AE31002 Aerospace Structural Dynamics
Rayleigh's Method

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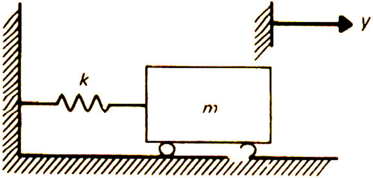
Principle of Conservation of Energy

If no external forces are acting on the system and there is no dissipation of energy due to damping, then the total energy of the system must remain constant during motion and consequently its derivative with respect to time must be equal to zero.

Let us use the the principle of conservation of energy to find out the dynamic equilibrium equation of the SDOF system.



The total energy in this case consists of the sum of kinetic energy and the potential energy of the spring.
In this case the kinetic energy is given by

$$T = \frac{1}{2} m\dot{y}^2$$

The force in the spring, when displaced y unit from the equilibrium position, is ky and the work done by this force on the mass for an additional displacement dy is -kydy (-ve because direction of force and displacements are opposite). However, by definition, the potential energy is the value of this work but with opposite sign.

$$V = \int_0^y ky \ dy = \frac{1}{2} ky^2$$

Now following the principle of conservation of energy

$$\frac{1}{2} m\dot{y}^2 + \frac{1}{2} ky^2 = C_0$$

Taking the time derivative

$$m\dot{y}\ddot{y} + ky\dot{y} = 0 \quad \Rightarrow \quad m\ddot{y} + ky = 0$$

Let us assume the equation of motion in the form
$y = C\ sin\ (\omega t + \alpha)$, leads to velocity as $\dot{y} = \omega C\ cos\ (\omega t + \alpha)$
At the neutral position (y = 0), there will be no force in the spring and the potential energy is zero. Consequently, the max kinetic energy

$$T_{max} = \frac{1}{2} m(\omega C)^2$$

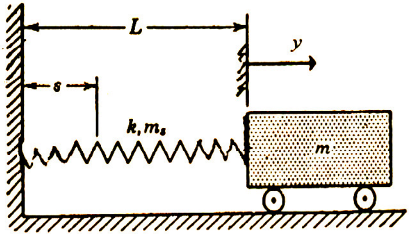
At the maximum displacement the velocity of the mass is zero and all the energy is then potential,

$$V_{max} = \frac{1}{2} kC^2$$

If no energy has been added or lost during the quarter cycle, the two energy expressions must equal.

$$\frac{1}{2}m(\omega C)^2 = \frac{1}{2}kC^2 \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m}}$$

The method, in which the natural frequency is obtained by equating maximum kinetic energy with maximum potential energy, is known as **Rayleigh's Method.**



We did not consider the self weight of the spring in our previous derivation. Let spring has a mass of m_s and its length is L. Displacement of an arbitrary section of the spring at a distance s from the support may now be considered as u = sy/L. Assuming that the motion of mass m is harmonic and given by

$y = C \ sin \ (\omega t + \alpha)$

$$u = \frac{s}{L} y = \frac{s}{L} C \ sin \ (\omega t + \alpha)$$

The max. potential energy of the uniformly stretched spring is

$$V_{max} = \frac{1}{2} k C^2$$

A differential element of the spring of length ds has mass equal to m_s ds/L and max. velocity $\dot{u}_{max} = \omega u_{max} = \omega s C/L$
Consequently the total max. kinetic energy is

$$T_{max} = \int_0^L \frac{1}{2}\frac{m_s}{L} ds \left(\omega \frac{s}{L} C\right)^2 + \frac{1}{2} m \omega^2 C^2$$

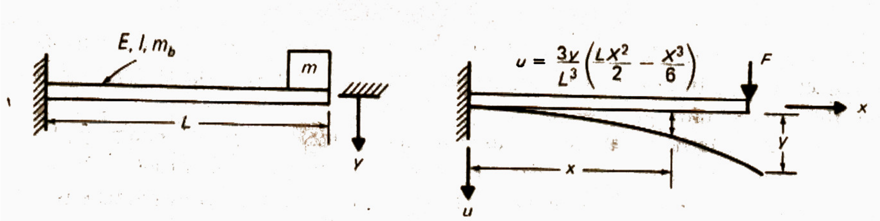
Now equating the max. potential and max. kinetic energy and after simplification

$$\frac{1}{2} k C^2 = \frac{1}{2}\omega^2 C^2 \left(m + \frac{m_s}{3}\right) \quad \Rightarrow \quad \omega = \sqrt{\frac{k}{m + m_s/3}}$$

Observations

- Rayleigh's method may also be used to determine the natural frequency of a continuous system provided that the deformed shape of the structure is described as a generalized coordinate.
- The success of the method depend on how close the assumed deformed shape will come to match the actual shape of the structure during vibration.
- However, if the deformed shape has been defined as the shape resulting from statically applied forces, it would be simpler to calculate the work done by the external force, instead of directly determining the potential energy. In this case the max. kinetic energy is equated to the work of the force applied statically.

Determine the natural frequency of vibration of the cantilever beam with a concentrated mass at its end. The beam has a total mass of m_b and length L.



For a static loading at the tip of a cantilever, the deflection at a distance x from the support is

$$u = \frac{3y}{L^3}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

where y is the deflection at the free end.

Considering harmonic motion tip as $y = C\ sin\ (\omega t + \alpha)$

$$u = \frac{3x^2L - x^3}{2L^3} C\ sin\ (\omega t + \alpha)$$

Potential energy due to the gradually increasing force F, is equal to the work done $= \frac{1}{2}$Fy. So, the max. potential energy

$$V_{max} = \frac{1}{2}FC = \frac{3EI}{2L^3}C^2 \qquad \left[since,\ y_{max} = C = \frac{FL^3}{3EI}\right]$$

The kinetic energy due to the distributed mass of the beam is given by

$$T_d = \int_0^L \frac{1}{2}\left(\frac{m_b}{L}\dot{u}^2\ dx\right)$$

substituting the value of u the max. kinetic energy becomes,

$$T_{max} = T_d + \frac{m\omega^2 C^2}{2} = \frac{m_b}{2L}\int_0^L \left(\frac{3x^2L - x^3}{2L^3}\omega C\right)^2 dx + \frac{m\omega^2 C^2}{2}$$

Now equating the energy quantities

$$\frac{3EI}{2L^3}C^2 = \frac{1}{2}\omega^2 C^2\left(m + \frac{33}{140}m_b\right)$$

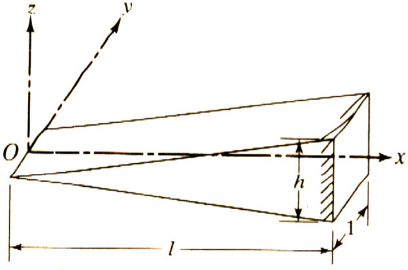
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3EI}{L^3\left(m + \frac{33}{140}m_b\right)}}$$

In practice it is may time approximated as

$$f = \frac{1}{2\pi}\sqrt{\frac{3EI}{L^3\left(m + \frac{m_b}{4}\right)}}$$

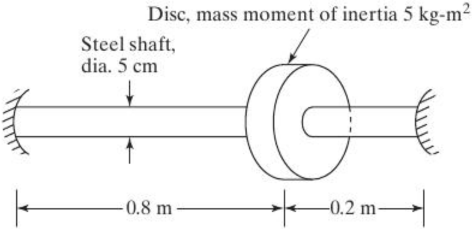
Check value of fundamental frequency with the analytical solution.
1) m $= 0$, 2) m$_b = 0$

Find the fundamental frequency of transverse vibration of the nonuniform cantilever beam shown below using the deflection shape $W(x) = (1 - x/l)^2$.



$$\omega = 1.5811 \left(\frac{Eh^2}{\rho l^4} \right)^{1/2}$$

\# Using Rayeigh's method, determine the fundamental natural frequency of the system



$$\omega^2 = 1923.0292 \times 10^5 \text{ rad/sec}$$