AE31002 Aerospace Structural Dynamics Vibrating Strings, Torsional and Axial Vibration of Rod

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What is continuous System?

* A system having continuous distribution of mass and elasticity.
* To specify the position of every point in the elastic body, an infinite number of coordinates is necessary.
* It posses an infinite number of degrees of freedom.
* The bodies are assumed to be homogeneous and isotropic and obeys Hooke's law.
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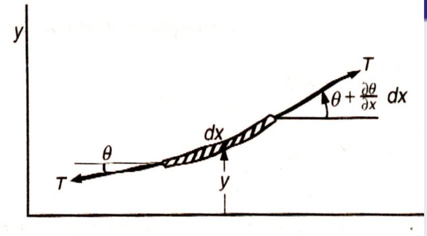
Vibrating String



Assumptions

* Mass of the spring is ρ per unit length.
* Lateral deflection y of the string is small.
* Change in tension with deflection is negligible (slope is small).

The equation of motion in y-direction

$$T\left(\theta + \frac{\partial\theta}{\partial x}dx\right) - T\theta = \rho\ dx\ \frac{\partial^2 y}{\partial t^2} \qquad \Rightarrow \frac{\partial\theta}{\partial x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad \textit{since} \quad \theta = \frac{\partial y}{\partial x} \quad \textit{and putting} \quad c = \sqrt{T/\rho}$$

The parameter c is known as the velocity of wave propagation along the string.

Let us assume the solution as

$$y(x,t) = Y(x)\, G(t) \qquad \Rightarrow \frac{1}{Y(x)}\frac{\partial^2 Y(x)}{\partial x^2} = \frac{1}{c^2}\frac{1}{G(t)}\frac{\partial^2 G(t)}{\partial t^2}$$

LHS is independent of t and RHS is independent of x $\Rightarrow$ both sides lead to a single constant. Let the constant be $-(\omega/c)^2$. So, we have

$$\frac{d^2 Y(x)}{dx^2} + \left(\frac{\omega}{c}\right)^2 \; Y(x) = 0$$

$$\frac{d^2 G(t)}{dt^2} + \omega^2 G(t) = 0$$

Corresponding general solutions are

$$Y(x) = A \; sin\frac{\omega}{c}x + B \; cos\frac{\omega}{c}x$$

$$G(t) = C \; sin \; \omega t + D \; cos \; \omega t$$

Constants **A, B, C, D** depend on the **boundary conditions** and the **initial conditions**.

Let us consider that the string is of length l and fixed at both ends. The BC will be y(0,t)=y(l,t)=0. The BC y(0,t)= 0, yields B=0 and the equation becomes

$$y = (C\ sin\ \omega t + D\ cos\ \omega t)\ sin\frac{\omega}{c}x$$

The other BC y(l,t)=0, leads to $sin\frac{\omega l}{c} = 0$, and the solution is

$$\frac{\omega l}{c} = \frac{2\pi l}{\lambda} = n\pi, \qquad n = 1, 2, 3, \ldots$$

where λ=c/f is the wavelength and f is the frequency of vibration. So, the frequencies related to the normal modes are

$$f_n = \frac{n}{2l}c = \frac{n}{2l}\sqrt{\frac{T}{\rho}} \qquad n = 1, 2, 3, \ldots$$

Frequency of a vibrating string is predominantly dependent on the tension in string T and the mass per unit length ρ.
The **mode shapes** of vibration is sinusoidal with distribution

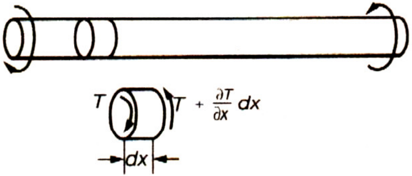
$$Y = sin\ (n\pi \frac{x}{l})$$

Following the concept of linear combination of normal modes, the general equation for the displacement can be

$$y(x,t) = \sum_{n=1}^{\infty} (C_n\, sin\, \omega_n t + D_n\, cos\, \omega_n t)\, sin\, \frac{n\pi x}{l},\ and\ \omega_n = \frac{n\pi c}{l}$$

The constants C_n and D_n can be determined can be determined using the BCs y(x,0) and $\dot{y}$(x,0).

Torsional Vibration of Rods



Let x is measured along the length of the rod. Angle of twist per unit length is

$$\frac{d\theta}{dx} = \frac{T}{I_p G}$$

where I_p is the polar moment of inertia of the cross-section of the rod and G is the shear modulus. The net force $\frac{\partial T}{\partial x} dx$ acting on the segment dx of the rod may be obtained from the above equation as

$$\frac{\partial T}{\partial x} dx = I_p G \frac{\partial^2 \theta}{\partial x^2} dx$$

This force must balance the inertia force when $\rho I_p dx$ is the mass moment of inertia and angular acceleration is $\partial^2\theta/\partial t^2$. i.e.,

$$\rho I_p dx \frac{\partial^2\theta}{\partial t^2} = I_p G \frac{\partial^2\theta}{\partial x^2} dx \qquad \Rightarrow \frac{\partial^2\theta}{\partial x^2} = \frac{\rho}{G}\frac{\partial^2\theta}{\partial t^2} = \frac{1}{c'}\frac{\partial^2\theta}{\partial t^2}$$

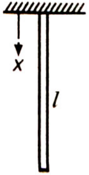
This is same as that of the string vibration equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ and the solution is (c'=$\sqrt{\frac{G}{\rho}}$)

$$\theta = \left(A\ sin\ \omega\sqrt{\frac{\rho}{G}}x + B\ cos\ \omega\sqrt{\frac{\rho}{G}}x\right)(C\ sin\ \omega t + D\ cos\ \omega t)$$

The equation can also be represented as

$$\theta = \left(A\ sin\ \omega\sqrt{\frac{\rho}{G}}x + B\ cos\ \omega\sqrt{\frac{\rho}{G}}x\right)\ \ sin\ (\omega t + \alpha)$$

Find the natural frequencies of the bar shown below.



The equation to be considered is

$$\theta = \left(A\ sin\ \omega\sqrt{\frac{\rho}{G}}x + B\ cos\ \omega\sqrt{\frac{\rho}{G}}x \right) \quad sin\ (\omega t + \alpha)$$

The BCs are

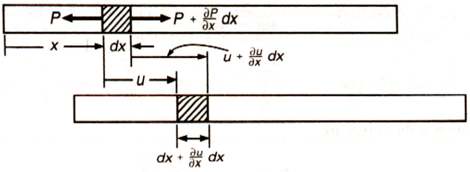
1. at x=0, the rotation due to torsion is zero, i.e., θ=0.
2. at x=l, i.e., at free end, applied torque T=0=$\partial\theta/\partial x$ because we are considering free vibration to find out the natural frequencies.

The first BC leads to B=0
The 2nd BC leads to $cos\ \omega\sqrt{\rho/G}l = 0$, and we get the frequencies of the free vibration.

$$\omega_n l\sqrt{\frac{\rho}{G}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \left(n + \frac{1}{2}\right)\pi \qquad \Rightarrow \omega_n = \left(n + \frac{1}{2}\right)\frac{\pi}{l}\sqrt{\frac{G}{\rho}}$$

Longitudinal (Axial) Vibration of Rods

If u is the displacement at x, the displacement at x+dx will be u+($\partial u/\partial x$)dx. It is evident then that the element dx in the new position has changed in length by an amount ($\partial u/\partial x$) dx, thus the unit strain is $\partial u/\partial x$



$$\frac{\partial u}{\partial x} = \frac{P}{AE} \Rightarrow AE\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x}$$

The inertia equation

$$\frac{\partial P}{\partial x}dx = \rho A dx \frac{\partial^2 u}{\partial t^2}$$

Leads to

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c''}\frac{\partial^2 u}{\partial t^2} \qquad with\ c'' = \sqrt{\frac{E}{\rho}}$$

The general solution is

$$u = \left(A \ sin \ \frac{\omega}{c''}x + B \ cos \ \frac{\omega}{c''}x\right)(C \ sin \ \omega t + D \ cos \ \omega t)$$

$$u = \left(A \ sin \ \omega\sqrt{\frac{\rho}{E}}x + B \ cos \ \omega\sqrt{\frac{\rho}{E}}x\right)(C \ sin \ \omega t + D \ cos \ \omega t)$$

Determine the natural frequencies and mode-shapes of a

free-free rod.

For this type of boundary condition stresses at the ends must be zero. So,

$$\frac{\partial u}{\partial x} = 0 \ at \ x = 0 \ and \ x = l$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = A\frac{\omega}{c}(C\ \sin\,\omega t + D\ \cos\,\omega t) = 0$$

The equation is true for any value of t. So, A =0.

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = \frac{\omega}{c''}\left(A\ \cos\ \frac{\omega l}{c''} + B\ \sin\ \frac{\omega l}{c''}\right)(C\ \sin\,\omega t + D\ \cos\,\omega t) = 0$$

To show the phenomenon of vibration B must be finite. The above equation is satisfied when

$$sin\ \frac{\omega l}{c''} = 0 \qquad \Rightarrow \frac{\omega_n l}{c''} = \omega_n l\sqrt{\frac{\rho}{E}} = \pi, 2\pi, 3\pi, \ldots, n\pi$$

$$\omega_n = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}, \qquad f_n = \frac{n}{2l}\sqrt{\frac{E}{\rho}}$$

$$u = u_0\ cos\ \frac{n\pi x}{l}\ sin\ \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}t \quad \text{with zero initial displacement}(D = 0)$$

Examples

* Rayleigh's Energy Method.
* Rayleigh-Ritz Mehod.
* FEM