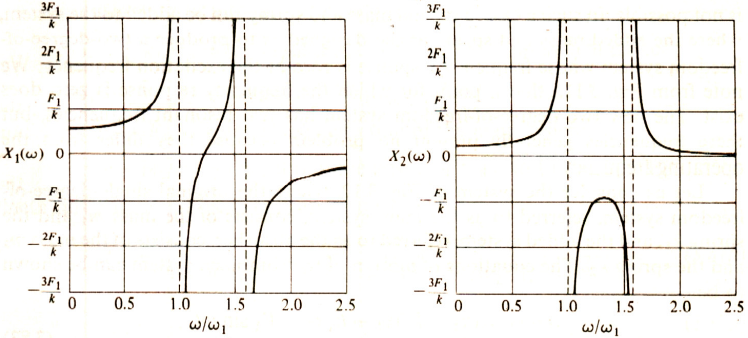
$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = \frac{2k - (5k/2m)m}{-k} = -0.5$$

$$\{u\}_1 = \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \; and \; \{u\}_2 = \left\{ \begin{matrix} 1 \\ -0.5 \end{matrix} \right\}$$

The mode shapes are



The response in terms of X_1 and X_2



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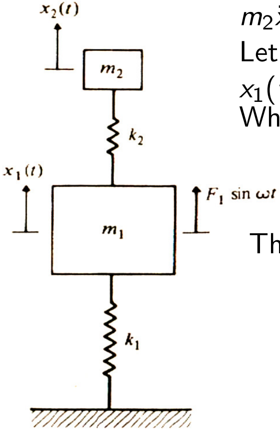
- We add one more spring mass system to make it 2DOFS.
- Let us refer the response plot in the previous slide. There is a point in the first figure, where the response of the 1st mass is zero.
- Our aim is to design a 2DOFS whose frequency response is zero at the excitation frequency.

let, the main system is with m_1 and k_1 and the the absorber system is with m_2 and k_2.



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \; sin \; \omega t$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

Let the solution be,

$$x_1(t) = X_1 \; sin \; \omega t \qquad x_2(t) = X_2 \; sin \; \omega t$$

Which leads to

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

The response becomes

$$X_1(\omega) = \frac{(k_2 - \omega^2 m_2)F_1}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2}$$

$$X_2(\omega) = \frac{k_2 F_1}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2}$$

If we introduce the terms

$$\omega_n = \sqrt{k_1/m_1} = \ main\ system\ natural\ frequency$$

$$\omega_a = \sqrt{k_2/m_2} = \ absorber\ system\ natural\ frequency$$

$$x_{st} = F_1/k_1 = \ main\ system\ static\ deflection$$

$$\mu = m_2/m_1 = ratio\ of\ absorber\ mass\ to\ main\ mass$$

The response becomes

$$X_1 = \frac{[1-(\omega/\omega_a)^2]x_{st}}{[1+\mu(\omega_a/\omega_n)^2-(\omega/\omega_n)^2][1-(\omega/\omega_a)^2]-\mu(\omega_a/\omega_n)^2}$$

$$X_2 = \frac{x_{st}}{[1+\mu(\omega_a/\omega_n)^2-(\omega/\omega_n)^2][1-(\omega/\omega_a)^2]-\mu(\omega_a/\omega_n)^2}$$

For $\omega = \omega_a$, $X_1 = 0$

For $\omega = \omega_a$, $X_1 = 0$ and

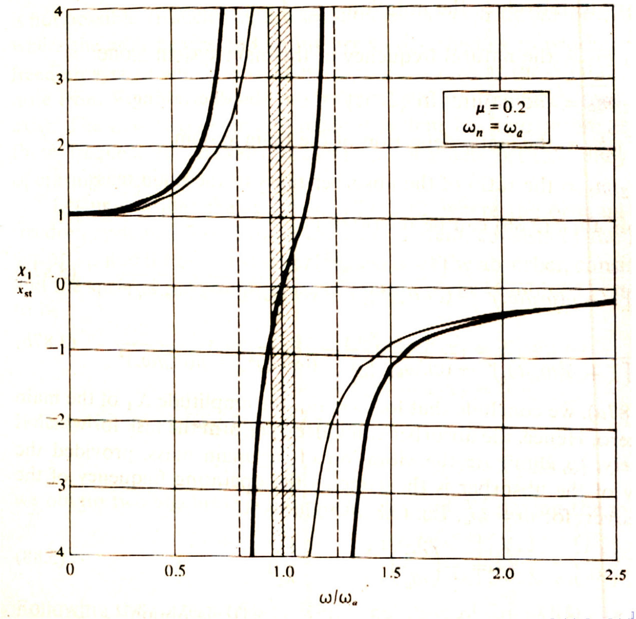
$$X_2 = -\left(\frac{\omega_n}{\omega_a}\right)^2 \frac{x_{st}}{\mu} = -\frac{F_1}{k_2} \Rightarrow \quad x_2(t) = -\frac{F_1}{k_2}\ sin\ \omega t$$

Force in the absorber spring

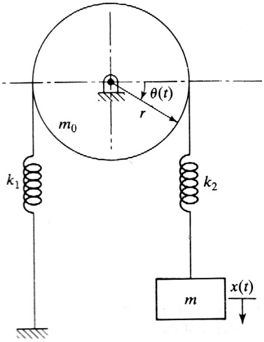
$$k_2 x_2(t) = -F_1\ sin\ \omega t$$

Observation

- There is a wide choice of absorber parameters.
- In most of the situations the actual choice is dictated by the amplitude X_2.
- The absorber can perform satisfactorily for operating frequencies that vary slightly from ω.


$\mu = 0.2$
$\omega_n = \omega_a$
$\frac{X_1}{x_{st}}$
ω/ω_a
4
3
2
1
0
−1
−2
−3
−4
0
0.5
1.0
1.5
2.0
2.5

Determine the natural frequencies of the system shown in the figure below assuming that the rope passing over the cylinder does not slip.



Equation of motion of mass m:

$$m\ddot{x} = -k_2(x - r\theta)$$

Equation of motion of cylinder of mass m_0 :

$$J_0\ddot{\theta} = -k_1 r^2\theta - k_2(r\theta - x)r$$

and the mass moment of inertia of the cylinder is $J_0 = \frac{1}{2}m_0 r^2$

Now assuming natural co-ordinate

$$x(t) = X\cos(\omega t + \phi)$$

$$\theta(t) = \Theta\cos(\omega t + \phi)$$

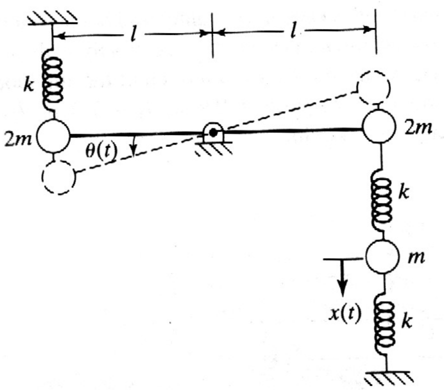
Now substituting the normal co-ordinates in the equation of motions, the frequency equation becomes

$$\begin{vmatrix} k_2 - \omega^2 m & -k_2 r \\ -k_2 r & -\frac{1}{2} m_0 r^2 \omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

$$\omega^4 - \omega^2 \left\{ \frac{k_2}{m} + \frac{2(k_1 + k_2)}{m_0} \right\} + \frac{2 k_1 k_2}{m_0 m} = 0$$

$$\omega_1^2, \omega_2^2 = \frac{k_2}{m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4} \left(\frac{k_2}{m} + \frac{2(k_1 + k_2)}{m_0} \right)^2 - \frac{2 k_1 k_2}{m m_0}}$$

A rigid rod of negligible mass and length 2l is pivoted at the middlw point and is constrained to move in the vertical plane by springs and masses, as shown. Find the natural frequencies and mode shaped of the system.



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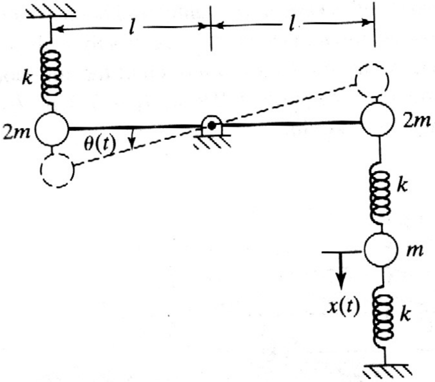
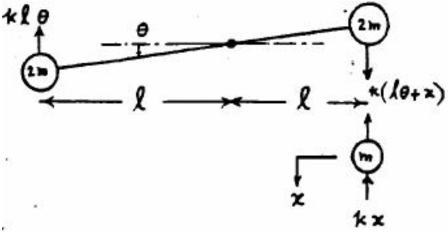


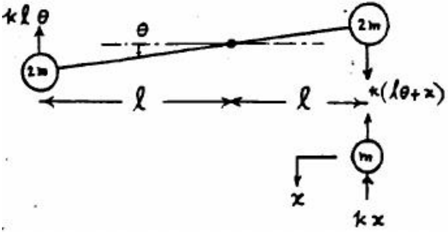


The system is having 3 springs and 3 lumped masses. If we choose the rotational co-ordinate θ and displacement in the vertical direction of the spring mass attached to the ground as the second co-ordinate the problem becomes a 2 DOF system.

Equation of motion in rotational direction and vertical direction:

$$4ml^2\ddot{\theta} = -kl\theta \cdot l - k(l\theta + x)l$$

$$m\ddot{x} = -kx - k(l\theta + x)$$



The eigen value equation becomes

Assuming natural co-ordinate similar to the previous example

$$\begin{vmatrix} -4ml^2\omega^2 + 2kl^2 & kl \\ kl & -m\omega^2 + 2k \end{vmatrix} = 0$$

$$4m^2\omega^4 - 10km\omega^2 + 3k^2 = 0$$

$$\omega^2 = \frac{k}{m}\left(\frac{5}{4} \mp \frac{\sqrt{13}}{4}\right) \Rightarrow \ \omega_1 = 0.5904\sqrt{\frac{k}{m}} \ \textit{and} \ \ \omega_2 = 1.4668\sqrt{\frac{k}{m}}$$

Amplitude ratios are

$$r_1 = \frac{X^{(1)}}{\Theta^{(1)}} = \frac{-4ml\omega_1^2 + 2kl^2}{-kl} = -0.6056l$$

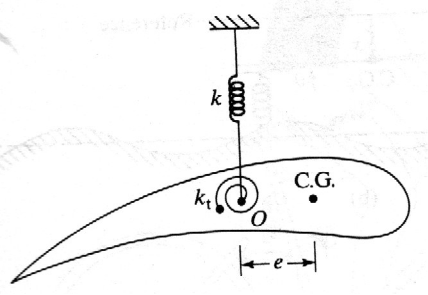
$$r_2 = \frac{X^{(2)}}{\Theta^{(2)}} = \frac{-4ml\omega_2^2 + 2kl^2}{-kl} = 6.6056l$$

Mode shapes are

$$\left\{ \begin{array}{c} \Theta^{(1)} \\ X^{(1)} \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ -0.6056l \end{array} \right\} \Theta^{(1)}$$

$$\left\{ \begin{array}{c} \Theta^{(2)} \\ X^{(2)} \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 6.6056l \end{array} \right\} \Theta^{(2)}$$

An airfoil of mass m is suspended by a linear spring of stiffness k and a torsional spring of stiffness k_1 in a wind tunnel. The C.G. is located at a distance of e from point O. The mass moment of inertia of the airfoil about an axis passing through point O is J_0. Find the natural frequencies of the airfoil.



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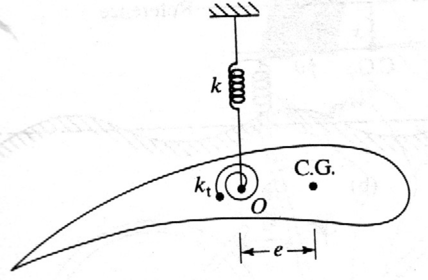
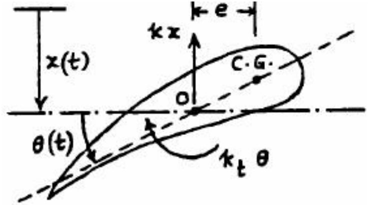
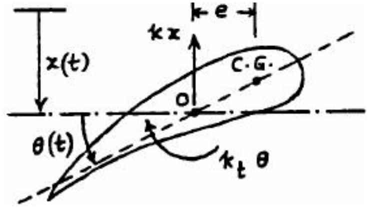




Equation of motion in vertical direction and rotational direction:

$$m(\ddot{x} - e\ddot{\theta}) = -kx$$

$$J_{CG}\ddot{\theta} = -k_t\theta - kxe$$



May be rewritten as

$$m\ddot{x} + kx - me\ddot{\theta} = 0$$

$$(j_0 - me^2)\ddot{\theta} + k_t\theta + kex = 0$$

The eigen value equation becomes

$$\begin{vmatrix} -m\omega^2 + k & me\omega^2 \\ ke & -(J_0 - me^2)\omega^2 + k_t \end{vmatrix} = 0$$