AE31002 Aerospace Structural Dynamics
Generalized Coordinate – Geometric Stiffness

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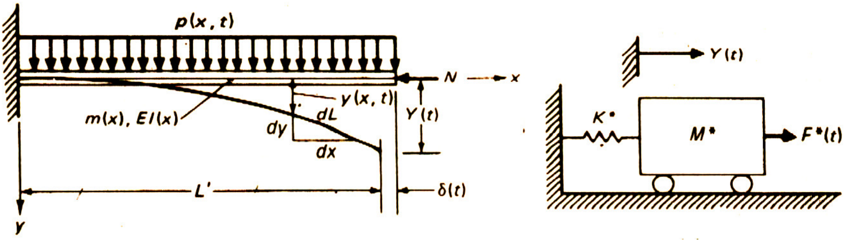
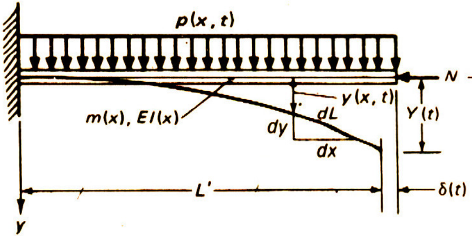


$$M^* = \int_0^L m(x)\ \phi^2(x)\ dx \quad \textit{Kinetic Energy Consideration}$$

$$K^* = \int_0^L EI(x)\left\{\phi''(x)\right\}^2 dx \quad \textit{Potential Energy Consideration}$$

$$F^*(t) = \int_0^L p(x,t)\phi(x)\ dx \quad \textit{Principle of Virtual Work}$$

$$C^* = \int_0^L c(x)\{\phi(x)\}^2\ dx \quad \textit{Principle of Virtual Work}$$



To find out the component of generalized stiffness with respect to the axial force, we need to consider the horizontal component of the motion $\delta(t)$ of the free end of the beam.

Considering differential element.

$$dL = (dy^2 + dx^2)^{1/2}$$

$$dL = \left(1 + (dy/dx)^2\right)^{1/2} dx$$

Now, integrating over the horizontal projection of the beam (L') and expanding in series the binomial expression,

$$L = \int_0^{L'} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx$$

$$L = \int_0^{L'} \left\{ 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 - \frac{1}{8}\left(\frac{dy}{dx}\right)^4 + \dots \right\} dx$$

Considering only two terms and L=L' for integration

$$L = L' + \int_0^{L} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$\delta(t) = L - L' = \int_0^{L} \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

Now we define a new stiffness coefficient to be called the generalized geometric stiffness, K_G^*, as the stiffness of the equivalent system required to store the same potential energy as the potential energy stored by the normal force N, i.e.,

$$\frac{1}{2}K_G^* Y(t)^2 = N\ \delta(t) = \frac{1}{2}N \int_0^{L} \left(\frac{dy}{dx}\right)^2 dx$$

$$K_G^* Y(t)^2 = N \int_0^L \left\{ Y(t) \frac{d\phi}{dx} \right\}^2 dx$$

$$K_G^* = N \int_0^L \left(\frac{d\phi}{dx} \right)^2 dx$$

For the case of an axial compressive force the potential energy in the beam decreases with a loss of stiffness in the beam. The opposite is true for a tensile axial force which results in an increase of the flexural stiffness of the beam. Customarily, the geaometric stiffness is determined for a compressive axial force. Consequently, the combined generalized stiffness is then given by

$$K_c^* = K^* - K_G^*$$

Differential equation for the equivalent system

$$M^* \ddot{Y}(t) + C^* \dot{Y}(t) + K_c^* Y(t) = F^*(t)$$

Buckling of Column

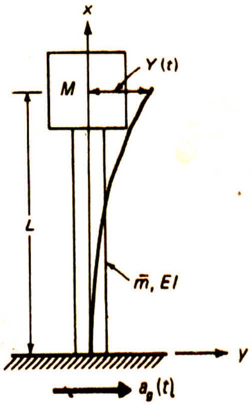
The critical buckling load N_{cr} is defined as the axial compressive load that reduces the combined stiffness to zero, i.e.,

$$K_c^* = K^* - K_G^* = 0$$

$$\Rightarrow \quad \int_0^L EI \left(\frac{d^2\phi}{dx^2}\right)^2 dx - N_{cr} \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx = 0$$

So the critical buckling load

$$N_{cr} = \frac{\int_0^L EI \left(\frac{d^2\phi}{dx^2}\right)^2 dx}{\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx}$$



Let us consider a case similar to a pylon mounted engine/store or may be a case of huge T tail. Let us assume that the pylon has a distributed mass $\bar{m}$ and stiffness EI along its length with a concentrated mass $M = \bar{m}L$ at the top. It is subjected to a support excitation of acceleration $a_g(t)$ and to an axial compressive load due to the weight of its distributed mass and concentrated mass at the top. Find the response of the system (neglect damping)

Let us assume that during the motion the shape of the tower is given by

$$\phi(x) = 1 - cos\frac{\pi x}{2L}$$

Considering the tip displacement Y(t) as the generalized coordinate as shown, the displacement at any point

$$y(x,t) = Y(t)\left(1 - cos\frac{\pi x}{2L}\right)$$

Generalized mass

$$M^* = \int_0^L m(x)\ \phi^2(x)\ dx$$

$$M^* = \bar{m}L + \bar{m}\int_0^L \left(1 - cos\frac{\pi x}{2L}\right)^2 dx = \frac{\bar{m}L}{2\pi}(5\pi - 8)$$

Generalized stiffness

$$K^* = \int_0^L EI(x)\left\{\phi''(x)\right\}^2 dx$$

$$K^* = \int_0^L EI\left(\frac{\pi}{2L}\right)^4 cos^2\frac{\pi x}{2L} dx = \frac{\pi^4 EI}{32L^3}$$

The axial force at any level x, will be due to the vertical portion of the structure (self weight) and the concentrated weight at the top.

$$N(x) = \bar{m}Lg\left(2 - \frac{x}{L}\right)$$

Then the geometric stiffness coefficient

$$K_G^* = \int_0^L \bar{m}Lg\left(2 - \frac{x}{L}\right)\left(\frac{\pi}{2L}\right)^2 \; sin^2\frac{\pi x}{2L}\; dx$$

$$K_G^* = \frac{\bar{m}g}{16}(3\pi^2 - 4)$$

So, combined stiffness

$$K_c^* = K^* - K_G^* = \frac{\pi^4 EI}{32L^3} - \frac{\bar{m}g}{16}(3\pi^2 - 4)$$

Considering u as the relative movement between the top mass and the support, i.e., $u(t) = Y(t) - y_s(t)$
The equation of motion in terms of generalized mass, stiffness and force becomes

$$M^*\ddot{u} + K_c^* u = F_{eff}^*(t)$$

Where the $F_{eff}^*(t)$ get modified due to the coordinate transformation, so

$$F_{eff}^*(t) = \int_0^L p_{eff}(x,t)\phi(x)\ dx - \bar{m}La_g(t)$$

and the $p_{eff}(x,t) = -\bar{m}a_g(t)$ is the inertia force of the continuous column due to the support movement make the generalized force as

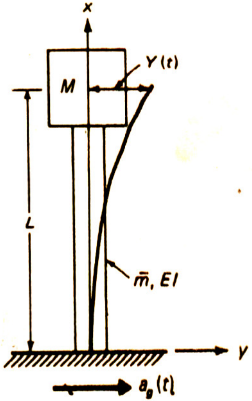
$$F_{eff}^*(t) = \int_0^L -\bar{m}a_g(t)\phi(x)\ dx - \bar{m}La_g(t)$$

Now substituting the value of $\phi(x)$ and after simplification

$$F^*_{eff}(t) = -\frac{2\bar{m}a_g(t)L}{\pi}(\pi - 1)$$

So, the steady state response in terms of relative displacement becomes

$$u(t) = \frac{F^*_{eff}/K^*_c}{1 - r^2}$$



Solve a problem for the following parameters.
$a_g(t) = 20 \sin 6.36\, t$ (in/sec^2)
$\bar{m} = 0.1$ k.sec^2/in per unit length,
$EI = 1.2 \times 10^{13}$ k.in^2,
$L = 100$ ft $= 1200$ in,
$\bar{\omega} = 6.36$ rad/sec.
Find steady state response u(t).

u = -0.217 sin 6.36 t in.