AE31002 Aerospace Structural Dynamics
Nonlinear Structural Response

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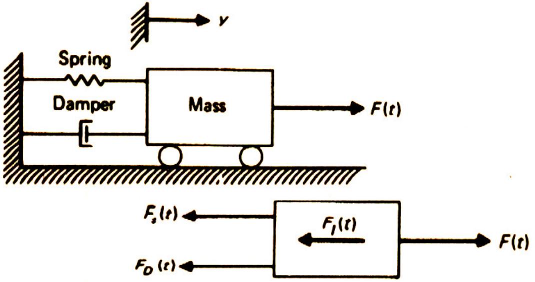
- Restoring force is proportional to the displacement.
- Damping force is proportional to the velocity.
- Mass in the model is always considered to be unchanging.

A linear second order ordinary differential equation.

$$m\ddot{y} + c\dot{y} + ky = F(t)$$

For harmonic forcing functions or some particular one it is simple to solve .
There are, however, physical situations for which this linear model does not adequately represent the dynamic characteristics of the structure. The analysis in such cases requires the introduction of model in which the spring force or the damping force may not remain proportional, respectively, to the displacement or to the velocity. Consequently, the resulting equation of motion will no longer be linear and its mathematical solution, in general, will have much greater complexity, often requiring a numerical procedure for its integration.



At time t_i the dynamic equilibrium equation is

$$F_I(t_i) + F_D(t_i) + F_s(t_i) = F(t_i) \tag{1}$$

at a short time Δt later

$$F_I(t_i + \Delta t) + F_D(t_i + \Delta t) + F_s(t_i + \Delta t) = F(t_i + \Delta t) \tag{2}$$

Subtracting the above equations we get differential equation of motion in terms of increments,

$$\Delta F_I + \Delta F_D + \Delta F_s = \Delta F_i \tag{3}$$

where

$$\begin{aligned}
\Delta F_I &= F_I(t_i + \Delta t) - F_I(t_i) \\
\Delta F_D &= F_D(t_i + \Delta t) - F_D(t_i) \\
\Delta F_s &= F_s(t_i + \Delta t) - F_s(t_i) \\
\Delta F_i &= F(t_i + \Delta t) - F(t_i)
\end{aligned} \tag{4}$$

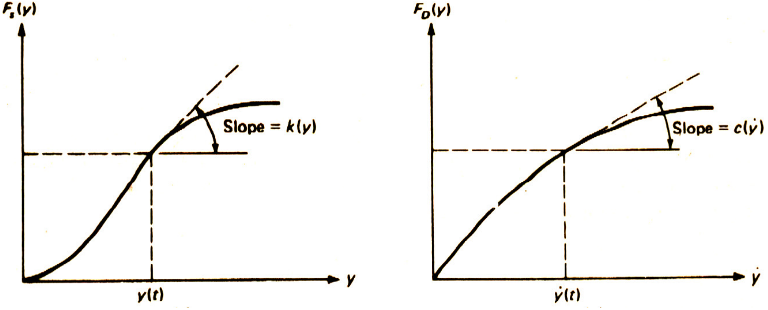


Figure : a) Nonlinear stiffness b) Nonlinear damping

Let us assume that the damping force spring force are nonlinear functions of time as shown, while the inertial force remains proportional to the acceleration.

$$\begin{aligned} \Delta F_I &= m\Delta\ddot{y}_i \\ \Delta F_D &= c_i\Delta\dot{y}_i \\ \Delta F_s &= k_i\Delta y_i \end{aligned} \tag{5}$$

where the incremental displacement, velocity and acceleration are

$$\begin{aligned} \Delta y_i &= y(t_i+\Delta t) - y(t_i) \\ \Delta\dot{y}_i &= \dot{y}_i(t_i+\Delta t) - \dot{y}_i(t_i) \\ \Delta\ddot{y}_i &= \ddot{y}_i(t_i+\Delta t) - \ddot{y}_i(t_i) \end{aligned} \tag{6}$$

The coefficient k_i and c_i are defined as the current evaluation for the derivative of the spring force and damping, respectively, with respect to the displacement, i.e., $k_i = \left(\frac{dF_s}{dy}\right)_{y=y_i}$ and $c_i = \left(\frac{dF_D}{d\dot{y}}\right)_{\dot{y}=\dot{y}_i}$ and the final equation to be solves is

$$m\Delta\ddot{y}_i + c_i\Delta\dot{y}_i + k_i\Delta y_i = \Delta F_i \tag{7}$$

where the coefficients k_i and c_i are assumed to remain constant during the increment of time Δt.

Step-by-Step Integration

- Response is evaluated at successive increments Δt of time usually taken of equal length of time for computational convenience.
- At the beginning of each interval, the condition of dynamic equilibrium is established.
- The response for a time increment Δt is evaluated approximately on the basis that stiffness and damping remain constant during the interval Δt.
- The non-linearity of the coefficients are considered in the analysis by reevaluating these coefficients at the beginning of each time interval.
- The response is then obtained using the displacement and velocity calculated at the end of the time interval as the initial condition for the next time step.

observations

- The nonlinear behavior of the of the system is approximated by a sequence of successively changing linear systems.
- The assumption of constant mass is unnecessary.
- Two most popular methods are a) constant acceleration method and b) linear acceleration method.
- Constant acceleration method is simple and less accurate.

Linear Acceleration Step-by-Step Method

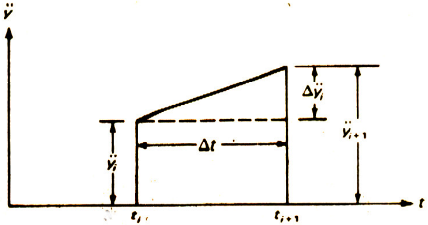


Figure : Assumed linear vibration of acceleration during time interval Δt.

$$\ddot{y}(t) = \ddot{y}_i + \frac{\Delta \ddot{y}_i}{\Delta t}(t - t_i) \tag{8}$$

where $\Delta \ddot{y}_i = \ddot{y}_i(t_i + \Delta t) - \ddot{y}_i(t_i)$.

Integrating twice w.r.t. time between t_i and t yields

$$\dot{y}(t) = \dot{y}_i + \ddot{y}_i(t - t_i) + \frac{1}{2}\frac{\Delta \ddot{y}_i}{\Delta t}(t - t_i)^2 \tag{9}$$

$$y(t) = y_i + \dot{y}_i(t - t_i) + \frac{1}{2}\ddot{y}_i(t - t_i)^2 + \frac{1}{6}\frac{\Delta \ddot{y}_i}{\Delta t}(t - t_i)^3 \tag{10}$$

The evaluation of the above equations at time $t = t_i + \Delta t$ gives

$$\Delta \dot{y}_i = \ddot{y}_i \Delta t + \frac{1}{2}\Delta \ddot{y}_i \Delta t \tag{11}$$

$$\Delta y_i = \dot{y}_i \Delta t + \frac{1}{2}\ddot{y}_i \Delta t^2 + \frac{1}{6}\Delta \ddot{y}_i \Delta t^2 \tag{12}$$

Where $\Delta y_i = y(t_i + \Delta t) - y(t_i)$ and $\Delta \dot{y}_i = \dot{y}_i(t_i + \Delta t) - \dot{y}_i(t_i)$

Now to use the incremental displacement Δy as the basic variable in the analysis, above equation is solved for incremental acceleration $\Delta \ddot{y}_i$ and substituted in the incremental velocity equation.

$$\Delta \ddot{y}_i = \frac{6}{\Delta t^2} \Delta y_i - \frac{6}{\Delta t} \dot{y}_i - 3\ddot{y}_i \tag{13}$$

$$\Delta \dot{y}_i = \frac{3}{\Delta t} \Delta y_i - 3\dot{y}_i - \frac{\Delta t}{2} \ddot{y}_i \tag{14}$$

The substitution of eqn. (13) and eqn. (14) in eqn. (7) leads to the following form

$$m\left\{\frac{6}{\Delta t^2} \Delta y_i - \frac{6}{\Delta t} \dot{y}_i - 3\ddot{y}_i\right\} + c_i\left\{\frac{3}{\Delta t} \Delta y_i - 3\dot{y}_i - \frac{\Delta t}{2} \ddot{y}_i\right\} + k_i \Delta y_i = \Delta F_i \tag{15}$$

The above equation simplifies to

$$\bar{k}_i \Delta y_i = \Delta \bar{F}_i \tag{16}$$

where

$$\bar{k}_i = k_i + \frac{6m}{\Delta t^2} + \frac{3c_i}{\Delta t} \tag{17}$$

$$\Delta \bar{F}_i = \Delta F_i + m\left\{\frac{6}{\Delta t}\dot{y}_i + 3\ddot{y}_i\right\} + c_i\left\{3\dot{y}_i + \frac{\Delta t}{2}\ddot{y}_i\right\} \tag{18}$$

The eqn. (16) can be solved with the known values of velocity and acceleration at t_i time.
To obtain the displacement $y_{i+1} = y(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$, this value of Δy_i is substituted in eqn. (6) to yield

$$y_{i+1} = y_i + \Delta y_i \tag{19}$$

The incremental velocity $\Delta \dot{y}_i$ can be found out from the eqn. (14) and the velocity at time $t_{i+1} = t_i + \Delta t$ from eqn. (6)

$$\dot{y}_{i+1} = \dot{y}_i + \Delta \dot{y}_i \tag{20}$$

Finally the acceleration $\ddot{y}_{i+1}$ is found out from the fist equation of motion eqn. (1), with the consideration that $F_I = m\ddot{y}_{i+1}$

$$\ddot{y}_{i+1} = \frac{1}{m} \{F(t_{i+1}) - F_{D,i+1} - F_{S,i+1}\} \tag{21}$$

where $F_{D,i+1}$ and $F_{S,i+1}$ are already evaluated for the i+1 time step.

Selection of time step Δt

1. The natural period of the structure.
2. the rate of variation of loading function and
3. the complexity of the stiffness and damping functions.
4. In general it has been found that sufficiently accurate results can be obtained if the time interval is taken to **no longer than one-tenth of the natural period of the structure**.