AE31002 Aerospace Structural Dynamics
Free-Free, Fixed and Centilever Beam

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Let the forcing term be zero, i.e., $p(x,t) = 0$, so, the equation reduce to a homogeneous equation

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0$$

$$y(x,t) = \Phi(x)\ f(t)$$

$$f(t) = A'\ cos\ \omega t + B'\ sin\ \omega t$$

$$\Phi(x) = A\ sin\ ax + B\ cos\ ax + C\ sinh\ ax + D\ cosh\ ax$$

A Free Free Beam

In this particular case the BCs are

$$M(0,t) = 0 \Rightarrow \Phi''(0) = 0, \qquad V(0,t) = 0 \Rightarrow \Phi'''(0) = 0,$$
$$M(L,t) = 0 \Rightarrow \Phi''(L) = 0, \qquad V(L,t) = 0 \Rightarrow \Phi'''(L) = 0,$$

So at the left end where L=0,

$$\Phi''(0) = a^2(-B + D) = 0 \qquad \Rightarrow B = D$$
$$\Phi'''(0) = a^3(-A + C) = 0 \qquad \Rightarrow A = C$$

Now for the other end i.e., x=L

$$\Phi''(L) = a^2(-A \sin aL - B \cos aL + C \sinh aL + D \cosh aL) = 0$$

$$\Phi'''(x) = a^3(-A \cos aL + B \sin aL + C \cosh aL + D \sinh aL) = 0$$

Rearrangement and use of result obtained from the first set of BCs.

$$A(sinh\ aL - sin\ aL) + B(cosh\ aL - cos\ aL) = 0$$

$$A(cosh\ aL - cos\ aL) + B(sinh\ aL + sin\ aL) = 0$$

For a nontrivial solution of the above equation, it is required that the determinant of the unknown coefficients A and B be equal to zero. So we get.

$$cos\ al \quad cosh\ al - 1 = 0 \qquad where \quad \omega_n = (a_n L)^2 \sqrt{\frac{EI}{\bar{m} L^4}}$$

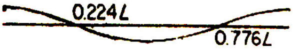
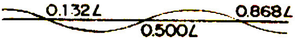
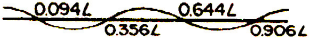
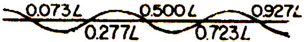
We need to find the roots by numerical methods.
Assuming the value of the constant A = 1 (since normal modes are determined only to a relative magnitude) a simplification leads to

$$\Phi_n(x) = cosh\ a_n x + cos\ a_n x - \sigma_n (sinh\ a_n x + sin\ a_n x)$$

$$where \qquad \sigma_n = \frac{cosh\ a_n L - cos\ a_n L}{sinh\ a_n L - sin\ a_n L}$$

TABLE 20.2 Natural Frequencies and Normal Modes for Free Beams.

Natural Frequencies	Normal Modes
$\omega_n = C_n \sqrt{\dfrac{EI}{\bar{m}L^4}}$	$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n(\sinh a_n x + \sin a_n x)$ $\sigma_n = \dfrac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L}$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shapes
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	

A Fixed Fixed Beam

The boundary conditions for a beam with both ends fixed are as follows. At x = 0 and x = L

$$y(0,t) = 0 \Rightarrow \Phi(0) = 0, \qquad y'(0,t) = 0 \Rightarrow \Phi'(0) = 0,$$
$$y(L,t) = 0 \Rightarrow \Phi(L) = 0, \qquad y'(L,t) = 0 \Rightarrow \Phi'(L) = 0,$$

So at the left end where x=0,

$$\Phi(0) = 0 \qquad \Rightarrow B = -D$$
$$\Phi'(0) = 0 \qquad \Rightarrow A = -C$$

Now for the other end i.e., x=L

$$\Phi(L) = 0 \Rightarrow (cos\ aL - cosh\ aL)B + (sin\ aL - sinh\ aL)A = 0$$

$$\Phi'(L) = 0 \Rightarrow -(sin\ aL + sinh\ aL)B + (cos\ aL - cosh\ aL)A = 0$$

The set of equation is same as of the previous case and it leads to the same frequency equation, i.e.,

$$cos\ al \quad cosh\ al - 1 = 0$$

$$A = -\frac{cos\ aL - cosh\ aL}{sin\ aL - sinh\ aL} B$$

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{\bar{m} L^4}}$$

The shape function equation become

$$\Phi_n(x) = cosh\ a_n x - cos\ a_n x - \sigma_n (sinh\ a_n x - sin\ a_n x)$$

$$where \qquad \sigma_n = \frac{cos\ a_n L - cosh\ a_n L}{sin\ a_n L - sinh\ a_n L}$$

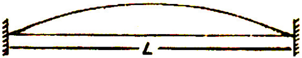
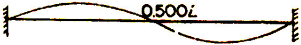
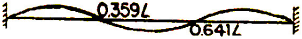
TABLE 20.3 Natural Frequencies and Normal Modes for Fixed Beams.

Natural Frequencies | **Normal Modes**

$$\omega_n = C_n \sqrt{\frac{EI}{\overline{m}L^4}}$$

$$\Phi_n(x) = \cosh a_n x - \cos a_n x - \upsilon_n(\sinh a_n x - \sin a_n x)$$

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	

A Cantilever Beam

The boundary conditions for a beam with one end fixed and the other end free are as follows. At x = 0 and x = L

$$y(0,t) = 0 \Rightarrow \Phi(0) = 0, \qquad y'(0,t) = 0 \Rightarrow \Phi'(0) = 0,$$
$$M(L,t) = 0 \Rightarrow \Phi''(L) = 0, \qquad V(L,t) = 0 \Rightarrow \Phi'''(L) = 0,$$

Following similar approach we have he frequency equation as

$$cos\ al \quad cosh\ al + 1 = 0, \qquad \omega_n = (a_n L)^2 \sqrt{\frac{EI}{\bar{m} L^4}}$$

The shape function equation become

$$\Phi_n(x) = cosh\ a_n x - cos\ a_n x - \sigma_n (sinh\ a_n x - sin\ a_n x)$$

$$where \qquad \sigma_n = \frac{cos\ a_n L + cosh\ a_n L}{sin\ a_n L + sinh\ a_n L}$$

TABLE 20.4 Natural Frequencies and Normal Modes for Cantilever Beams.

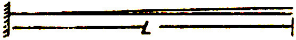
Natural Frequencies

$$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$$

Normal Modes

$$\Phi_n = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x)$$

$$\sigma = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$$

n	$C_n = (a_n L)^2$	σ_n	[illegible]	Shape
1	3.5160	0.734096	0.7830	
2	22.0345	1.018466	0.4340	
3	61.6972	0.999225	0.2589	
4	120.0902	1.000033	0.0017