AE31002 Aerospace Structural Dynamics
Approximate Methods

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We have seen in two examples that if the transverse deflection of a beam is given as the multiplication of a shape function and a sinusoidal trigonometric time function, like, w(x,t) = W(x) cos ωt, the kinetic energy becomes

$$T = \frac{1}{2}\int_0^l \dot{w}^2 \; dm = \frac{1}{2}\int_0^l \dot{w}^2 \rho A(x) \; dx$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho A(x) W^2(x) \; dx$$

The potential energy of the beam V is the same as the work done in deforming the beam. Disregarding the work done by shear force,

$$V = \frac{1}{2}\int_0^l M \; d\theta$$

Substituting the value of $M = EI\frac{\partial^2 w}{\partial x^2}$ and $\theta = \frac{\partial w}{\partial x}$

The potential energy become

$$V = \frac{1}{2}\int_0^l EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \, dx \qquad\qquad and$$

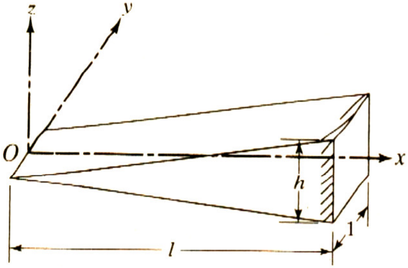
$$V_{max} = \frac{1}{2}\int_0^l EI(x)\left(\frac{d^2 W(x)}{dx^2}\right)^2 \, dx$$

Now equating the max energy expressions we obtain the **Rayleigh's quotient**

$$R(\omega) = \omega^2 = \frac{\displaystyle\int_0^l EI(x)\left(\frac{d^2 W(x)}{dx^2}\right)^2 \, dx}{\displaystyle\int_0^l \rho A(x) W^2(x) \, dx}$$

$$R(\omega) = \omega^2 = \frac{\displaystyle\int_0^{l_1} E_1 I_1 \left(\frac{d^2 W(x)}{dx^2}\right)^2 \, dx + \int_{l_1}^{l_2} E_2 I_2 \left(\frac{d^2 W(x)}{dx^2}\right)^2 \, dx + \dots}{\displaystyle\int_0^{l_1} \rho A_1(x) W^2(x) \, dx + \int_{l_1}^{l_2} \rho A_2(x) W^2(x) \, dx + \dots}$$

Find the fundamental frequency of **transverse vibration** of the nonuniform cantilever beam shown below using the deflection shape W(x) = $(1 - x/l)^2$.



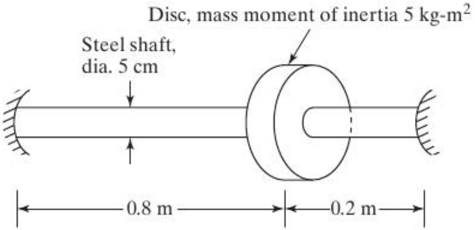
The cross sectional area and the moment of inertia of the transverse cross section about centroidal axis are

$$A(x) = \frac{hx}{l} \qquad and \qquad I(x) = \frac{1}{12}\left(\frac{hx}{l}\right)^3$$

From the Rayleigh's quotient

$$\omega^2 = \frac{\displaystyle\int_0^l E\frac{1}{12}\left(\frac{hx}{l}\right)^3\left(\frac{2}{l^2}\right)^2\, dx}{\displaystyle\int_0^l \rho\left(\frac{hx}{l}\right)\left(1-\frac{x}{l}\right)^4\, dx} = 2.5\frac{Eh^2}{\rho l^4}$$

Using Rayleigh's method, determine the fundamental natural frequency of the system



For a shaft under torsion, the shear stress τ at a distance r from the center of the shaft is given by

$$\tau = \frac{M_t(x)\ r}{J}$$

Potential energy in terms of strain energy is

$$V = \frac{1}{2}\int \left(\frac{\tau}{G}\cdot \tau dA\right)\ dx = \frac{1}{2}\int \left(\frac{\tau^2}{G} dA\right)\ dx$$

As we know from the torsion of circular shaft

$$M_t(x) = GJ\frac{\partial \theta}{\partial x} \quad \Rightarrow \quad V = \frac{1}{2}\int_0^l GJ\left(\frac{\partial \theta}{\partial x}\right)^2\ dx$$

Kinetic energy of the shaft can be written as

$$T = \frac{1}{2}\int_0^l \rho J\left(\frac{\partial \theta}{\partial t}\right)^2\ dx$$

Let us assume a harmonic variation of rotational displacement function θ(x,t) as

$$\theta(x,t) = \Theta(x)\ \cos(\omega t)$$

Now equating V_{max} and T_{max}

$$\omega^2 = \frac{\displaystyle\int_0^l GJ\left(\frac{\partial\Theta(x)}{\partial x}\right)^2\ dx}{\displaystyle\int_0^l \rho J\ (\Theta(x))^2\ dx}$$

For a steel shaft G $= 80 \times 10^9$ N/m^2; and ρ g $= 75.9$ kN/m^3

$$J = \frac{\pi d^4}{32} = \frac{\pi}{32}(0.05)^4 = 61.3594 \times 10^{-8} m^4$$

Let us assume that Θ(x) varies linearly on either side of the lumped mass.

$$\Theta(x) = \frac{\theta_0 x}{0.8} \quad and \quad \frac{\partial \Theta}{\partial x} = \frac{\theta_0}{0.8} \quad ; \qquad 0 \leq x \leq 0.8$$

$$\Theta(x) = \frac{\theta_0(1-x)}{0.2} \quad and \quad \frac{\partial \Theta}{\partial x} = -\frac{\theta_0}{0.2} \quad ; \quad 0.8 \leq x \leq 1$$

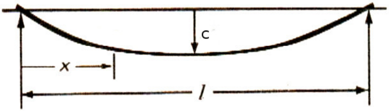
$$\int_0^l GJ \left(\frac{\partial \Theta(x)}{\partial x}\right)^2 \; dx = 306797\theta_0^2$$

$$\int_0^l \rho J \; (\Theta(x))^2 \; dx = 159.54 \times 10^{-5}\theta_0^2$$

$$\omega^2 = 1923.0292 \times 10^5 \text{ rad/sec}$$

Find the fundamental frequency of a simply supported beam using the deflection pattern

$$W(x) = \left(C\ sin\ \frac{\pi x}{l}\right) sin\ \omega t$$



The second derivative of the deflection function

$$\frac{d^2W(x)}{dx^2} = -\left(\frac{\pi}{l}\right)^2 C\ sin\ \frac{\pi x}{l}\ sin\ \omega t$$

From the Rayleigh quotient

$$\omega^2 = \frac{EI\left(\frac{\pi}{l}\right)^4 \int_0^l sin^2 \frac{\pi x}{l} dx}{m \int_0^l sin^2 \frac{\pi x}{l} dx} = \pi^4 \frac{EI}{ml^4}$$

$$\omega = \pi^2 \sqrt{\frac{EI}{ml^4}} \qquad exact\ solution$$

It may be proved that for inconsistent assumption of deflected shape the estimated fundamental frequency becomes higher than the exact natural frequency.

- Rayleigh-Ritz method is an extension of the Rayleigh's energy method.
- Here we assume more shape functions or approximate deflection/representation of the structure to get more accurate frequencies and mode shapes.
- An arbitrary number of functions can be used to obtain that many number of frequencies and mode shapes.
- It also increases computation cost.
- If n arbitrary functions are chosen to describe the transverse vibration of beam the generalized deflection becomes

$$W(x) = c_1 w_1(x) + c_2 w_2(x) + \cdots + c_n w_n(x)$$

  where $w_1(x), \; w_2(x), \; \ldots \; w_n(x)$ are the **admissible function of the spatial variable x, which satisfy the boundary condition of the structure.**
- The constants c_i are the arbitrary constants to be determined to have best possible mode shapes in combination of $w_i(x)$.

- To obtain the constants c_i, natural frequency is made stationary at natural modes.
- **The partial derivative of Rayleig quotient w.r.t. constants** c_i are made to zero.

$$\frac{\partial(\omega^2)}{\partial c_i} = 0, \qquad i = 1, 2, 3, \ldots, n$$

- The above equation denotes a set of n linear algebraic equation in the coefficients $c_1, c_2, \ldots, c_n$ and also contains the undetermined quantity ω^2.
- It is an eigenvalue problem to yield n natural frequencies and n natural modes.
- The i-th mode with respect to the i-th natural frequency

$$\{C^{(i)}\} = \left\{ c_1^{(i)} \quad c_2^{(i)} \quad c_3^{(i)} \quad \ldots \quad c_n^{(i)} \right\}^T$$

Find the natural frequencies of transverse vibration of the nonuniform cantilever beam shown below using the deflection shapes

$$w_1(x) = \left(1 - \frac{x}{l}\right)^2 \; and \; w_2(x) = \frac{x}{l}\left(1 - \frac{x}{l}\right)^2$$

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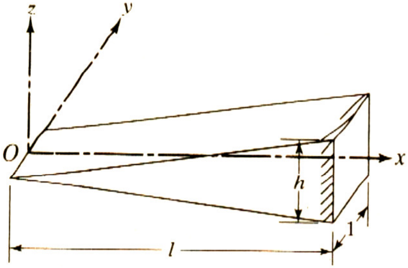

The cross sectional are and the moment of inertia of the transverse cross section about centroidal axis are

$$A(x) = \frac{hx}{l} \qquad and \qquad I(x) = \frac{1}{12}\left(\frac{hx}{l}\right)^3$$

$$W(x) = c_1 \left(1 - \frac{x}{l}\right)^2 + c_2 \frac{x}{l}\left(1 - \frac{x}{l}\right)^2$$

Rayleigh's quotient

$$R(W(x)) = \omega^2 = \frac{\int_0^l EI(x)\left(\frac{d^2W(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A(x) W^2(x)\, dx} = \frac{X}{Y}$$

The condition that make ω^2 or R(W(x)) stationary are

$$\frac{\partial(\omega^2)}{\partial c_1} = \frac{Y\frac{\partial X}{\partial c_1} - X\frac{\partial Y}{\partial c_1}}{Y^2} = 0$$

$$\frac{\partial(\omega^2)}{\partial c_2} = \frac{Y\frac{\partial X}{\partial c_2} - X\frac{\partial Y}{\partial c_2}}{Y^2} = 0$$

May be rewritten as

$$\frac{\partial X}{\partial c_1} - \frac{X}{Y}\frac{\partial Y}{\partial c_1} = \frac{\partial X}{\partial c_1} - \omega^2\frac{\partial Y}{\partial c_1} = 0$$

$$\frac{\partial X}{\partial c_2} - \frac{X}{Y}\frac{\partial Y}{\partial c_2} = \frac{\partial X}{\partial c_2} - \omega^2\frac{\partial Y}{\partial c_2} = 0$$

Evaluating the X and Y,

$$X = \frac{Eh^3}{3l^3}\left(\frac{c_1^2}{4} + \frac{c_2^2}{10} + \frac{c_1 c_2}{5}\right) \text{ and } Y = \rho h l\left(\frac{c_1^2}{30} + \frac{c_2^2}{280} + \frac{2c_1 c_2}{105}\right)$$

The algebraic equations become

$$\begin{bmatrix} \left(\frac{1}{2} - \bar{\omega}^2 \frac{1}{15}\right) & \left(\frac{1}{5} - \bar{\omega}^2 \frac{2}{105}\right) \\ \left(\frac{1}{5} - \bar{\omega}^2 \frac{2}{105}\right) & \left(\frac{1}{5} - \bar{\omega}^2 \frac{2}{140}\right) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where, $\bar{\omega}^2 = \frac{3\omega^2 \rho l^4}{Eh^2}$ By setting determinant equal to zero

$$\frac{1}{8820}\bar{\omega}^4 - \frac{13}{1400}\bar{\omega}^2 + \frac{3}{50} = 0$$

$$\bar{\omega}_1 = 2.6599 \quad \Rightarrow \quad \omega_1 \simeq 1.5367 \left(\frac{Eh^2}{\rho l^4}\right)^{1/2}$$

$$\bar{\omega}_1 = 8.6492 \quad \Rightarrow \quad \omega_1 \simeq 4.9936 \left(\frac{Eh^2}{\rho l^4}\right)^{1/2}$$