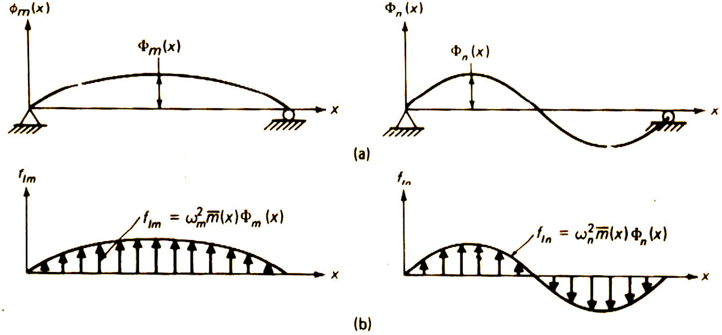
AE31002 Aerospace Structural Dynamics
Forced Vibration of Beam

Anup Ghosh



Let a beam subjected to the inertial forces from the vibrations of two different modes, $\Phi_m(x)$ and $\Phi_n(x)$. Corresponding inertial forces are shown in the figure. The inertial force is obtained from the multiplication of mass per unit length and the acceleration amplitude, i.e., $f_{In} = \bar{m}(x)\omega_n^2\Phi_n(x)$

Betti's Theorem

for a linear elastic structure subject to two sets of forces $\{P_i\}$ and $\{Q_j\}$, the work done by the set P through the displacements produced by the set Q is equal to the work done by the set Q through the displacements produced by the set P.

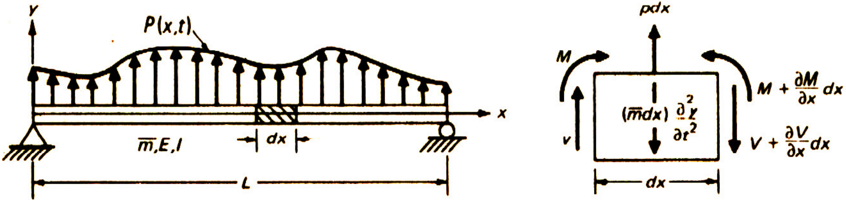
Accordingly, the work done by the inertial force, f_{In}, acting on the displacements of mode m is equal to the work of the inertial force, f_{In}, acting on the displacement of mode n, i.e.,

$$\int_0^L \Phi_m(x)\ f_{In}\ dx = \int_0^L \Phi_n(x)\ f_{Im}\ dx$$

$$\omega_n^2 \int_0^L \Phi_m(x)\ \bar{m}(x)\Phi_n(x)\ dx = \omega_m^2 \int_0^L \Phi_n(x)\ \bar{m}(x)\Phi_m(x)\ dx$$

$$(\omega_n^2 - \omega_m^2) \int_0^L \Phi_m(x)\ \bar{m}(x)\ \Phi_n(x)\ dx = 0$$

Since $\omega_n \neq \omega_m$ The modes are orthogonal to each other.



Following small deflection theory and the force terms, shear and moments, as function of both x & t, equation of motion perpendicular to the beam is

$$V - \left(V + \frac{\partial V}{\partial x}\right) + p(x,t)dx - \bar{m}\ dx\ \frac{\partial^2 y}{\partial t^2} = 0$$

where $\bar{m}$ is mass/unit length, so the relation simplifies to,

$$\frac{\partial V}{\partial x} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t)$$

From the simple bending theory and following the present direction of co-ordinate and forcing, the moment curvature relation is

$$M = EI\frac{\partial^2 y}{\partial x^2} \quad \textit{and} \quad V = \frac{\partial M}{\partial x} \quad \Rightarrow \quad V = EI\frac{\partial^3 y}{\partial x^3}$$

Now substituting this in the equation of motion, we have

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t)$$

A partial differential equation of fourth order, considering only transverse flexural deflection. (The deflection associated with the shear force and associated rotary moment of inertia is not considered here – Timoshenko's beam theory)

Let us assume that the general solution of the of this equation is

$$y(x,t) = \sum_{n=1}^{\infty} \Phi_n(x) z_n(t)$$

The normal modes $\Phi_n(x)$ satisfy the basic differential equation $\Phi^{IV}(x) - a^4\Phi(x) = 0$ and since $a^4 = \frac{\bar{m}\omega^2}{EI}$.

$$EI\ \Phi_n^{IV} = \bar{m}\omega_n^2\Phi_n(x), \qquad n = 1, 2, 3, \ldots$$

To satisfy the force boundary condition, let us substitute it to the previous equation

$$EI \sum_n \Phi_n^{IV}(x)\ z_n(t) = p(x,t) - \bar{m} \sum_n \Phi_n(x)\ \ddot{z}_n(t)$$

$$\sum_n \bar{m}\omega_n^2\Phi_n(x)\ z_n(t) = p(x,t) - \bar{m} \sum_n \Phi_n(x)\ \ddot{z}_n(t)$$

Multiplying both sides of the equation by $\Phi_m(x)\ dx$ and integrating between 0 to L and using the orthogonality condition, we have

$$\omega_m^2 z_m(t) \int_0^L \bar{m}\Phi_m^2\ dx = \int_0^L \Phi_m(x) p(x,t)\ dx - \ddot{z}_m(t) \int_0^L \bar{m}\Phi_m^2\ dx$$

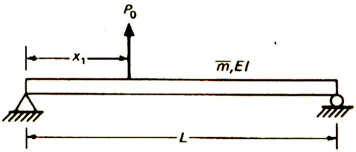
Now in a generalized form

$$M_n \ddot{z}_n(t) + \omega_n^2 M_n z_n(t) = F_n(t), \qquad n = 1, 2, 3, \ldots$$

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{F_n(t)}{M_n}$$

where, $M_n = \int_0^L \bar{m}\Phi_n^2\ dx$ is known as the **modal mass** and $F_n(t) = \int_0^L \Phi_n(x) p(x,t)\ dx$ is known as the **modal force.**

Consider a simply supported uniform beam subjected to a concentrated constant force suddenly applied at a section x_1 units from the left support. Determine the response using modal analysis.



The mode shapes of a simply supported beam are

$$\Phi_n = sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

and the modal force is

$$F_n(t) = \int_0^L \Phi_n(x)p(x,t)\ dx$$

In this problem p(x,t) = p_0 and x = x_1 otherwise p(x,t) = 0.

$$F_n(t) = P_0 \Phi_n(x_1) \quad \Rightarrow \quad P_0 \ sin \frac{n\pi x_1}{L}$$

The modal mass is

$$M_n = \int_0^L \bar{m}\Phi_n^2 \ dx = \int_0^L \bar{m} \ sin^2 \frac{n\pi x}{L} \ dx = \frac{\bar{m}L}{2}$$

Substituting the value of modal force in the equation $\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{F_n(t)}{M_n}$ we get

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{P_0 \ sin \frac{n\pi x_1}{L}}{\bar{m}L/2}$$

Following the standard solution of the undamped SDOF sytem for suddenly applied load,

$$z_n = (z_{st})_n (1 - cos \ \omega_n t)$$

$$Here \qquad (z_{st})_n = \frac{2P_0 \ sin\frac{n\pi x_1}{L}}{\omega_n^2 \bar{m} L}$$

$$So, \qquad z_n = \frac{2P_0 \ sin\frac{n\pi x_1}{L}}{\omega_n^2 \bar{m} L}(1 - cos \ \omega_n t)$$

The modal deflection at any section of beam is

$$y_n(x,t) = \Phi_n(x) z_n(t)$$

$$y_n(x,t) = \frac{2P_0 \ sin\frac{n\pi x_1}{L}}{\omega_n^2 \bar{m} L}(1 - cos \ \omega_n t) \ sin\frac{n\pi x}{L}$$

The total deflection is then

$$y(x,t) = \frac{2P_0}{\bar{m}L} \sum_n \left[\frac{1}{\omega_n^2} \ sin\frac{n\pi x_1}{L}(1 - cos \ \omega_n t) \ sin\frac{n\pi x}{L} \right]$$

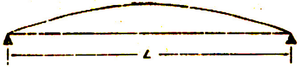
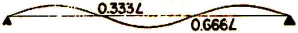
$x_1 = L/2$

#
$$y_n(x,t) = \frac{2P_0}{\bar{m}L}\sum_n \left[\frac{1}{\omega_n^2}\ sin\frac{n\pi}{2}(1 - cos\ \omega_n t)\ sin\frac{n\pi x}{L}\right]$$

No even mode contribute to the deflection of the beam.
Position of the excitation is one of the nodes of even modes, so those do not get excited.
Amplitude of modal displacement is a measure of a certain mode.
Amplitude is dependent on dynamic load factor $(1 - cos\ \omega_n t)$ (max value 2) and $1/\omega_n^2$, i.e., 1; 1/81; 1/625 etc.

Natural Frequencies and Normal Modes for Simply Supported Beams.

Natural Frequencies	Normal Modes
$\omega_n = C_n \sqrt{\dfrac{EI}{\overline{m}L^4}}$	$\Phi_n = \sin \dfrac{n\pi x}{L}$

n	C_n	I_n^*	Shapes
1	π^2	$4/\pi$	
2	$4\pi^2$	0	
3	$9\pi^2$	$4/3\pi$	
4	$16\pi^2$	0	
5	$25\pi^2$	$4/5\pi$	

$^*I_n = \int_0^L \Phi_n(x)\,dx \Big/ \int_0^L \Phi_n^2(x)\,dx$