AE31002 Aerospace Structural Dynamics
Free Vibration of Simply Supported Beam

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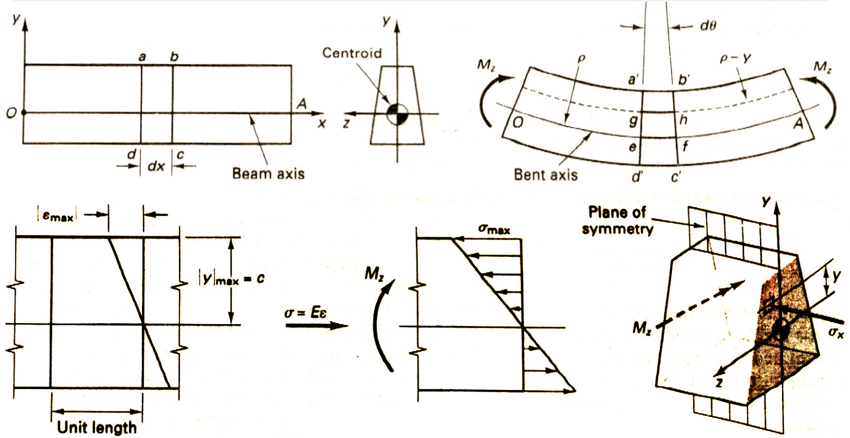
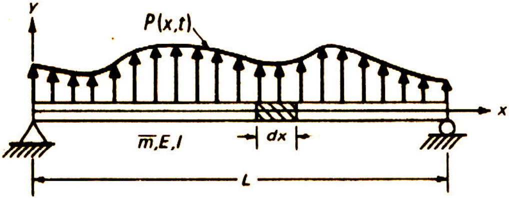
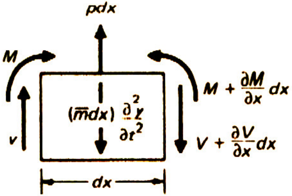
Assumption of Bernoulli-Euler Beam

* Deflection of the beam is small.
* Plane section through a beam taken normal to its axis remain plane after the beam is subject to bending.







Following small deflection theory and the force terms, shear and moments, as function of both x & t, equation of motion perpendicular to the beam is

$$V - \left(V + \frac{\partial V}{\partial x}\right) + p(x,t)dx - \bar{m}\ dx\ \frac{\partial^2 y}{\partial t^2} = 0$$

where $\bar{m}$ is mass/unit length, so the relation simplifies to,

$$\frac{\partial V}{\partial x} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t)$$

From the simple bending theory and following the present direction of co-ordinate and forcing, the moment curvature relation is

$$M = EI\frac{\partial^2 y}{\partial x^2} \quad and \quad V = \frac{\partial M}{\partial x} \quad \Rightarrow \quad V = EI\frac{\partial^3 y}{\partial x^3}$$

Now substituting this in the equation of motion, we have

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t)$$

A partial differential equation of fourth order, considering only transverse flexural deflection. (The deflection associated with the shear force and associated rotary moment of inertia is not considered here – Timoshenko's beam theory)

Let the forcing term be zero, i.e., $p(x,t) = 0$, and the equation reduce to a homogeneous equation

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0$$

Following the standard method of solution, method of separation of variable.

$$y(x,t) = \Phi(x)\ f(t)$$

Substituting in the above equation we have

$$EI\ f(t)\frac{\partial^4 \Phi(x)}{\partial x^4} + \bar{m}\ \Phi(x)\frac{\partial^2 f(t)}{\partial t^2} = 0$$

$$\Rightarrow \frac{EI}{\bar{m}}\frac{\Phi^{IV}(x)}{\Phi} = -\frac{\ddot{f}(t)}{f(t)}$$

LHS is a function of x and RHS is a function of t, so it is a constant and let us consider it to be ω^2

So, we have

$$\Phi^{IV}(x) - a^4\Phi(x) = 0 \quad \textit{and} \quad \ddot{f}(t) + \omega^2 f(t) = 0,$$

where, $a^4 = \frac{\bar{m}\omega^2}{EI}$ and it is convenient to find ω from the standard form given below.

$$\omega = C\sqrt{\frac{EI}{\bar{m}L^4}} \qquad \textit{in which} \quad C = (aL)^2$$

The time dependent equation is of standard form, what we have already found out, so,

$$f(t) = A' \; cos \; \omega t + B' \; sin \; \omega t$$

For the equation with x, we may assume as

$$\Phi(x) = C' e^{sx} \quad \Rightarrow \; (s^4 - a^4)C' e^{sx} = 0$$

For the nontrivial solution,

$$s^4 - a^4 = 0$$

and the roots are

$$\begin{aligned} s_1 &= a, & s_3 &= ai, \\ s_2 &= -a, & s_4 &= -ai \end{aligned}$$

$$\Phi(x) = C_1 e^{ax} + C_2 e^{-ax} + C_3 e^{iax} + C_4 e^{-iax}$$

With use of the trigonometric function

$$\begin{aligned} e^{\pm ax} &= cosh\ ax \pm sinh\ ax \\ e^{\pm iax} &= cos\ ax \pm i\ sin\ ax \end{aligned}$$

$$\Phi(x) = A\ sin\ ax + B\ cos\ ax + C\ sinh\ ax + D\ cosh\ ax$$

The constants A, B, C, D are to be determined from the BCs and these responsible for defining the amplitude and mode shapes of vibration.

Now we need to implement the boundary condition in the above generalized equation to find out the frequencies and mode-shapes. In this particular case the BCs are

$$y(0,t) = 0 \Rightarrow \Phi(0) = 0, \qquad M(0,t) \Rightarrow \Phi''(0) = 0,$$
$$y(L,t) = 0 \Rightarrow \Phi(L) = 0, \qquad M(L,t) \Rightarrow \Phi''(L) = 0,$$

So at the left end where L=0,

$$\Phi(0) = A\times 0 + B\times 1 + C\times 0 + D\times 1 = 0 \qquad \Rightarrow B + D = 0$$
$$\Phi''(0) = a^2(-A\times 0 - B\times 1 + C\times 0 + D\times 1) = 0 \quad \Rightarrow -B + D = 0$$

So, leads to B=0=D

Now with the implementation of the BCs at the right end or x=L,

$$\Phi(L) = A\ sin\ aL + C\ sinh\ aL = 0$$

$$\Phi''(L) = a^2(-A\ sin\ aL + C\ sinh\ aL) = 0$$

So we have 2C sinh al $= 0 \Rightarrow$ C $= 0$, since the hyperbolic sine function can not vanish except a zero argument. So the above boundary conditions leads to $A\ sin\ al = 0$.

Excluding the trivial solution (A=0), we get the frequency equation

$$sin\ al = 0 \Rightarrow a_n L = n\pi, \qquad n = 0, 1, 2, \ldots$$

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{\bar{m}L^4}} \ \ and \qquad \Phi_n(x) = A\ sin\ \frac{n\pi x}{L}$$

Since the function $\Phi(x)$ represent the shape only, the amplitude A can be assumed as 1. So the response of individual mode becomes

$$y_n(x,t) = \Phi_n(x)\ f_n(t)$$

$$y_n(x,t) = sin\ \frac{n\pi x}{L}\ [A_n\ cos\ \omega_n t + B_n\ sin\ \omega_n t]$$

The general solution for any time instant become

$$y(x,t) = \sum_{n=1}^{\infty} sin\ \frac{n\pi x}{L}\ [A_n\ cos\ \omega_n t + B_n\ sin\ \omega_n t]$$

The constants A_n and B_n can be found out from the initial condition w.r.t. time. Let, the displacement and velocity at the time t = 0 and for the domain $0 \leq x \leq L$, are

$$y(x,0) = \rho(x) = \sum_{n=1}^{\infty} A_n \; sin \; \frac{n\pi x}{L}$$

and

$$\frac{\partial y(x,0)}{\partial t} = \psi(x) = \sum_{n=1}^{\infty} B_n \; \omega_n \; sin \; \frac{n\pi x}{L}$$

Now using the Fourier series description

$$A_n = \frac{2}{L} \int_0^L \rho(x) \; sin \; \frac{n\pi x}{L} dx$$

$$B_n = \frac{2}{\omega_n L} \int_0^L \psi(x) \; sin \; \frac{n\pi x}{L} dx$$

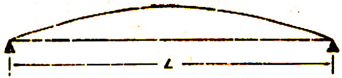
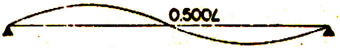
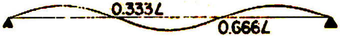
Natural Frequencies and Normal Modes for Simply Supported Beams.

Natural Frequencies

$$\omega_n = C_n \sqrt{\frac{EI}{\overline{m}L^4}}$$

Normal Modes

$$\Phi_n = \sin \frac{n\pi x}{L}$$

n	C_n	I_n^*	Shapes
1	π^2	$4/\pi$	
2	$4\pi^2$	0	
3	$9\pi^2$	$4/3\pi$	
4	$16\pi^2$	0