AE31002 Aerospace Structural Dynamics
Modal Analysis

Anup Ghosh

A free vibration equation of a multiple degree of freedom system may be defined as

$$[M]\{\ddot{x}\} + [K]\{x\} = 0$$

let us assume the displacement vector as

$$\{x\} = \{a\}\sin(\omega t - \alpha)$$

Since ω_i are found out from the non trivial solution of a set of homogeneous equations, only the ratio for the components of the modal vectors a_{i1}/a_{11}, a_{i2}/a_{12} ... ... a_{in}/a_{1n} can be found out. These vectors are known as the modal or normal or natural modes of vibration.

The equation of equilibrium for the free vibration

$$[K]\{a\} = \omega^2[M]\{a\}$$

Let us consider the mode shapes as the static deflection resulting from the forces on the right hand side of the equation as shown below

System -I

$$\begin{array}{lll} Force & \omega_1^2 a_{11} m_1 & \omega_1^2 a_{21} m_2 \\ Displacements & a_{11} & a_{21} \end{array}$$

System -II

$$\begin{array}{lll} Force & \omega_2^2 a_{12} m_1 & \omega_2^2 a_{22} m_2 \\ Displacements & a_{12} & a_{22} \end{array}$$

Applying **Betti's theorem**: *For a structure acted upon by two systems of loads and corresponding displacements, the work done by the first system of loads moving through the displacements of the second system is equal to the work done by this second system of loads undergoing the displacements produced by the fist load system.*

$$\omega_1^2 a_{11} m_1 a_{12} + \omega_1^2 a_{21} m_2 a_{22} = \omega_2^2 a_{12} m_1 a_{11} + \omega_2^2 a_{22} m_2 a_{21}$$
$$(\omega_1^2 - \omega_2^2)(m_1 a_{11} a_{12} + m_2 a_{21} a_{22}) = 0$$
$$since \quad \omega_1^2 \neq \omega_2^2, \quad \Rightarrow \quad m_1 a_{11} a_{12} + m_2 a_{21} a_{22} = 0$$

The above condition is the condition of orthogonality with respect to mass. or,

$$\sum_{k=1}^{m} m_k a_{ki} a_{kj} = 0, \qquad for \ i \neq j$$

$$\{a_i\}^T[M]\{a_j\} = 0, \quad for\ i \neq j$$

Where $\{a_i\}$ and $\{a_j\}$ are any two modal vectors and [M] is the mass matrix of the system.

As mentioned before, the amplitude of vibration in a normal mode are only relative values which may be scaled or normalized to some extent as a matter of choice. The following is an especially convenient normalization for a general system.

$$u_{ij} = \frac{a_{ij}}{\sqrt{\{a_i\}^T[M]\{a_j\}}}$$

for a system with diagonal mass matrix

$$u_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^{n} m_k a_{kj}^2}}$$

Under the said normalization the eigen vectors satisfies the following equations

$$\{u\}_i^T[M]\{u\}_i = 1, \qquad i = 1, 2, 3, ..., n$$

$$\{u\}_i^T[M]\{u\}_j = 0, \qquad for \quad i \neq j$$

$$[u]^T[M][u] = [I]$$

$$[u]^T[K][u] = [\omega^2]$$

Let the equation of motion for n-th degree freedom system is

$$[m]\{\ddot{q}(t)\} + [k]\{q(t)\} = \{Q(t)\}$$

where [m] and [k] are the n x n symmetric mass and stiffness matrices of the system and {q(t)} & {Q(t)} are n-dimensional generalised coordinate and force vectors, respectively.

We need to find out the eigenvalues of the system of equations considering the following equations.

$$[m][\omega^2][u] = [k][u]$$

where [u] is the modal matrix and $[\omega^2]$, a diagonal matrix of natural frequencies squared.
We have proved that modal matrix follow the following properties

$$[u]^T[m][u] = [1] \quad and \quad [u]^T[k][u] = [\omega^2]$$

let us consider the linear transformation

$$\{q(t)\} = [u]\{\eta(t)\}$$

where {q(t)} and $\{\eta$(t)} represent two different sets of generalised coordinates. Now using this and premultiplying the 1st equation by $[u]^T$ and considering the condition of orthogonality with respect to mass and stiffness, we have

$$\{\ddot{\eta}(t)\}+[\omega^2]\{\eta(t)\} = \{N(t)\}, \quad where\ \{N(t)\} = [u]^T\{Q(t)\}$$

{N(t)} is the generalized force. The above equation may be represented as n independent equations of the form

$$\ddot{\eta}_r(t) + \omega_r^2\ \eta_r(t) = N_r(t), \qquad r = 1, 2, \ldots, n$$

where $\eta_r(t)$ are known as the **normal coordinates** of the system.

We have already found out the response of a damped single degree of freedom system as

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\xi\omega_n(t-\tau)}\ sin\ \omega_d(t-\tau)\ d\tau$$

$$+\frac{x_0}{(1-\xi^2)^{1/2}}e^{-\xi\omega_n t}\ cos\ (\omega_d t-\psi)+\frac{v_0}{\omega_d}e^{-\xi\omega_n t}\ sin\ \omega_d t$$

where $\psi = tan^{-1}\frac{\xi}{(1-\xi^2)^{1/2}}$
So the complete solution with m=1, ξ =0 and $\omega_d = \omega_r$

$$\eta_r(t) = \frac{1}{\omega_r}\int_0^t N_r(\tau)\ sin\ \omega_r(t-\tau)\ d\tau$$

$$+\eta_r(0)\ cos\ \omega_r t+\frac{\dot{\eta}_r(0)}{\omega_r}\ sin\ \omega_r t, \qquad r=1,2,\ldots,n$$

where $\eta_r(0)$ and $\dot{\eta}_r(0)$ are the initial generalized displacements and velocities respectively.
The response in terms of modal coordinates

$$\{q(t)\} = [u]\{\eta(t)\} = \sum^n \{u\}_r\ \eta_r(t)$$

Initial conditions, at t=0,

$$\{q(0)\} = [u]\{\eta(0)\} = \sum_{r=1}^{n} \{u\}_r \ \eta_r(0)$$

multiplying the above equation by $\{u\}_r^T[m]$ and considering the orthogonality with respect to mass

$$\eta_r(0) = \{u\}_r^T[m]\{q(0)\}, \qquad r = 1, 2, \ldots, n$$

similarly the modal initial velocity is

$$\dot{\eta}_r(0) = \{u\}_r^T[m]\{\dot{q}(0)\}, \qquad r = 1, 2, \ldots, n$$

$$[m]\{\ddot{q}(t)\} + [c]\{\dot{q}(t)\} + [k]\{q(t)\} = \{Q(t)\}$$

let us consider the linear transformation

$$\{q(t)\} = [u]\{\eta(t)\}$$

where {q(t)} and $\{\eta$(t)} represent two different sets of generalised coordinates. Now using this and premultiplying the result by $[\mathrm{u}]^T$ and considering the condition of orthogonality with respect to mass and stiffness, we have

$$\{\ddot{\eta}(t)\}+[C]\{\dot{\eta}(t)\}+[\omega^2]\{\eta(t)\} = \{N(t)\}, \quad where\,[C] = [u]^T[c][u]$$

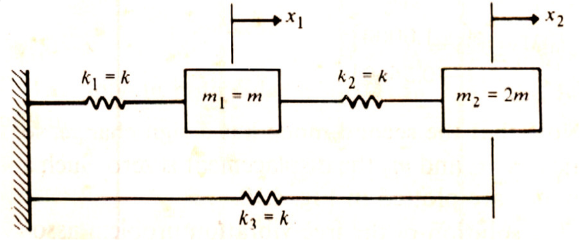
Following **proportional damping**, i.e., $[c] = \alpha[m] + \beta[k]$, where α and β are constants to be determined
$[C] = \alpha[1] + \beta[\omega^2] = [2\xi\omega]$ and the response becomes

$$\ddot{\eta}_r(t) + 2\xi_r\omega_r\dot{\eta}(t) + \omega_r^2\ \eta_r(t) = N_r(t), \qquad r = 1, 2, \ldots, n$$

Response for m=1 and $\omega = \omega_{dr}$

$$\begin{aligned}\eta_r(t) &= \frac{1}{\omega_d}\int_0^t N_r(\tau)e^{-\xi_r\omega_r(t-\tau)}\ sin\ \omega_{dr}(t-\tau)\ d\tau \\ &\quad + e^{-\xi_r\omega_r t}[\frac{\eta_0}{(1-\xi_r^2)^{1/2}}\ cos\ (\omega_{dr}t-\psi_r) + \frac{\dot{\eta}_0}{\omega_{dr}}\ sin\ \omega_{dr}t] \\ &\qquad r = 1, 2, \ldots, n\end{aligned}$$

where $\omega_{dr} = (1-\xi_r^2)^{1/2}\omega_r$ and $\psi = tan^{-1}\frac{\xi_r}{(1-\xi_r^2)^{1/2}}$



Let the force vector is

$$F(t) = \left\{ \begin{array}{c} F_1(t) \\ F_2(t) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ F_0 f(t) \end{array} \right\}$$

where $f(t)$ is a unit step function.

The equations of equilibrium are

$$m\ddot{x}_1(t) + 2kx_1(t) - kx_2(t) = 0$$

$$2m\ddot{x}_2(t) - kx_1(t) + 2kx_2(t) = F_0 f(t)$$

$$[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad [k] = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\omega_1 = 0.7962\sqrt{\frac{k}{m}} \qquad \{u\}_1 = \frac{1}{\sqrt{m}}\left\{\begin{array}{c} 0.4597 \\ 0.6280 \end{array}\right\}$$

$$\omega_2 = 1.5382\sqrt{\frac{k}{m}} \qquad \{u\}_2 = \frac{1}{\sqrt{m}}\left\{\begin{array}{c} 0.8881 \\ -0.3251 \end{array}\right\}$$

$$[u] = \frac{1}{\sqrt{m}}\left\{\begin{array}{cc} 0.4597 & 0.8881 \\ 0.6280 & -0.3251 \end{array}\right\}$$

Now using the transformation

$$\{x(t)\} = [u]\{\eta(t)\}$$

$$\{N(t)\} = [u]^T\{F(t)\} = \frac{1}{\sqrt{m}}\left\{\begin{array}{cc} 0.4597 & 0.6280 \\ 0.8881 & -0.3251 \end{array}\right\}\left\{\begin{array}{c} 0 \\ F_0 f(t) \end{array}\right\}$$

$$= \frac{F_0}{\sqrt{m}}\left\{\begin{array}{c} 0.6280 \\ -0.3251 \end{array}\right\} f(t)$$

Now the normal coordinates

$$\begin{aligned}
\eta_1(t) &= 0.6280\frac{F_0}{\sqrt{m}}\frac{1}{\omega_1}\int_0^t f(\tau)\ sin\ \omega_1(t-\tau)\ d\tau \\
& \quad since\ \ f(\tau) = 1\ constant \\
&= 0.6280\frac{F_0}{\omega_1^2\sqrt{m}}(1 - cos\ \omega_1 t)
\end{aligned}$$

$$\begin{aligned}
\eta_2(t) &= -0.3251\frac{F_0}{\sqrt{m}}\frac{1}{\omega_2}\int_0^t f(\tau)\ sin\ \omega_2(t-\tau)\ d\tau \\
&= -0.3251\frac{F_0}{\omega_2^2\sqrt{m}}(1 - cos\ \omega_2 t)
\end{aligned}$$

Now use the

$$\{x(t)\} = [u]\{\eta(t)\}$$

to find out response.