AE31002 Aerospace Structural Dynamics
Generalized Coordinate

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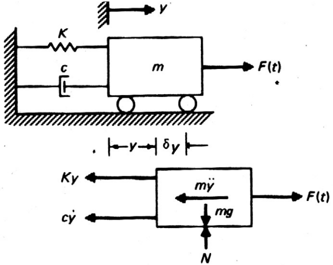
This principle is particularly useful for relatively complex structural systems which contain many interconnection parts. It is also used in indeterminate structures. This principle was originally stated for a system in equilibrium. This can also be applied to dynamic system in equilibrium after consideration of inertial force in the system following D'Alemberts Principle.

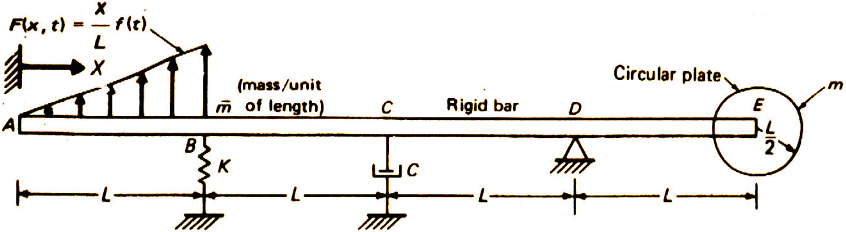
In Principle of Virtual Work: a system that is in equilibrium, the work done by all the forces during an assumed displacement (virtual displacement) which is compatible with the system constrains, is equal to zero.



$$m\ddot{y}\delta y + c\dot{y}\delta y + ky\delta y - F(t)\delta y = 0$$
$$\{m\ddot{y} + c\dot{y} + ky - F(t)\}\delta y = 0$$
$$m\ddot{y} + c\dot{y} + ky - F(t) = 0$$

Generalized Single Degree of Freedom System

The number of independent displacement quantities, sufficient to specify the position of all parts of the system are known as Generalized Coordinates.

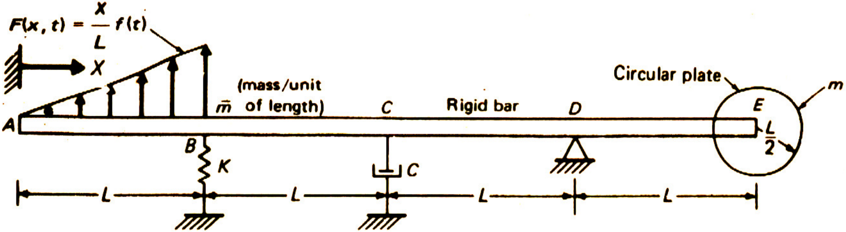


Assumptions:

1. The bar is rigid with uniform cross-section.
2. The hinge is friction less.
3. The bar is supporting a plate fixed at its end.

Can we consider this system as a SDOF system?

The displacement quantity $\theta(t)$ may be the only assumption. Let us use the principle of virtual work to find out the dynamic equilibrium equation.



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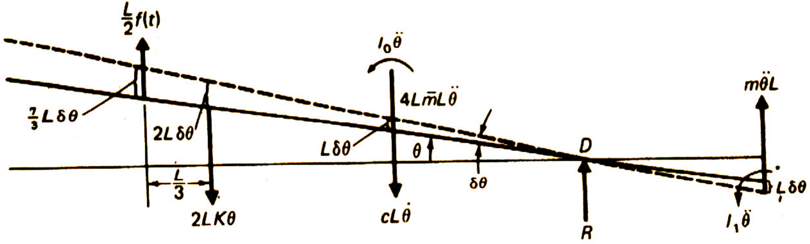


Let us consider a virtual displacement $d\theta$. The total work done by the forces during the virtual displacement

$$\delta\theta[I_0\ddot{\theta} + I_1\ddot{\theta} + 4L^3\bar{m}\ddot{\theta} + mL^2\ddot{\theta} + cL^2\dot{\theta} + 4kL^2\theta - \frac{7}{6}L^2f(t)] = 0$$

since $\delta\theta \neq 0$ The above equation simplifies to

$$(I_0 + I_1 + 4L^3\bar{m} + mL^2)\ddot{\theta} + cL^2\dot{\theta} + 4kL^2\theta - \frac{7}{6}L^2 f(t) = 0$$

$$I_0 = \frac{1}{12}(4\bar{m}L)(4L)^2 \qquad \textit{mass moment of inertia of the rod}$$

$$I_1 = \frac{1}{2}m\left(\frac{L}{2}\right)^2 \qquad \textit{mass moment of inertia of the circular plate}$$

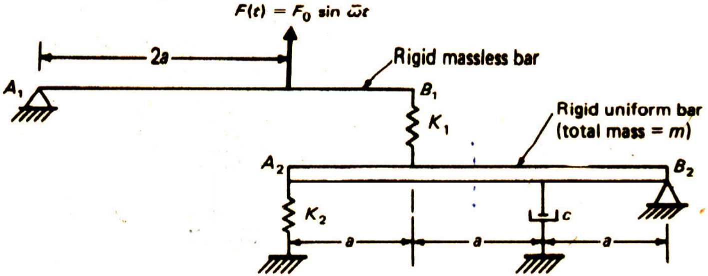
$$I^*\ddot{\theta} + C^*\dot{\theta} + K^*\theta = F^*(t)$$

$$I^* = I_0 + I_1 + 4L^3\bar{m} + mL^2,$$

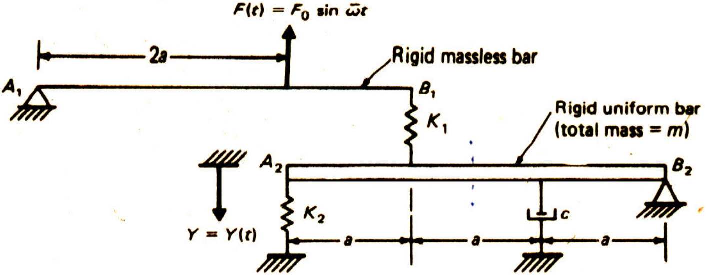
$$C^* = cL^2, \quad K^* = 4kL^2, \quad F^*(t) = \frac{7}{6}L^2 f(t)$$

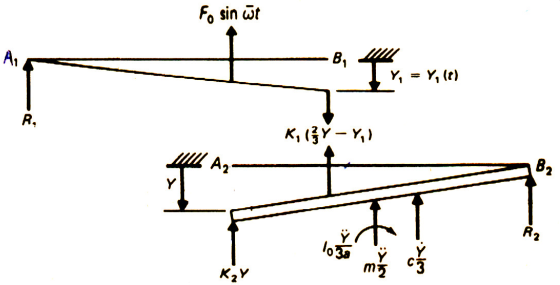
The starred quantities are known as the **generalized inertia, generalized damping, generalized stiffness and generalized load** for this system.

For the system shown, determine the generalized physical properties $M^*,\ C^*,\ K^*,\ F^*(t)$



Let Y(t) at the point A_2 be the generalized coordinate of the system.





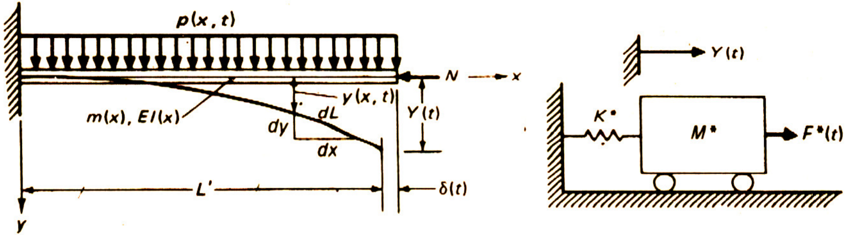
$$k_1(\frac{2}{3}Y - Y_1) \times 3a = F_0 \; sin \; \bar{\omega} t \times 2a$$

$$\frac{I_0}{3a}\ddot{Y} + m\frac{\ddot{Y}}{2} \times \frac{3}{2}a + \frac{a}{3}c\dot{Y} + k_1(\frac{2}{3}Y - Y_1) \times 2a + k_2 Y \times 3a = 0$$

$$\Rightarrow M^* \ddot{Y}(t) + C^* \dot{Y}(t) + K^* Y(t) = F^*(t)$$

$$M^* = \frac{I_0}{3a^2} + \frac{3}{4}m, \quad C^* = \frac{c}{3}, \quad K^* = 3k_2, \quad F^*(t) = -\frac{4}{3}F_0 \; sin \; \bar{\omega} t$$

If the bars are not rigid? Infinite degrees of freedom.



Let us consider that the flexural stiffness is EI(x), mas per unit length is m(x) and it is under two types of loads p(x,t) (transverse) and N (axial).
We need a single shape function to describe the deflected shape during motion.
Let $\phi(x)$ be the function describing this shape and a generalized coordinate Y(t) for the tip of the beam. Displacement at any point

$$y(x,t) = \phi(x)Y(t) \quad where \quad \phi(L) = 1$$

To find out the equation of equilibrium, we will find out the generalized mass, stiffness and damping from the considerations of kinetic energy, potential along with strain energy and virtual displacement.

Kinetic energy

$$T = \int_0^L \frac{1}{2} m(x) \{\phi(x) \dot{Y}(t)\}^2 \, dx$$

Now equating the kinetic energy with the equivalent SDOF system i.e., $(\frac{1}{2} M^* \dot{Y}(t)^2)$

$$M^* = \int_0^L m(x) \; \phi^2(x) \; dx$$

Strain energy due to bending (M(x)) of the system

$$U = \int_0^L \frac{M(x)^2}{2EI(x)}\, dx = \int_0^L \frac{1}{2} EI(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx$$

$$U = \int_0^L \frac{1}{2} EI(x) \left\{ \phi''(x) Y(t) \right\}^2 dx$$

Now equating with potential energy $(\frac{1}{2} K^* Y(t)^2)$ of the equivalent system,

$$K^* = \int_0^L EI(x) \left\{ \phi''(x) \right\}^2 dx$$

Similar manner the work of the distributed external force p(x,t) due to this virtual displacement is given by

$$W = \int_0^L p(x,t)\delta y \; dx = \int_0^L p(x,t)\phi(x)\delta Y \; dx \;\; where \;\; \delta y = \phi(x)\delta Y$$

Equating with the work of the generalized force, i.e., $F^*(t)\delta Y$

$$F^*(t) = \int_0^L p(x,t)\phi(x) \; dx$$

Similarly from the work done of damping forces in both systems, we have

$$C^*\dot{Y}\delta Y = \int_0^L c(x)\dot{y}\delta y \; dx \quad \Rightarrow \quad C^* = \int_0^L c(x)\{\phi(x)\}^2 \; dx$$